논리연구 13-1(2010) pp. 1~20

A new proof of standard completeness for the uninorm logic **UL***

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【Abstract】 This paper investigates a new proof of standard completeness (i.e. completeness on the real unit interval [0, 1]) for the uninorm (based) logic **UL** introduced by Metcalfe and Montagna in [15]. More exactly, standard completeness is established for **UL** by using nuclear completions method introduced in [8, 9].

【Key words】 (substructural) fuzzy logic, fuzzy logic, uninorm (based) logic

* 접수완료: 2009. 11. 15 심사 및 수정완료: 2010. 2. 19

1. Introduction

In this paper we investigate a new proof of standard completeness (i.e., completeness on the real unit interval [0, 1]) of the uninorm logic **UL**. For this, we first recall briefly some historical facts associated with fuzzy logic, which are mentioned in [22].

Many-valued logics with truth values in the real unit interval [0, 1] have a long and distinguished history, and the well-known examples are the infinite-valued systems **Ł** (Łukasiewicz logic), **G** (Gödel-Dummett logic), and Π (Product logic). In particular, Hájek [11] introduced **BL** (Basic fuzzy logic) and showed that **Ł**, **G**, and Π are its extensions. **BL** is the most important logic of *continuous* t-norms, and **Ł**, **G**, and Π are emerging in this respect as fundamental examples of logics based on continuous t-norms. Esteva and Godo further [5] introduced the logic of *left-continuous* t-norms **MTL** (Monoidal t-norm logic), which copes with the logic of *left-continuous* t-norms, as a weakening of **BL**. This is the most basic t-norm logic known to us. In this approach, (multiplicative) conjunction connectives are interpreted by t-norms (see [11]), which are commutative, associative, monotonic binary functions with identity 1.

Although t-norms play an important role in fuzzy logic (theory), these operators do not admit a compensating behavior. As Detyniecki [3] mentioned, t-norms do not allow low values to be compensated by high values (see [19]). For this reason, Yager and Rybalov [21] introduced *uninorms* as a generalization of t-norms. These operators have identity lying anywhere in [0, 1] rather than at 1 as t-norms. After their introducing uninorms, many interesting properties of uninorms and their applications such as full reinforcement, compensation behavior, bipolar problems, etc., have been studied (see e.g. [1, 7, 13, 17, 19, 20]). Furthermore, several uninorm (based) logics have been recently introduced. For instance, Metcalfe (and Montagna) [14, 15] introduced the uninorm (based) logics **UL**, **IUL** (Involutive uninorm logic), **UML** (Uninorm mingle logic), and **IUML** (Involutive uninorm mingle logic) as substructural fuzzy logics based on *uninorms*. In particular, **UL** is the most basic uninorm logic, which is the logic of conjunctive *left-continuous* uninorms.

Notice that all of the systems above are complete (so called standard complete) w.r.t. algebras with lattice reduct [0, 1]. One method introduced in [6, 12] for **MTL** and its axiomatic extensions (calling it *Jenei and Montagna's method, briefly JM method*), consists of showing that countable linearly ordered algebras of a given variety can be embedded into linearly and *densely* ordered members of the same variety, which can in turn be embedded into algebras with lattice reduct [0, 1]. (Notice that the present author showed that standard completeness for some axiomatic extensions of **UL** using JM method in [22].) But this method seems to fail with associativity for **UL**, and so appears not to work in general for weakening-free fuzzy logics such as **UL** based on uninorms. Because of this negative fact Metcalfe and Montagna [15] instead introduced a new approach for proving standard completeness of uninorm logics (calling it *Metcalfe and Montagna's method, briefly MM method*), consisting of the following two steps: 1. after extending logics with density rule, showing that such systems are complete w.r.t. linearly and densely ordered algebras, and for particular extensions are complete w.r.t. those algebras with lattice reduct [0, 1]; 2. giving a syntactic elimination of density rule (as a rule of the corresponding hypersequent calculus), i.e., showing that if ϕ is derivable in a uninorm logic **L** extended with density rule, then it is also derivable in **L**.

The starting point for the current work is the observation that MM method is unnecessarily complicate. Namely, MM method may be simplified. To verify this, we shall provide a way to simplify MM method by eliminating the step extending logics with density rule. More exactly, we establish a new proof of standard completeness for **UL** by means of a way requiring dense theory in place of density rule. For this we further use nuclear completions method introduced in [8, 9], generalizing Dedekind-McNeille completions.

