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AN ESTIMATE OF THE SOLUTIONS FOR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS†

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Abstract. In this paper, we give an estimate on the difference between $x^n(t)$ and $x(t)$ and it clearly shows that one can use the Picard iteration procedure to the approximate solutions to stochastic functional differential equations with infinite delay at phase space $BC((-\infty,0]:R^d)$ which denotes the family of bounded continuous R^d-valued functions φ defined on $(-\infty,0]$ with norm $\|\varphi\|=\sup_{-\infty<\theta\leq 0}|\varphi(\theta)|$ under non-Lipschitz condition being considered as a special case and a weakened linear growth condition.

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1. Introduction

Stochastic differential equations(SDEs in short) are well known to model problems from many areas of science and engineering, wherein quite often the future state of such systems depends not only on the present state but also on its past history(delay) leading to stochastic functional differential equations(SFDEs in short) with delay rather than SDEs. In the recent years, there is an increasing interest in stochastic evolution equations with finite delay under less restrictive conditions than Lipschitz condition; on this topic, one can see Boukfaoui and Erraoui [3], Govindan [4], Halidias [5], Henderson and Plaschko [6], Liu [7], Taniguchi [10], Wei and Wang [11], and references therein for details.

Mao[8] showed the existence and uniqueness of the solution to the following SFDEs under uniform Lipschitz condition and linear growth condition on the coefficients:

$$\mathrm{d}X(t) = f(X_t,t)\mathrm{d}t + g(X_t,t)\mathrm{d}B(t), \quad t_0 \leq t \leq T,$$

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where $X_t = \{X(t+\theta) : -\tau \le \theta \le 0\}$ could be considered as a $C([-\tau, 0]; R^d)$-value stochastic process, $f : C((-\tau, 0]; R^d) \times [t_0, T] \to R^d$ and $g : C((-\tau, 0]; R^d) \times [t_0, T] \to R^{d\times m}$ be Borel measurable.

Recently, Ren et al [9] considered one such class of the so-called stochastic functional differential equations with infinite delay (ISFDEs in short) at phase space $BC((-\infty, 0]; R^d)$ to be described below:

$$\mathrm{d}X(t) = f(X_t, t)\mathrm{d}t + g(X_t, t)\mathrm{d}B(t), \quad t_0 \le t \le T, \tag{1}$$

where $X_t = \{X(t+\theta) : -\infty \le \theta \le 0\}$ could be considered as a $BC((-\infty, 0]; R^d)$-value stochastic process and the initial value was proposed as follows:

$$X_{t_0} = \xi = \{\xi(\theta) : -\infty \le \theta \le 0\} \quad \text{is an}\, \mathcal{F}_{t_0} - \text{measurable} \tag{2}$$
$$BC((-\infty, 0]; R^d) - \text{value random variable such that}\, \xi \in \mathcal{M}^2((-\infty, 0]; R^d).$$

Now we recall the following the existence and uniqueness theorem to (1) with initial data (2) under the non-Lipschitz condition and the weakened linear growth condition proved by Ren et al.

Theorem 1.1 ([9]). *Assume that (H1) and (H2) hold.*
(H1) For any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, it follows that

$$|f(\varphi, t) - f(\psi, t)|^2 \vee |g(\varphi, t) - g(\psi, t)|^2 \le \kappa(\|\varphi - \psi\|^2),$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$.
(H2) For any $t \in [t_0, T]$, it follows that $f(0, t), g(0, t) \in L^2$ such that

$$|f(0, t)|^2 \vee |g(0, t)|^2 \le K,$$

where $K > 0$ is a constant. Then, there exist a unique solution to (1) with initial data (2). □

Motivated by the above works, in this paper we will give an estimate on the difference between $x^n(t)$ and $x(t)$ and it clearly shows that one can use the Picard iteration procedure to the approximate solutions to ISDEs under the non-Lipschitz condition and the weakened linear growth condition.

2. Preliminary

Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T ; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\text{trace}(A^T A)}$. Let t_0 be a positive constant and $(\Omega, \mathcal{F}, P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets). Assume that $B(t)$ is an m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$. Let $BC((-\infty, 0]; R^d)$ denote the family of bounded continuous R^d-value functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\| = \sup_{-\infty<\theta\le 0} |\varphi(\theta)|$. We denote by

$\mathcal{M}^2((-\infty,T];R^d)$ the family of all $\mathcal{F}_{t_0}$-measurable, $\mathbb{R}^d$-valued process $\psi(t)=\psi(t,w), t\in(-\infty,0]$, such that $\int_{-\infty}^0|\psi(t)|^2dt<\infty$.

