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SOME EXAMPLES OF ALMOST GCD-DOMAINS

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Abstract. Let D be an integral domain, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D. We show that D is an almost weakly factorial PvMD if and only if $D + XD_S[X]$ is an integrally closed almost GCD-domain for each (saturated) multiplicative subset S of D, if and only if $D + XD_1[X]$ is an integrally closed almost GCD-domain for any t-linked overring D_1 of D, if and only if $D_1 + XD_2[X]$ is an integrally closed almost GCD-domain for all t-linked overrings $D_1 \subseteq D_2$ of D.

1. Introduction

Let D be an integral domain, K be the quotient field of D, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D. Let $c(f)$ denote the ideal of D generated by the coefficients of a polynomial $f \in D[X]$. An overring of D means a ring between D and K.

1.1. Definitions

Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For any $A \in \mathbf{F}(D)$, let $A^{-1} = \{x \in K | xA \subseteq D\}$, $A_v = (A^{-1})^{-1}$, and $A_t = \cup\{I_v | I \subseteq A$ is a nonzero finitely generated fractional ideal of $D\}$. An $A \in \mathbf{F}(D)$ is called a *divisorial ideal* (resp., *t-ideal*) if $A_v = A$ (resp., $A_t = A$), while an integral t-ideal A is a *maximal t-ideal* if A is maximal among proper integral t-ideals of D. One can easily show that each maximal t-ideal is a prime ideal; each proper integral t-ideal is contained in a maximal t-ideal; and D has at least one maximal t-ideal if D is not a field. An $I \in \mathbf{F}(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$. The *class group* of D is an ableian group $Cl(D) = T(D)/Prin(D)$, where $T(D)$ is the group of t-invertible fractional t-ideals of D under the t-multiplication $I * J = (IJ)_t$ and $Prin(D)$ is the subgroup of $T(D)$ of nonzero principal fractional ideals.

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An overring R of D is said to be *t-linked over* D if $I^{-1} = D$ for a finitely generated ideal I of D implies $(IR)^{-1} = R$. It is known that R is t-linked over D if and only if $(Q \cap D)_t \subset D$ for each prime t-ideal Q of R [14, Proposition 2.1], if and only if $R[X]_{N_v} \cap K = R$, where $N_v = \{f \in D[X] | c(f)_v = D\}$ [12, Lemma 3.2]. We know that if R is an overring of D, then $R[X]_{N_v} \cap K$ is the smallest t-linked overring of D containing R [12, Remark 3.3].

We say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. Following [15], we say that D has the tQR-property if each t-linked overring of D is a ring of fractions of D. Clearly, if D has the tQR-property, then D is a PvMD. Also, if D is a PvMD with $Cl(D)$ torsion, then D has the tQR-property [14, Theorem 1.3].

As in [18], we say that D is an *almost GCD-domain* (AGCD-domain) if for each $0 \neq a, b \in D$, there is an integer $n \geq 1$ such that $a^nD \cap b^nD$ is principal. Clearly, a GCD-domain is an AGCD-domain, but $\mathbb{Z}_2[X^2, X^3]$ is an AGCD-domains that is not a GCD-domain (cf. [11, Lemma 3.2]). It is well known that an integrally closed domain D is an AGCD-domain if and only if D is a PvMD and $Cl(D)$ is torsion, if and only if for any $0 \neq a, b \in D$, there is an integer $n = n(a, b) \geq 1$ such that $(a^n, b^n)_v$ is principal, if and only if $D[X]$ is an AGCD-domain [18, Theorems 3.9 and 5.6]. In particular, each t-linked overring of an integrally closed AGCD-domain D is a quotient ring of D, because $Cl(D)$ is torsion.

Let $X^1(D)$ be the set of height-one prime ideals of D. We say that D is a *weakly Krull domain* if (i) $D = \cap_{P \in X^1(D)} D_P$ and (ii) the intersection $D = \cap_{P \in X^1(D)} D_P$ is locally finite, i.e., each nonzero nonunit of D is contained in only a finite number of prime ideals in $X^1(D)$. A nonzero element $a \in D$ is said to be *primary* if aD is a primary ideal. As in [5], we will call D a *weakly factorial domain* (WFD) if each nonzero nonunit of D is a product of primary elements. It is known that D is a WFD if and only if D is a weakly Krull domain and $Cl(D) = 0$ [7, Theorem]. As in [6], D is called an *almost weakly factorial domain* (AWFD) if for each nonzero nonunit $x \in D$, there is an integer $n = n(x) \geq 1$ such that x^n is a product of primary elements. It is known that D is an AWFD if and only if D is a weakly Krull domain and $Cl(D)$ is torsion [6, Theorem 3.4]. We know that any localization of a weakly Krull domain (resp., WFD, AWFD) is a weakly Krull domain (resp., WFD, AWFD) [9, Lemma 2.1].