For convenience, we shall adopt the notation and terminology similar to those in [2, 5, 6, 11, 15], and assume being familiar with them (together with results found in them).

2. Syntax

We base the uninorm logic **UL** on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$, and
df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define ϕ^n_t as ϕ_t & $\cdots$ & ϕ_t, n factors, where $\phi_t := \phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of **UL** as a (substructural) fuzzy logic.

Definition 2.1 **UL** consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)
A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ ($\wedge$-elimination, $\wedge$-E)
A3. $((\phi\rightarrow\psi)\wedge(\phi\rightarrow\chi)) \rightarrow (\phi\rightarrow(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)
A4. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)
A5. $((\phi\rightarrow\chi)\wedge(\psi\rightarrow\chi)) \rightarrow ((\phi\vee\psi)\rightarrow\chi)$ ($\vee$-elimination, $\vee$-E)
A6. $\phi \rightarrow \mathbf{T}$ (verum ex quolibet, VE)
A7. $\mathbf{F} \rightarrow \phi$ (ex falso quadlibet, EF)
A8. $(\phi \,\&\, \psi) \rightarrow (\psi \,\&\, \phi)$ (&-commutativity, &-C)
A9. $(\phi \,\&\, \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)
A10. $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$ (suffixing, SF)
A11. $(\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\phi \,\&\, \psi) \rightarrow \chi)$ (residuation, RE)

A12. for each n, $(\phi \rightarrow \psi)^n_t \vee (\psi \rightarrow \phi)^n_t$ ($^n{}_t$-prelinearity, PL^n_t).

$\phi \rightarrow \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj)

Proposition 2.2 **UL** proves:

(1) $(\phi \,\&\, (\psi \,\&\, \chi)) \leftrightarrow ((\phi \,\&\, \psi) \,\&\, \chi)$ (&-associativity, AS).

In **UL**, **f** can be defined as ~**t** and vice versa. A *theory* over **UL** is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of **UL** or a member of T or follows from some preceding members of the sequence using the rules (mp) and (adj). $T \vdash \phi$, more exactly $T \vdash_{UL} \phi$, means that ϕ is *provable* in T w.r.t. **UL**, i.e., there is a **UL**-proof of ϕ in T. The local **t**-deduction theorem (LDT_t) for **UL** is as follows:

Proposition 2.3 Let T be a theory, and ϕ, ψ formulas. $T \cup \{\phi\} \vdash_{UL} \psi$ iff there is n such that $T \vdash_{UL} \phi^n_t \rightarrow \psi$.

Proof: See [16]. □

A theory T is *inconsistent* if $T \vdash$ **F**; otherwise it is *consistent*.

For convenience, "~", "∧", "∨", and "→" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

3. Semantics

Suitable algebraic structures for **UL** are obtained as a subvariety of the variety of commutative residuated lattices in the sense of e.g. [8].

Definition 3.1 A *pointed bounded commutative residuated lattice* is a structure **A** = (A, $\top$, $\bot$, $\top_t$, $\bot_f$, $\wedge$, $\vee$, *, $\rightarrow$) such that:

(I) (A, $\top$, $\bot$, $\wedge$, $\vee$) is a bounded lattice with top element $\top$ and bottom element $\bot$.

(II) (A, *, $\top_t$) satisfies for some $\top_t$ and for all x, y, z $\in$ A,

(a) x * y = y * x (commutativity)

(b) $\top_t$ * x = x (identity)

(c) x * (y * z) = (x * y) * z (associativity).

(III) y $\leq$ x$\rightarrow$z iff x * y $\leq$ z, for all x, y, z $\in$ A (residuation).

(A, *, $\top_t$) satisfying (II-b, c) is a *monoid*. Thus (A, *, $\top_t$) satisfying (II-a, b, c) is a commutative monoid. To define the above lattice we may take in place of (III) a family of equations as in [11]. Using $\rightarrow$ and $\bot_f$ we can define $\top_t$ as $\bot_f \rightarrow \bot_f$, and $\sim$ as in (df1). Then, UL-algebra whose class characterizes **UL** is defined as follows.

Definition 3.2 (UL-algebra) A *UL-algebra* is a pointed bounded commutative residuated lattice satisfying the condition: for all x, y, and for each n ($\geq$ 1),

(pl$_t$) $\top_t \leq (x \to y)^n{}_{\top t} \vee (y \to x)^n{}_{\top t}$.

