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Estimation of the Parameters of the New Generalized Weibull Distribution

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Abstract. Recently, Zaindin and Sarhan (2009) introduced a new distribution named new generalized Weibull distribution. This paper deals with the problem of estimating the parameters of this distribution in the case where the data is grouped and censored. We use both the maximum likelihood and Bayes techniques. The results obtained are illustrated on a set of real data.

Key Words : *Modified Weibull distribution, Maximum likelihood, Bayes, Type I censoring.*

1. INTRODUCTION

In analyzing lifetime data one often uses the exponential, Rayleigh, linear failure rate or generalized exponential distributions. It is known that exponential can have only constant hazard function whereas Rayleigh, linear failure rate and generalized exponential distribution can have only monotone (increasing in case of Rayleigh or linear failure rate and increasing/ decreasing in case of generalized exponential distribution) hazard functions. Unfortunately, in practice often one needs to consider non-monotonic function such as bathtub shaped hazard function also, see, for example, Lai *et al.* (2001).

Mudholkar and Srivastava (1993) presented a generalization of the Weibull family called the exponentiated-Weibull family. They showed that this generalization not only includes distributions with bathtub and unimodal hazard rates but provides a broader class of monotone hazard rates. Nadarajah and Kotz (2006) introduced four exponentiated type distributions: the exponentiated gamma, exponentiated Weibull, exponentiated Gumbel, and the exponentiated Frchet distribution. They provided a treatment of the mathematical properties for each distribution. Sarhan and Kundu (2008) presented a generalization of the linear hazard rate distribution called the generalized linear hazard rate distribution. They explained that this distribution can have increasing, decreasing and bathtub shaped hazard rate functions which are quite desirable for data analysis purposes. Recently, Sarhan and Zaindin (2009) presented a new generalization of the traditional Weibull distribution

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called the Modified Weibull Distribution and denoted as MWD(α, β, γ). The cumulative distribution function, say CDF, of MWD(α, β, γ) takes the following form

$$F(x) = 1 - \exp\{-\alpha x - \beta x^{\gamma}\}, \qquad x \geq 0, \tag{1.1}$$

where $\alpha, \beta, \gamma \geq 0$ such that $\alpha + \beta\gamma > 0$. This condition is made on the parameter space to insure that the hazard rate function is positive. The MWD(α, β, γ) generalizes the following distributions:

1. Exponential distribution, Johnson and Kotz (1970): by setting $\beta = 0$ or $\alpha = 0, \gamma = 1$.
2. Weibull distribution, Johnson and Kotz (1970): by setting $\alpha = 0$.
3. Rayleigh distribution, Johnson and Kotz (1970): by setting $\alpha = 0$, $\gamma = 2$.
4. Linear Failure Rate distribution, Johnson and Kotz (1970): by setting $\gamma = 2$.

It is known that the exponential distribution has a constant hazard function whereas the Rayleigh can have only monotone increasing hazard function and the modified Weibull distribution can have either constant or monotone increasing hazard function. Unfortunately, in practice often one needs to consider non-monotonic functions such as bathtub shaped hazard functions, see for example Lai et al. (2001). Several papers studied the statistical inference of the different models. Among these are Pandey et al. (1993), Sen and Bhattacharyya (1995), and Sarhan (2004).

Recently, Sarhan and Zaindin (2009) introduced a new four-parameter distribution which may have bathtub shaped hazard function, called as new generalized Weibull distribution denoted by NGWD$(\alpha, \beta, \gamma, \lambda)$.

It is observed that NGWD has increasing, decreasing and bathtub shaped hazard functions. The CDF of the NGWD$(\alpha, \beta, \gamma, \lambda)$ takes the following form

$$F(x; \alpha, \beta, \gamma, \lambda) = \left[1 - e^{-\alpha x - \beta x^{\gamma}}\right]^{\lambda}, \qquad x \geq 0, \tag{1.2}$$

where $\gamma, \lambda > 0$, $\alpha, \beta \geq 0$ such that $\alpha + \beta > 0$.