Let S be a saturated multiplicative subset of D, and let $N(S) = \{0 \neq x \in D | (x, s)_v = D$ for all $s \in S\}$. We say that S is a *splitting set* if for each $0 \neq d \in D$, we can write $d = sa$ for some $s \in S$ and $a \in N(S)$.

The S is called an *almost splitting set* of D if for each $0 \neq d \in D$, there is an integer $n \geq 1$ such that $d^n = sa$ for some $s \in S$ and $a \in N(S)$. Clearly, a splitting set is an almost splitting set. It is known that each saturated multiplicative subset of D is a splitting set (resp., an almost splitting set) if and only if D is a WFD (resp., an AWFD) [7, Theorem] (resp., [4, Theorem 2.11]). The notation and terminology used in this paper are standard as in [16] or [17].

1.2. Motivations and results

Let $A \subseteq B$ be an extension of integral domains and X be an indeterminate over B. Clearly, $A + XB[X]$ is a subring of the polynomial ring $B[X]$. It is known that $A+XB[X]$ is a GCD-domain if and only if A is a GCD-domain and $B = A_S$ for a splitting set S of A [10, Theorem 2.10]. Also, $D[X]$ is a WFD if and only if D is a weakly factorial GCD-domain [5, Theorem 17]. We know that every saturated multiplicative subset of D is a splitting set if and only if D is a WFD. So D is a weakly factorial GCD-domain if and only if $D + XD_S[X]$ is a GCD-domain for every saturated multiplicative subset S of D [3, Theorem 10].

In this paper, we study when $D + XD_S[X]$ is an integrally closed AGCD-domain for every saturated multiplicative subset S of D. Precisely, we show that $D[X]$ is an integrally closed AWFD if and only if D is an almost weakly factorial PvMD, if and only if $D+XD_S[X]$ is an integrally closed AGCD-domain for each multiplicative subset S of D, if and only if $D + XD_1[X]$ is an integrally closed AGCD-domain for each t-linked overring D_1 of D, if and only if $D_1 + XD_2[X]$ is an integrally closed AGCD-domain for all t-linked overrings $D_1 \subseteq D_2$ of D.

2. Almost weakly factorial AGCD-Domain

Let D be an integral domain, X be an indeterminate over D, $D[X]$ be a polynomial ring over D. Obviously, if $D_1 \subseteq D_2$ are overrings of D, then $D_1 + XD_2[X]$ is an overring of $D[X]$.

Lemma 2.1. 1. *If D' is an overring of D, then $D + XD'[X]$ is integrally closed if and only if D and D' are integrally closed.*
2. *D is a PvMD if and only if $D + XD_1[X]$ is integrally closed for each t-linked overring D_1 of D.*

Proof. (1) [2, Theorem 2.7]. (2) This follows from (1), because D is a PvMD if and only if each t-linked overring of D is integrally closed [14, Theorem 2.10]. □

An integral domain D is called a *generalized Krull domain* if (i) $D = \cap_{P \in X^1(D)} D_P$, (ii) D_P is a valuation domain for each $P \in X^1(D)$, and (iii) the intersection $D = \cap_{P \in X^1(D)} D_P$ is locally finite. It is clear that D is a generalized Krull domain if and only if D is a weakly Krull PvMD.

We next give the main result of this paper.

THEOREM 2.2. *The following statements are equivalent for an integral domain D.*

1. *$D[X]$ is an integrally closed AWFD.*
2. *D is an almost weakly factorial PvMD.*
3. *$D + XD_S[X]$ is an integrally closed AGCD-domain for each saturated multiplicative subset S of D.*
4. *For each t-linked overring D_1 of D, $D + XD_1[X]$ is an integrally closed AGCD-domain.*
5. *For any t-linked overrings $D_1 \subseteq D_2$ of D, $D_1 + XD_2[X]$ is an integrally closed AGCD-domain.*
6. *D is a generalized Krull domain and $Cl(D)$ is torsion.*

Proof. If D is a field, then the result is clear. So we assume that D is not a field.

(1) $\Leftrightarrow$ (2) [8, Theorem 3.3].

(2) $\Rightarrow$ (3) We first note that S is an almost splitting set [4, Theorem 2.11], because D is an AWFD. Next, note that D and D_S are integrally closed AGCD-domain, and so $D_S[X]$ is an AGCD-domain [18, Theorem 5.6]. Thus $D+XD_S[X]$ is an integrally closed AGCD-domain by Lemma 2.1 and [4, Theorem 3.10].