UL-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function v : FOR $\to \mathcal{A}$ satisfying:

$v(\phi \to \psi) = v(\phi) \to v(\psi)$,
$v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$,
$v(\phi \vee \psi) = v(\phi) \vee v(\psi)$,
$v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$,
$v(\mathbf{F}) = \perp$,
$v(\mathbf{f}) = \perp_f$,

(and hence $v(\sim\phi) = \sim v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = \top_t$).

Definition 3.4 Let $\mathcal{A}$ be a UL-algebra, T a theory, ϕ a formula, and K a class of UL-algebras.

(i) (Tautology) ϕ is a *$\top_t$-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq \top_t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq \top_t$ for each $\phi \in$ T. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by T $\vDash_K \phi$, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ $\{\phi\}$, $\mathcal{A}$) for each $\mathcal{A} \in$ K.

Definition 3.5 (UL-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is a ***UL-algebra*** iff whenever ϕ is UL-provable in T (i.e. T $\vdash_{UL} \phi$), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. T$\vDash_{\{A\}} \phi$), $\mathcal{A}$ a UL-algebra. By *$MOD^{(l)}$(UL)*, we denote the class of (linearly ordered) **UL**-algebras. Finally, we write T $\vDash^{(l)}_{UL} \phi$ in place of T $\vDash_{MOD^{(l)}(UL)} \phi$.

Note that since each condition for the UL-algebra has a form of equation or can be defined in equation (exercise), it can be ensured that the class of all UL-algebras is a variety.

Let **A** be a UL-algebra. We first show that classes of provably equivalent formulas form a UL-algebra. Let T be a fixed theory over **UL**. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that T $\vdash_{UL} \phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \to [\psi]_T = [\phi \to \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \& \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $\bot = [\mathbf{F}]_T$, $\top = [\mathbf{T}]_T$, $\top_\mathbf{t} = [\mathbf{t}]_T$, and $\bot_\mathbf{f} = [\mathbf{f}]_T$. By A_T, we denote this algebra.

Proposition 3.6 For T a theory over **UL**, $\mathbf{A}_T$ is a **UL**-algebra.

Proof: Note that A1 to A7 ensure that $\wedge$ and $\vee$ satisfy (I) in Definition 3.1; that AS, A8, A9 ensure that & satisfies (II); that

A11 and A12 ensure that (III) and (pl^n_t) hold. It is obvious that $[\phi]_T \leq [\psi]_T$ iff $T \vdash_{UL} \phi \leftrightarrow (\phi \wedge \psi)$ iff $T \vdash_{UL} \phi \to \psi$. Finally recall that $\mathbf{A}_T$ is a **UL**-algebra iff $T \vdash_{UL} \psi$ implies $T \vDash_{UL} \psi$, and observe that for ϕ in T, since $T \vdash_{UL} t \to \phi$, it follows that $[\mathbf{t}]_T \leq [\phi]_T$. Thus it is a **UL**-algebra. □

We next note that the nomenclature of the prelinearity condition is explained by the subdirect representation theorem below.

Proposition 3.7 ([18]) Each UL-algebra is a subdirect product of linearly ordered UL-algebras.

Theorem 3.8 (Strong completeness) Let T be a theory, and ϕ a formula. $T \vdash_{UL} \phi$ iff $T \vDash_{UL} \phi$ iff $T \vDash^l_{UL} \phi$.

Proof: (i) $T \vdash_{UL} \phi$ iff $T \vDash_{UL} \phi$. Left to right follows from definition. Right to left is as follows: from Proposition 3.6, we obtain $\mathbf{A}_T \in MOD(L)$, and for $\mathbf{A}_T$-evaluation v defined as $v(\psi) = [\psi]_T$, it holds that $v \in Mod(T, \mathbf{A}_T)$. Thus, since from $T \vDash_{UL} \phi$ we obtain that $[\phi]_T = v(\phi) \geq \top_t$, $T \vdash_{UL} \mathbf{t} \to \phi$. Then, since $T \vdash_{UL} \mathbf{t}$, by (mp) $T \vdash_{UL} \phi$, as required.

(ii) $T \vDash_{UL} \phi$ iff $T \vDash^l_{UL} \phi$. It follows from Proposition 3.7.

4. Uninorms and their residua

In this section, using *1, 0, and some* 1_t, *and* 0_f in the real unit interval [0, 1], we shall express ⊤, ⊥, $\top_t$, and $\bot_f$, respectively. We also define standard UL-algebras and uninorms on [0, 1].