With all the above preparation, consider a d-dimensional stochastic functional differential equations:

$$\mathrm{d}x(t)=f(x_t,t)\mathrm{d}t+g(x_t,t)\mathrm{d}B(t),\quad t_0\le t\le T, \tag{3}$$

where $x_t=\{x(t+\theta):-\infty<\theta\le 0\}$ can be considered as a $BC((-\infty,0];R^d)$-value stochastic process, where $f:BC((-\infty,0];R^d)\times[t_0,T]\to R^d$ and $g:BC((-\infty,0];R^d)\times[t_0,T]\to R^{d\times m}$ be Borel measurable. Next, we give the initial value of (1) as follows:

$$\begin{aligned}&x_{t_0}=\xi=\{\xi(\theta):-\infty<\theta\le 0\}\quad \text{is an}\,\mathcal{F}_{t_0}-\text{measurable}\\&BC((-\infty,0];R^d)-\text{value random variable such that}\,\xi\in\mathcal{M}^2((-\infty,0];R^d).\end{aligned}\tag{4}$$

In order to find a solution of initial-value problem for the equation satisfying the initial data, we define the solution of equation. The definition is followed:

Definition 2.1 ([8])**.** *R^d-value stochastic process $x(t)$ defined on $-\infty<t\le T$ is called the solution of (3) with initial data (4), if $x(t)$ has the following properties:*

(i) $x(t)$ is continuous and $\{x(t)\}_{t_0\le t\le T}$ is $\mathcal{F}_t$-adapted;

(ii) $\{f(x_t,t)\}\in\mathcal{L}^1([t_0,T];R^d)$ and $\{g(x_t,t)\}\in\mathcal{L}^2([t_0,T];R^{d\times m})$;

(iii) $x_{t_0}=\xi$, for each $t_0\le t\le T$,

$$x(t)=\xi(0)+\int_{t_0}^t f(x_s,s)ds+\int_{t_0}^t g(x_s,s)dB(s)\quad \text{a.s.}$$

The $x(t)$ is called as a unique solution, if any other solution $\overline{x}(t)$ is distinguishable with $x(t)$, that is

$$P\{x(t)=\overline{x}(t),\ \text{for any}\ -\infty<t\le T\}=1.\quad\square$$

3. The Approximate Solutions

In order to obtain an estimate of the solutions to (3) with initial data (4), we define $x_{t_0}^0=\xi$ and $x^0(t)=\xi(0)$, for $t_0\le t\le T$. Let $x_{t_0}^n=\xi, n=1,2,...$ and define the Picard sequence:

$$x^n(t)=\xi(0)+\int_{t_0}^t f(x_s^{n-1},s)\,\mathrm{d}s+\int_{t_0}^t g(x_s^{n-1},s)\,\mathrm{d}B(s),\quad t_0\le t\le T. \tag{5}$$

Now we begin to establish the approximate solutions for (3) with initial data (4) under the non-Lipschitz condition and the weakened linear growth condition. We first prepare some lemmas.

Lemma 3.1 (Gronwall's inequality). *Let $u(t)$ and $b(t)$ be nonnegative continuous functions for $t \geq \alpha$, and let*

$$u(t) \leq a + \int_{\alpha}^{t} b(s)u(s)\mathrm{d}s, \quad t \geq \alpha,$$

where $a \geq 0$ is a constant. Then

$$u(t) \leq a \exp\biggl(\int_{\alpha}^{t} b(s)\mathrm{d}s\biggr), \quad t \geq \alpha.$$

Lemma 3.2 ([8], p.39). *If $p \geq 2, g \in \mathcal{M}^2([0,T]; R^{d\times m})$ such that*

$$E\int_0^T |g(s)|^p \,\mathrm{d}s < \infty,$$

then

$$E\biggl|\int_0^T g(s)\,\mathrm{d}B(s)\biggr|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p\,\mathrm{d}s.$$

In particular, for $p = 2$, there is equality.

Lemma 3.3 ([8], p.40). *Under the same assumptions as Lemma 3.2,*

$$E\biggl(\sup_{0\leq t\leq T}\biggl|\int_0^t g(s)dB(s)\biggr|^p\biggr) \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p d(s).$$

In [9], they have shown that the Picard iterations $x^n(t)$ converge to the unique solution $x(t)$ of equation. The following theorem gives an estimate on the difference between $x^n(t)$ and $x(t)$, and it clearly shows that one can use the Picard iteration procedure to obtain the approximate solutions to equation.