The NGWD$(\alpha, \beta, \gamma, \lambda)$ generalizes several distributions. Among these distributions are

1. Generalized exponential distribution, Gupta and Kundu (1999): by setting either $\beta = 0$ or $\gamma = 1$ or $\alpha = 0, \gamma = 1$.

2. Modified Weibull distribution, Sarhan and Zaindin (2009): by setting $\lambda = 1$.
3. Generalized Weibull distribution, Mudholkar and Srivastava (1993): by setting $\alpha = 0$.
4. Generalized Rayleigh distribution, Surles and Padgett (2005): by setting $\alpha = 0$, $\gamma = 2$.
5. Generalized linear failure rate distribution, Sarhan and Kundu (2008): by setting $\gamma = 2$.

Sarhan and Zaindin (2009) studied different statistical properties of this distribution and some physical interpretations. Also, they used a simple random sample to obtain the maximum likelihood estimates (MLEs) of the NGWD$(\alpha, \beta, \gamma, \lambda)$. In this paper, we use grouped and censored data to estimate the parameters of the NGWD$(\alpha, \beta, \gamma, \lambda)$. The maximum likelihood and Bayes methods are used to derive the point and confidence interval estimates of the parameters. Further, we study whether this distribution fits a set of real data better than the modified Weibull distribution the MWD(α, β, γ). Two criteria are used for this purpose. These are the Kolmogorov-Smirnov test statistic and the values of the log-likelihood function.

The rest of this paper is organized as follows. Some properties of the NGWD$(\alpha, \beta, \gamma, \lambda)$ are presented in Section 2. Section 3 presents the model assumptions and notations. Section 4 gives the parameter estimations using both maximum likelihood and Bayes techniques. We use a set of real data in Section 5 as an application.

2. THE NGWD

The survival function of the NGWD$(\alpha, \beta, \gamma, \lambda)$ is

$$S(x; \alpha, \beta, \gamma, \lambda) = 1 - \left[1 - e^{-\alpha x - \beta x^{\gamma}}\right]^{\lambda}, \qquad x \geq 0, \tag{2.1}$$

The probability density function, say pdf, of NGWD$(\alpha, \beta, \gamma, \lambda)$ is

$$f(x; \alpha, \beta, \gamma, \lambda) = \lambda \left(\alpha + \beta \gamma x^{\gamma - 1}\right) e^{-\alpha x - \beta x^{\gamma}} \left[1 - e^{-\alpha x - \beta x^{\gamma}}\right]^{\lambda - 1}, \quad x \geq 0, \tag{2.2}$$

Figure 2.1 shows some patterns of the pdf of NGWD$(\alpha, \beta, \gamma, \lambda)$, which may have a single mode or no mode at all.