Definition 4.1 A UL-algebra is *standard* iff its lattice reduct is [0, 1].

In standard UL-algebras the monoid operator * is a uninorm.

Definition 4.2 A *uninorm* is a function ○ : $[0, 1]^2 \rightarrow [0, 1]$ such that for some $1_t \in [0, 1]$ and for all x, y, z ∈ [0, 1]:

(a) x ○ y = y ○ x (commutativity),
(b) x ○ (y ○ z) = (x ○ y) ○ z (associativity),
(c) x ≤ y implies x ○ z ≤ y ○ z (monotonicity), and
(d) 1_t ○ x = x (identity).

The function ○ satisfying (1-identity) $1_t = 1$ is a *t-norm*. ○ is *residuated* iff there is → : $[0, 1]^2 \rightarrow [0, 1]$ satisfying (residuation) on [0, 1]. A uninorm is called *conjunctive* if 0 ○ 1 = 0, and *disjunctive* if 0 ○ 1 = 1.

The left-continuity property of conjunctive uninorms is important in the sense that it gives a residuated implication and so plays an important role in standard completeness proof of **UL** as in t-norm based logics such as **MTL**. Given a uninorm $\bigcirc$, *residuated implication* $\rightarrow$ determined by $\bigcirc$ is defined as $x \rightarrow y := \sup\{z: x \bigcirc z \leq y\}$ for all x, y $\in$ [0, 1]. Then, we can show that for any uninorm $\bigcirc$, $\bigcirc$ and its residuated implication $\rightarrow$ form a residuated pair iff $\bigcirc$ is conjunctive and left-continuous in both arguments (see Proposition 5.4.2 [10]).

5. Standard completeness

We here provide *standard* completeness results for **UL** using nuclear completions in [8, 9]. We shall call these completions method *nuclear completions method.*

A linear theory T is said to be *dense* if for each pair ϕ, ψ of formulas, $T \nvdash \phi \rightarrow \psi$ implies that there is a propositional variable p not occurring in T, ϕ, or ψ such that $T \nvdash \phi \rightarrow p$ and $T \nvdash p \rightarrow \psi$.

Proposition 5.1 Let T be a theory over **UL** and ϕ a formula. $T \vdash_{UL} \phi$ iff for every linearly densely ordered **UL**-algebra and evaluation v, if $v(\psi) \geq \top_t$ for each $\psi \in T$, then $v(\phi) \geq \top_t$.

Proof: Left to right is by induction on the height of a proof for $T \vdash_{UL} \phi$. As an example we prove the rule mp. Suppose toward contradiction that there is a linearly and densely ordered **L**-algebra and evaluation v such that $v(\alpha) \geq \top_t$ for each $\alpha \in T$

and $\top_t \leq v(\phi \rightarrow \psi)$, $v(\phi)$ but $v(\psi) < \top_t$. Since $v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$, $\top_t \leq v(\phi \rightarrow \psi) = v(\phi) \rightarrow v(\psi)$ and so $v(\phi) \leq v(\psi)$. This implies that $\top_t \leq v(\psi)$, a contradiction.

We prove right to left contrapositively. We extend the language (if necessary) with countably many new variables not occurring in T or ϕ. We then fix an enumeration (ϕ_n, ψ_n), $n \in \omega$, of all pairs of formulas of the extended language. For a theory T over **UL** such that $T \nvdash_{UL} \phi$, we define a sequence of sets T_n by induction as follows:

$T_1 = \{\phi' : T \vdash_{UL} \phi'\}$.

$T_{i+1} = T \cup \{\phi_i \rightarrow \psi_i\}$ if $T, \phi_i \rightarrow \psi_i \nvdash_{UL} \phi$,

$T \cup \{\psi_i \rightarrow \phi_i\}$ otherwise,

where $T_{i+1} \vdash_{UL} \phi_i \rightarrow \psi_i$ iff for every q_i not in $T_{i+1} \cup \{\phi_i, \psi_i\}$, $T_{i+1} \vdash_{UL} \phi_i \rightarrow q_i$ or $T_{i+1} \vdash_{UL} q_i \rightarrow \psi_i$.