Theorem 3.4. *Assume that there exists a positive number K such that*
(i) For any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, it follows that

$$|f(\varphi,t) - f(\psi,t)|^2 \vee |g(\varphi,t) - g(\psi,t)|^2 \leq \kappa(\|\varphi - \psi\|^2), \tag{6}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying the following $\kappa(0) = 0, \kappa(u) > 0$ for $u > 0$.
(ii) For any $t \in [t_0, T]$, it follows that $f(0,t), g(0,t) \in L^2$ such that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K. \tag{7}$$

Let $x(t)$ be the unique solution of equation (3) with initial data (4) and $x^n(t)$ be the Prcard iterations defined by (5). Then, for all $n \geq 1$,

$$E\biggl(\sup_{t_0\leq t\leq T}|x^n(t) - x(t)|^2\biggr) \leq \frac{2CM_1M^{n-1}(T-t_0)^n}{n!} e^{2M_1(T-t_0)}, \tag{8}$$

where $C = 4(T-t_0)(T-t_0+1)(K + b(E\|\xi\|^2))$, $M = 2b(T-t_0+1), M_1 = 2b(T-t_0+4)$, and b is some positive constant.

Proof. From the Picard sequence and the definition of the solution of equation (3), we have

$$\begin{aligned} & x^n(t) - x(t) \\ &= \int_{t_0}^t [f(x_s^{n-1}, s) - f(x_s, s)]ds + \int_{t_0}^t [g(x_s^{n-1}, s) - g(x_s, s)]dB(s). \end{aligned}$$

Using the elementary inequality $|u+v|^2 \le 2(|u|^2 + |v|^2)$ and Hölder inequality, we can derive that

$$\begin{aligned} & |x^n(t) - x(t)|^2 \\ & \le 2(t-t_0)\int_{t_0}^t \kappa(\|x_s^{n-1} - x_s\|^2)ds + 2\left|\int_{t_0}^t [g(x_s^{n-1}, s) - g(x_s, s)]dB(s)\right|^2. \end{aligned}$$

It also follows that

$$\begin{aligned} & \sup_{t_0 \le s \le t} |x^n(s) - x(s)|^2 \\ & \le 2(t-t_0)\int_{t_0}^t \kappa(\|x_s^{n-1} - x_s\|^2)ds + 2\sup_{t_0 \le s \le t}\left|\int_{t_0}^s [g(x_r^{n-1}, r) - g(x_r, r)]dB(r)\right|^2. \end{aligned}$$

Taking the expectation and using Lemma 3.3, we find that

$$\begin{aligned} & E\Big(\sup_{t_0 \le s \le t} |x^n(s) - x(s)|^2\Big) \qquad (9) \\ & \le 2(T-t_0)\int_{t_0}^t E\Big(\sup_{t_0 \le r \le s} |x^n(r) - x(r)|^2\Big)ds + 8\int_{t_0}^t E\Big(\sup_{t_0 \le r \le s} |x^n(r) - x(r)|^2\Big)ds \\ & \le 2M_1\int_{t_0}^t E\Big(\sup_{t_0 \le r \le s} |x^n(r) - x^{n-1}(r)|^2\Big)ds + 2M_1\int_{t_0}^t E\Big(\sup_{t_0 \le r \le s} |x^n(r) - x(r)|^2\Big)ds, \end{aligned}$$

where $M_1 = 2b(T - t_0 + 4)$. We now claim that for $n \ge 0$,

$$E\left(\sup_{t_0 \le s \le t} |x^{n+1}(s) - x^n(s)|^2\right) \le \frac{C[M(t-t_0)]^n}{n!}. \qquad (10)$$

We shall show this by induction. From the Picard sequence and Hölder inequality, we have

$$E|x^1(t) - x^0(t)|^2 \le 2E\left|\int_{t_0}^t f(x_s^0, s)ds\right|^2 + 2E\left|\int_{t_0}^t g(x_s^0, s)dB(s)\right|^2.$$

Next, using Lemma 3.2 and elementary inequality $(u+v)^2 \le 2(u^2+v^2)$, we derive that

$$\begin{aligned} E|x^1(t) - x^0(t)|^2 \quad & \le \quad 2(t-t_0)E\int_{t_0}^t |f(x_s^0, s) - f(0,s) + f(0,s)|^2 ds \\ & \quad\quad +2E\int_{t_0}^t |g(x_s^0, s) - g(0,s) + g(0,s)|^2 dB(s) \\ & \le \quad 2(t-t_0)E\int_{t_0}^t 2[|f(x_s^0, s) - f(0,s)|^2 + |f(0,s)|^2]ds \end{aligned}$$

$$+2E\int_{t_0}^{t}2[|g(x_s^0,s)-g(0,s)|^2+|g(0,s)|^2]dB(s).$$

From the condition (6) and (7), we have

$$E|x^1(t)-x^0(t)|^2 \quad\leq\quad 4(T-t_0+1)\int_{t_0}^{t}E[K+\kappa(\|X_s^0\|^2)]\,\mathrm{d}s.$$