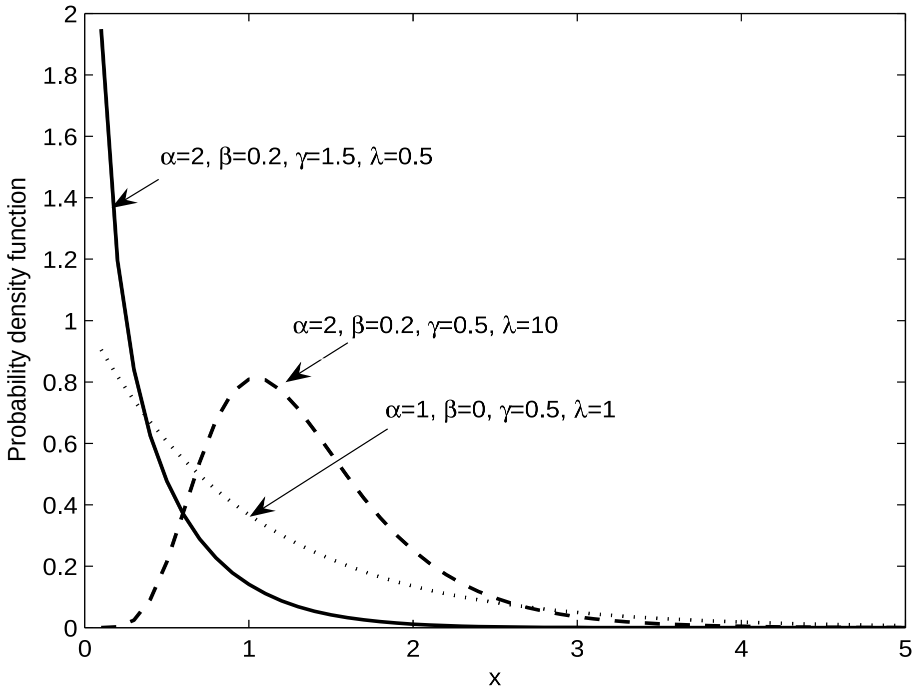

Figure 2.1. Different patterns of the probability density function.

and the hazard function of NGWD$(\alpha, \beta, \gamma, \lambda)$ is

$$h(x; \alpha, \beta, \gamma, \lambda) = \frac{\lambda \left(\alpha + \beta\gamma x^{\gamma-1}\right) e^{-\alpha x - \beta x^{\gamma}} \left[1 - e^{-\alpha x - \beta x^{\gamma}}\right]^{\lambda-1}}{1 - \left[1 - e^{-\alpha x - \beta x^{\gamma}}\right]^{\lambda}} \quad x \geq 0. \tag{2.3}$$

Figure 2.2 shows the failure rate function of NGWD$(\alpha, \beta, \gamma, \lambda)$ for different parameter values. From this figure, it is immediate that the hazard functions can be increasing, decreasing or bathtub shaped.

One can easily verify that:

1. when $\gamma = 1$, then the hazard function is either increasing (if $\lambda > 1$) or constant (if $\lambda = 1$) or decreasing (if $\lambda < 1$);

2. when $\gamma < 1$, then the hazard function is either decreasing if $\lambda \leq 1$ or bathtub shaped if $\lambda > 1$;

3. when $\gamma > 1$, then the hazard function is either increasing (if $\lambda \geq 1$) or bathtub shaped (if $\lambda < 1$).

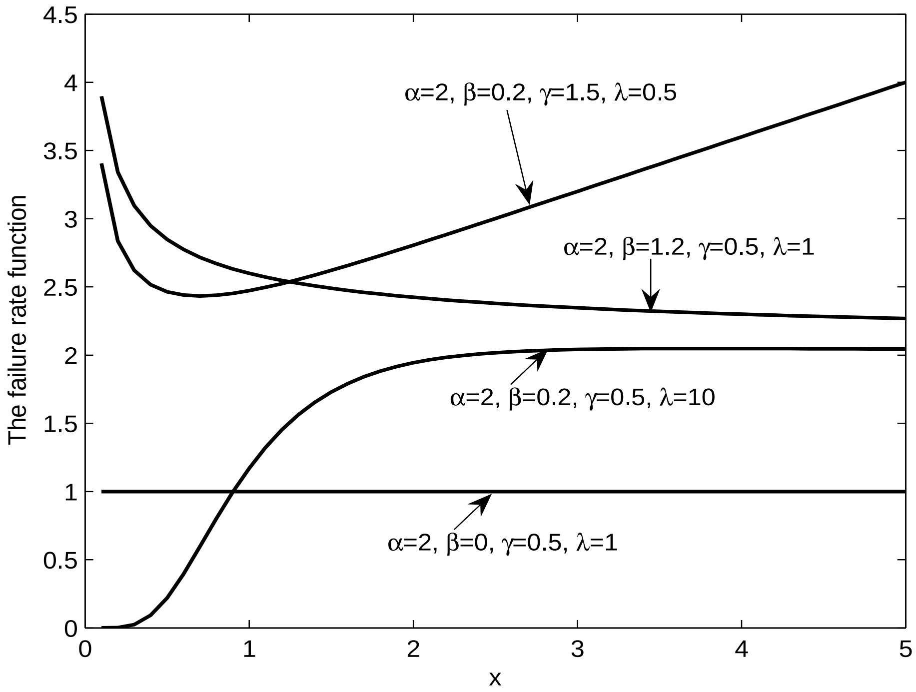

Figure 2.2. Different patterns of the hazard rate function.