Let T' be the union of all these T_n's. By Proposition 3.6, $\mathbf{A}_{T'}$ is a **UL**-algebra. Moreover, $\mathbf{A}_{T'}$ is linearly and densely ordered. For this we show that T' is linearly and densely ordered. For linearity, it suffices to note that having $T_n \nvdash_{UL} \phi$ observe that $T, \phi_i \rightarrow \psi_i \nvdash_{UL} \phi$ or $T, \psi_i \rightarrow \phi_i \nvdash_{UL} \phi$. Otherwise, $T, \phi_i \rightarrow \psi_i \vdash_{UL} \phi$ and $T, \psi_i \rightarrow \phi_i \vdash_{UL} \phi$. Then by LDT_t, there are m, n such that $T \vdash_{UL} (\phi_i \rightarrow \psi_i)^m{}_t \rightarrow \phi$ and $T \vdash_{UL} (\psi_i \rightarrow \phi_i)^n{}_t \rightarrow \phi$. Since $(\phi_t \,\&\, \phi_t) \rightarrow \phi_t$, without loss of generality we may assume that $m \leq n$ and so $T \vdash_{UL} (\phi_i \rightarrow \psi_i)^n{}_t \rightarrow \phi$ and $T \vdash_{UL} (\psi_i \rightarrow \phi_i)^n{}_t \rightarrow \phi$. Then, by adj, $T \vdash_{UL}((\phi_i$

$\rightarrow \psi_i)^n_t \rightarrow \phi) \wedge ((\psi_i \rightarrow \phi_i)^n_t \rightarrow \phi)$, and so by A5 and mp, $T \vdash_{UL} ((\phi_i \rightarrow \psi_i)^n_t \vee (\psi_i \rightarrow \phi_i)^n_t) \rightarrow \phi$. But then by A12, $T \vdash_{UL} \phi$, a contradiction. For density, we just note that it follows from the definition that if $T' \nvdash_{UL} \phi_n \rightarrow \psi_n$, then $T' \nvdash_{UL} \phi_n \rightarrow q_n$ and $T' \nvdash_{UL} q_n \rightarrow \psi_n$; and if $T' \nvdash_{UL} \psi_n \rightarrow \phi_n$, then $T' \nvdash_{UL} \psi_n \rightarrow q_n$ and $T' \nvdash_{UL} q_n \rightarrow \phi_n$.

Hence, defining an evaluation v such that $v(p) = [p]_{T'}$ for all propositional variables p, we obtain that $v(\psi) = [\psi]_{T'} \geq \top_t$ for each $\psi \in T'$, but $v(\phi) = [\phi]_{T'} < \top_t$, as desired. □

A *partially-ordered monoid* (*po-monoid* for brevity) is a structure $\mathbf{A} = (A, \leq, *)$ such that $*$ is a binary operation on A, $\leq$ is a partial order on A, and $*$ is order preserving, i.e., monotone. A (commutative) residuated lattice is a po-monoid. A *nucleus* on a po-monoid $\mathbf{A}$ is a map $g : A \rightarrow A$ such that g is a closure operator on $(A, \leq)$ and for all $x, y \in A$,

(nuc) $g(x) * g(y) \leq g(x * y)$.

Using nuclear completions we show that **UL** is standard complete.

Theorem 5.2 Every countable linearly and densely ordered **UL**-algebra can be embedded into a standard **UL**-algebra.

Proof: Its proof is analogous to that of Theorem 28 in [15]. We first recall that any (bounded and pointed) residuated lattice

A can be embedded into a complete residuated lattice $\mathbf{A}^+$ by means of the nuclear completion (see [8]). The lattice $\mathbf{A}^+$ is defined as follows:

1. For every $X \subseteq \mathbf{A}$, let $C(X)$ denote the intersection of all sets Z such that: (1) $X \subseteq Z$, (2) Z is closed downward, and (3) for all $Y \subseteq Z$, if $\sup(Y)$ exists in **A**, then $\sup(Y) \in Z$. Then it follows that C is a closure operator. The domain of $\mathbf{A}^+$ is $\{X: X \subseteq \mathbf{A}$ such that $C(X) = X\}$.
2. The operations of $\mathbf{A}^+$ are: $X \circ Y = C(X * Y)$, where, letting $*$ be the monoid operator of **A**, $X * Y = \{x * y: x \in X$ and $y \in Y\}$; $X \wedge Y = X \cap Y$; $X \vee Y = C(X \cup Y)$; and $X \rightarrow Y = \{z \in \mathbf{A}: \forall x \in X, z * x \in Y\}$. Then it follows from the definition that C is a nucleus on $(\mathbf{A}^+, \subseteq)$ because $C(X) \circ C(Y) = X \circ Y = C(X \circ Y)$ for $X, Y \in \mathbf{A}^+$.
3. The constants in $\mathbf{A}^+$ are: $\top^+ = \mathbf{A}$, $\bot^+ = \{\bot\}$, $\top_t^+ = \{z \in \mathbf{A}: z \leq \top_t\}$, and $\bot_f^+ = \{z \in \mathbf{A}: z \leq \bot_f\}$.