(3) $\Rightarrow$ (2) If $D+XD_S[X]$ is an integrally closed AGCD-domain, then S is an almost splitting set and D is an integrally closed AGCD-domain by Lemma 2.1 and [4, Theorem 3.10]. Hence D is an AWFD [4, Theorem 2.11] and D is a PvMD.

(2) $\Rightarrow$ (5) Let $D_1 \subseteq D_2$ be t-linked overrings of D. Note that an integrally closed AGCD-domain has the tQR-property; so $D_i = D_{N_i}$ for some multiplicative subsets N_i of D. Also, note that $a^n D_{N_1} \cap b^n D_{N_1} = (a^n D \cap b^n D) D_{N_1}$ for any $a, b \in D$ and an integer $n \geq 1$ (so D_{N_1} is an AGCD-domain); D_{N_1} is an integrally closed AWFD by (2); and $D_2 = D_{N_2} = (D_{N_1})_{N_2} = (D_1)_{N_2}$. Thus $D_1 + XD_2[X]$ is an integrally closed AGCD-domain by the implication (2) $\Rightarrow$ (3) above.

(5) $\Rightarrow$ (4) This is clear, because D is a t-linked overring of D itself.

(4) $\Rightarrow$ (3) Let S be a multiplicative subset of D. Then D_S is t-linked over D [14, Proposition 2.2], and thus $D + XD_S[X]$ is an integrally closed AGCD-domain by (4).

(2) $\Leftrightarrow$ (6) This follows, because D is an AWFD if and only if D is a weakly Krull domain and $Cl(D)$ is torsion. $\square$

Let $A \subset B$ be an extension of integral domains, and let X be an indeterminate over B. Then $R = A + XB[X]$ is an integrally closed AGCD-domain and $\mathrm{char}(A) \neq 0$ if and only if $A + X^2B[X]$ is an AGCD-domain with integral closure R [11, Corollary 3.5]. Thus, by Theorem 2.2, we have

Corollary 2.3. *Let D be an almost weakly factorial PvMD with char$(D) \neq 0$. If $D_1 \subseteq D_2$ are t-linked overrings of D, then $D_1 + X^2D_2[X]$ is an AGCD-domain.*

We next give three interesting examples.

Example 2.4. Let $A \subset B$ be an extension of integral domains, $R = A + XB[X]$, $\mathbb{Z}$ be the ring of integers, and $\mathbb{Q}$ be the field of rational numbers. For $m \in \mathbb{Z}$, let E be the integral closure of $\mathbb{Z}[\sqrt{m}]$ in $\mathbb{Q}(\sqrt{m})$ such that E is not a principal ideal domain (for example, $m = -17, -15, -14, -13, -10, -5, 10, 15, 26$ [1, pages 325-326]).

(1) We know that E is a Dedekind domain with $Cl(E)$ torsion. So E is an almost weakly Krull PvMD, and thus $E[X]$ is an integrally closed AWFD by Theorem 2.2.

(2) It is known that if R is a weakly Krull domain, then $qf(A) \cap B = A$, where $qf(A)$ is the quotient field of A, [9, Theorem 3.4]. So in Theorem 2.2, if $D_1 \subset D_2$, then $D_1 + XD_2[X]$ is an integrally closed AGCD-domain but not an AWFD. For example, if $A = E$ and $B = \mathbb{Q}(\sqrt{m})$, then R is an integrally closed AGCD-domain but R is not an AWFD.

(3) Let $\{X_\alpha\}$ be a nonempty set of indeterminates over A, $N = \{f \in A[\{X_\alpha\}] | c(f)_v = A\}$, and $B = A[\{X_\alpha\}]_N$. Then R is an integrally closed AWFD if and only if A is an integrally closed almost weakly factorial AGCD-domain by Lemma 2.1(1) and [9, Corollary 3.10]. Thus if $A = E$, then R is an integrally closed AWFD but R is not an AGCD-domain.

We end this paper by studying the property of the ring $D + XD_1[X]$ for each t-linked overring D_1 of a GCD-domain D. To do this, we recall that an element x of D is *primal* if whenever x divides y_1y_2, with $y_1, y_2 \in D$, then $x = z_1z_2$ where z_1 divides y_1 and z_2 divides y_2. An integrally closed domain in which each element is primal is called a *Schreier domain*. The notion of Schreier domains was introduced by Cohen [13]. It was shown that a GCD-domain is a Schreier domain, and if D is Schreier, then $D[X]$ is also Schreier [13, Theorems 2.4 and 2.7].