3. MODEL ASSUMPTION AND NOTATION

Throughout this paper, we use the following assumptions.

1. n independent and identical experimental units are put on a life test at time zero.
2. The lifetime of each unit follows a NGWD$(\alpha, \beta, \gamma, \lambda)$ with CDF given by (1.2).
3. The inspection times $0 < t_1 < t_2 < \cdots < t_k < \infty$ are predetermined.
4. The test is terminated at the predetermined time t_k. That is, the data is of Type-I censoring.

5. $t_0 = 0$ and $t_{k+1} = \infty$.

6. The numbers of failures in $(t_i, t_{i+1}]$ are recorded.

The data collected from the above test scheme consist of the number n_i of failures in the interval $(t_{i-1},\, t_i]$, $i = 1, 2, ...k$ and the number n_{k+1} of units tested without failing up to time t_k (censored units).

4. PARAMETER ESTIMATION

In this section, we use the maximum likelihood and Bayes procedures to derive point and interval estimates of the unknown parameters of the NGWD$(\alpha, \beta, \gamma, \lambda)$. Let τ denote the set of available observations, $\tau = \{t_1, \cdots, t_k; n_1, \cdots, n_k, n_{k+1}\}$ and let $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha, \beta, \gamma, \lambda)$.

4.1. Maximum likelihood procedure

In this subsection, we use the maximum likelihood procedure to derive the point and interval estimates of the parameters.

4.1.1. Point estimators

Based on the Type-I censored data, the likelihood function is given by

$$L(\tau; \theta) = C \prod_{i=1}^{k} \left[P\{t_{i-1} < T \le t_i\}\right]^{n_i} \, \left[P\{T > t_k\}\right]^{n_{k+1}} \,, \tag{4.1}$$

where $C = \frac{n!}{\prod_{\ell=1}^{k+1} n_\ell !}$ is independent of θ. Since,

$$P\{t_{i-1} < T \le t_i\} = F(t_i) - F(t_{i-1})\,,$$

therefore, expression (4.1) can be rewritten as

$$\begin{aligned} L(\tau; \theta) \;\; = \;\; & C \left\{1 - \left[1 - e^{-\left(\alpha t_k + \beta t_k^{\gamma}\right)}\right]^{\lambda}\right\}^{n_{k+1}} \\ & \prod_{i=1}^{k} \left\{\left[1 - e^{-\left(\alpha t_i + \beta t_i^{\gamma}\right)}\right]^{\lambda} - \left[1 - e^{-\left(\alpha t_{i-1} + \beta t_{i-1}^{\gamma}\right)}\right]^{\lambda}\right\}^{n_i} . \end{aligned} \tag{4.2}$$

Thus, the log-likelihood function is

$$\begin{aligned}\mathcal{L}(\tau;\theta) &= \ln C + n_{k+1}\ln\left\{1-\left[1-e^{-(\alpha t_k+\beta t_k^{\gamma})}\right]^{\lambda}\right\} \\ &+\sum_{i=1}^{k} n_i \ln\left\{\left[1-e^{-(\alpha t_i+\beta t_i^{\gamma})}\right]^{\lambda}-\left[1-e^{-(\alpha t_{i-1}+\beta t_{i-1}^{\gamma})}\right]^{\lambda}\right\}. \qquad (4.3)\end{aligned}$$

Let for $i = 1, \cdots, k$,

$$D_i(\theta) = \left[1 - A_i(\alpha,\beta,\gamma)\right]^{\lambda} - \left[1 - A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda},$$

where

$$A_i(\alpha,\beta,\gamma) = \begin{cases} 1, & i = 0, \\ e^{-(\alpha t_i+\beta t_i^{\gamma})}, & i = 1,\cdots,k, \\ 0, & i = k+1. \end{cases}$$