Given that $\kappa(\cdot)$ is concave and $\kappa(0)=0$, we can find a positive constant b such that $\kappa(u)\leq bu$ for $u\geq 0$. So, we have

$$E|x^1(t)-x^0(t)|^2\leq 4(T-t_0+1)(t-t_0)(K+b(E\|\xi\|^2)).$$

One further obtains that

$$E\Big(\sup_{t_0\leq s\leq t}|x^1(s)-x^0(s)|^2\Big) \quad\leq\quad 4(T-t_0+1)(T-t_0)(K+b(E\|\xi\|^2))=C.$$

By the same ways as above, we obtain

$$\begin{aligned}
&E|x^2(t)-x^1(t)|^2\\
&\leq 2E\left|\int_{t_0}^{t}(f(x_s^1,s)-f(x_s^0,s))ds\right|^2+2E\left|\int_{t_0}^{t}(g(x_s^1,s)-g(x_s^0,s))dB(s)\right|^2\\
&\leq 2(t-t_0)E\int_{t_0}^{t}|f(x_s^1,s)-f(X_s^0,s)|^2ds+2E\int_{t_0}^{t}|g(x_s^1,s)-g(x_s^0,s)|^2ds,
\end{aligned}$$

thus we derive that

$$E\Big(\sup_{t_0\leq s\leq t}|x^2(s)-x^1(s)|^2\Big) \quad\leq\quad 2(T-t_0+1)E\int_{t_0}^{t}\kappa(\|x_s^1-x_s^0\|^2)\,\mathrm{d}s.$$

From the definition of $\kappa(\cdot)$, we have

$$E\Big(\sup_{t_0\leq s\leq t}|x^2(s)-x^1(s)|^2\Big) \quad\leq\quad 2(T-t_0+1)\int_{t_0}^{t}b(E\sup_{t_0\leq r\leq s}|x^1(r)-x^0(r)|^2)\,\mathrm{d}s.$$

One further obtains that

$$E\Big(\sup_{t_0\leq s\leq t}|x^2(s)-x^1(s)|^2\Big) \quad\leq\quad MC(t-t_0),$$

where $M=2b(T-t_0+1)$. When $n=0,1$, the inequality (10) holds. We suppose that (10) holds for some n, now to check (10) for $n+1$. In fact,

$$\begin{aligned}
&E\Big(\sup_{t_0\leq s\leq t}|x^{n+2}(s)-x^{n+1}(s)|^2\Big)\\
&\leq 2(T-t_0+1)\int_{t_0}^{t}\kappa\Big(E\sup_{t_0\leq r\leq s}|x^{n+1}(r)-x^n(r)|^2\Big)\,\mathrm{d}s\\
&\leq M\int_{t_0}^{t}E\Big(\sup_{t_0\leq r\leq s}|x^{n+1}(r)-x^n(r)|^2\Big)\,\mathrm{d}s\\
&\leq M\int_{t_0}^{t}\frac{C[M(s-t_0)]^n}{n!}\,\mathrm{d}s
\end{aligned}$$

$$\leq \frac{C[M(t-t_0)]^{n+1}}{(n+1)!}.$$

It is easy to see that (10) holds for $n+1$, therefore, by induction, (10) holds for $n \geq 0$.

Substituting (10) into inequality (9) yields that

$$\begin{aligned} &E\Big(\sup_{t_0\leq s\leq t}|x^n(s)-x(s)|^2\Big)\\ &\leq 2M_1\int_{t_0}^{T}\frac{C[M(s-t_0)]^{n-1}}{(n-1)!}ds+2M_1\int_{t_0}^{t}E\Big(\sup_{t_0\leq r\leq s}|x^n(r)-x(r)|^2\Big)ds\\ &\leq \frac{2CM_1M^{n-1}[(T-t_0)]^n}{n!}ds+2M_1\int_{t_0}^{t}E\Big(\sup_{t_0\leq r\leq s}|x^n(r)-x(r)|^2\Big)ds. \end{aligned}$$

The required inequality (8) now follows by applying the Gronwall inequality. The proof is complete. □

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