Proposition 2.5. *The following statements are equivalent for a Schreier domain D.*

1. *D is a GCD-domain.*
2. *D has the tQR-property.*
3. *D is a PvMD.*
4. *$D + XD_1[X]$ is a Schreier domain for each t-linked overring D_1 of D.*
5. *$D + XD_1[X]$ is integrally closed for each t-linked overring D_1 of D.*

Proof. (1) $\Rightarrow$ (2) This follows from [15, Theorem 1.3], because a GCD-domain D is a PvMD with $Cl(D) = 0$.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (1) Assume that I is a nonzero finitely generated ideal of D. Then I is t-invertible, and hence $I^{-1} = J_v$ for some nonzero finitely generated ideal J of D. Thus I_v is principal [19, Corollary 3.7].

(3) $\Leftrightarrow$ (5) This follows from Lemma 2.1(2).

(2) $\Rightarrow$ (4) Let D_1 be a t-linked overring of D. Then $D_1 = D_S$ for some multiplicative subset S of D, and since D is a GCD-domain by the (1) $\Leftrightarrow$ (2) above, $D + XD_1[X]$ is a Schreier domain [20, Proposition 4.5].

(4) $\Rightarrow$ (5) This follows, because a Schreier domain is integrally closed. □

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References

[1] S. Alaca and K. S. Williams, *Introductory Algebraic Number Theory*, Cambridge Univ. Press, Cambridge, 2004.

[2] D. D. Anderson, D.F. Anderson, and M. Zafrullah, *Rings between $D[X]$ and $K[X]$*, Houston J. Math. **17** (1991), 109-129.

[3] D. D. Anderson, D.F. Anderson, and M. Zafrullah, *A generalization of unique factorization*, Bollettino U.M.I. (7) 9-A (1995), 401-413.

[4] D. D. Anderson, T. Dumitrescu, and M. Zafrullah, *Almost splitting sets and AGCD domains*, Comm. Algebra **32** (2004), 147-158.

[5] D. D. Anderson and L.A. Mahaney, *On primary factorization,* J. Pure Appl. Algebra **54** (1988), 141-154.

[6] D. D. Anderson, J. Mott and M. Zafrullah, *Finite character representations for integral domains*, Bollettino U.M.I. **6** (1992), 613-630.

[7] D. D. Anderson and M. Zafrullah, *Weakly factorial domains and groups of divisibility*, Proc. Amer. Math. Soc. **109** (1990), 907-913.

[8] D. F. Anderson, G.W. Chang, and J. Park, *Generalized weakly factorial domains*, Houston J. Math. **29** (2003), 1-13.

[9] D. F. Anderson, G.W. Chang, and J. Park, *Weakly Krull and related domains of the form $D+M$, $A+XB[X]$, and $A+X^2B[X]$*, Rocky Mountain J. Math. **36** (2006), 1-22.

[10] D. F. Anderson and D. El Abidine, *The $A+XB[X]$ and $A+XB[[X]]$ constructions from GCD-domains*, J. Pure Appl. Algebra **159** (2001), 15-24.

[11] G. W. Chang, *Almost splitting sets in integral domains*, J. Pure Appl. Algebra **197** (2005), 279-292.

[12] G. W. Chang, *Strong Mori domains and the ring $D[X]_{N_v}$*, J. Pure Appl. Algebra **197** (2005), 293-304.

[13] P. M. Cohn, *Bezout rings and their subrings*, Proc. Cambridge Phil. Soc. **64** (1968), 251-264.

[14] D. Dobbs, E. Houston, T. Lucas, and M. Zafrullah, *t-linked overrings and Prüfer v-multiplication domains*, Comm. Algebra **17** (1989), 2835-2852.

[15] D. Dobbs, E. Houston, T. Lucas, and M. Zafrullah, *t-linked overrings as intersections of localizations*, Proc. Amer. Math. Soc. **109** (1990), 637-646.

[16] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.

[17] I. Kaplansky, *Commutative rings, Revised Ed.*, Univ. of Chicago, Chicago, 1974.

[18] M. Zafrullah, *A general theory of almost factoriality*, Manuscripta Math. **51** (1985), 29-62.

[19] M. Zafrullah, *On a property of pre-Schreier domains*, Comm. Algebra **15** (1987), 1895-1920.

[20] M. Zafrullah, *Various facets of rings between $D[X]$ and $K[X]$*, Comm. Algebra **31** (2003), 2497-2540.

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