First note that $\mathbf{A}^+$ is the nucleus retraction of **A**. The embedding h of **A** into $\mathbf{A}^+$ is defined by $h(x) = \{z \in \mathbf{A}: z \leq x\}$. Notice that for $X \in \mathbf{A}^+$, we have $X = \sup\{h(x): x \in X\}$, i.e., every element of $\mathbf{A}^+$ is the supremum of a set of elements of **A**. Furthermore, the suprema and infima existing in **A** are preserved by h, and for $X, Y \in \mathbf{A}^+$,

(1) $X \circ Y = \sup\{h(x) \circ h(y): x \in X, y \in Y\}$.

Since C-closed sets are closed downwards and so C is a downward nucleus, if **A** is linearly ordered, so is $\mathbf{A}^+$ by inclusion. Hence, if **A** is a linearly ordered **UL**-algebra, so is $\mathbf{A}^+$. Note that if **A** is densely ordered, the image of **A** under h is dense in $\mathbf{A}^+$, i.e., for every $X \subset Y \in \mathbf{A}^+$, there is $z \in \mathbf{A}$ such that $X \subset h(z) \subset Y$. Hence, if **A** is a countable linearly and densely ordered **UL**-algebra, it is order isomorphic to $\mathbf{Q} \cap [0, 1]$, and its nuclear completion, be completely and densely ordered, is isomorphic to [0, 1]. Since by (1), the monoid operation $\circ$ on $\mathbf{A}^+$ is left-continuous, it follows that $\mathbf{A}^+$ is a standard **UL**-algebra. □

Theorem 5.3 (Strong standard completeness) $T \vdash_{UL} \phi$ iff for every standard **UL**-algebra and evaluation v, if $v(\psi) \geq \top_t$ for each $\psi \in T$, then $v(\phi) \geq \top_t$.

Proof: It follows from Proposition 5.1 and Theorem 5.2. □

Remark 5.4 Recall that any (bounded and pointed) residuated lattice **A** can be embedded into a complete residuated lattice $\mathbf{A}^+$ by means of the Dedekind-McNeille completion (see [8]). This implies that we can prove standard completeness of **UL** using Dedekind-McNeille completion in place of nuclear completion. We here just note that Theorem 5.2 can be proved using Dedekind-McNeille completion (see Theorem 28 in [15]), and this gives a standard completeness of **UL** using Dedekind-McNeille completion.

6. Concluding remark

We here investigated (not merely algebraic completeness but also) standard completeness for **UL**. This work can be generalized to the systems, which are the axiomatic extensions of **UL** introduced in [15]. We shall investigate this in some subsequent paper.

To some readers it will be interesting to say that **IUML**, an extension of **UL**, is **R**-mingle (**RM**) plus (FP) **t** $\leftrightarrow$ **f** and so **IUML** can be regarded not merely as fuzzy logic but also as relevance logic. Dunn (see e.g. [4]) provided a Kripke-style semantics for **RM** and Yang (see [23]) has recently studied Kripke-style semantics for some neighbors of **R**. Kripke-style semantics seems to be provided for **UL** and its axiomatic extensions, in particular, **IUML**. We shall consider this in another subsequent paper.

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부록	Abstracts

ARTICLE ABSTRACTS

Uninorm 논리 UL을 위한 새로운 표준 완전성 증명

양은석

이 논문에서 우리는 멧칼페와 몬테그나에 의해 [15]에서 소개된 uninorm에 바탕을 두고 있는 논리 **UL**을 위한 표준 완전성 즉 단위 실수 [0, 1] 위에서의 완전성의 새로운 증명을 연구한다. 즉 [8, 9]에서 소개된 nuclear completion을 사용하여 **UL**이 표준적으로 완전하다는 것을 보인다.

주요어 : (준구조)퍼지 논리, 퍼지 논리, 유니놈 논리