The log-likelihood function (4.3) becomes

$$\begin{aligned}\mathcal{L}(\tau;\theta) &= \ln C + \sum_{i=1}^{k+1} n_i \ln\left\{\left[1 - A_i(\alpha,\beta,\gamma)\right]^{\lambda} - \left[1 - A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda}\right\} \\ &= \ln C + \sum_{i=1}^{k+1} n_i \ln D_i(\theta). \qquad (4.4)\end{aligned}$$

To derive the MLE of the vector of unknown parameters θ, we need to compute the first partial derivatives of the log-likelihood function $\mathcal{L}(\tau;\theta)$ with respect to each parameter. For ease of notation, we will denote, for any function $f(x_1, x_2, x_3, x_4)$, the first partial derivatives by f_{x_i} and the second partial derivatives by $f_{x_i x_j}$.

Using the following derivatives, for $i = 1, \cdots, k$,

$$\frac{\partial A_i(\alpha,\beta,\gamma)}{\partial\alpha} = -t_i A_i(\alpha,\beta,\gamma),$$

$$\frac{\partial A_i(\alpha,\beta,\gamma)}{\partial\beta} = -t_i^{\gamma} A_i(\alpha,\beta,\gamma),$$

$$\frac{\partial A_i(\alpha,\beta,\gamma)}{\partial\gamma} = -\beta t_i^{\gamma}\, ln(t_i) A_i(\alpha,\beta,\gamma),$$

we get the first partial derivatives of $\mathcal{L}(\tau;\theta)$ with respect to θ_j, $j=1,2,3,4$ as

$$\mathcal{L}_{\theta_j} = \sum_{i=1}^{k+1} n_i \frac{\mathcal{D}_{\theta_j}^{(i)}}{D_i(\theta)}, \tag{4.5}$$

where

$$\mathcal{D}_{\alpha}^{(i)} = \lambda \begin{cases} t_i A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}, & i=1, \\ t_i A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1} \\ -t_{i-1} A_{i-1}(\alpha,\beta,\gamma)\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-1}, & i=2,\cdots,k+1, \end{cases}$$

$$\mathcal{D}_{\beta}^{(i)} = \lambda \begin{cases} t_i^{\gamma} A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}, & i=1, \\ t_i^{\gamma} A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1} \\ -t_{i-1}^{\gamma} A_{i-1}(\alpha,\beta,\gamma)\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-1}, & i=2,\cdots,k+1, \end{cases}$$

$$\mathcal{D}_{\gamma}^{(i)} = \lambda \begin{cases} t_i^{\gamma}\beta\ln(t_i) A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}, & i=1, \\ t_i^{\gamma}\beta\ln(t_i) A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1} \\ -t_{i-1}^{\gamma}\beta\ln(t_{i-1}) A_{i-1}(\alpha,\beta,\gamma)\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-1}, & i=2,\cdots,k+1, \end{cases}$$

$$\mathcal{D}_{\lambda}^{(i)} = \begin{cases} \left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda}\ln\left[1-A_i(\alpha,\beta,\gamma)\right], & i=1, \\ \left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda}\ln\left[1-A_i(\alpha,\beta,\gamma)\right] \\ -\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda}\ln\left[1-A_{i-1}(\alpha,\beta,\gamma)\right], & i=2,\cdots,k+1. \end{cases}$$

Setting $\mathcal{L}_{\theta_j}=0$, the likelihood equations become

$$0 = \sum_{i=1}^{k+1} n_i \frac{\mathcal{D}_{\theta_j}^{(i)}}{D_i(\theta)}, \quad j=1,2,3,4. \tag{4.6}$$

The MLE of the parameters α,β,γ and λ are the solution of the system of nonlinear equations (4.6). As it seems, this system has no closed form solution in α,β,γ and λ. Therefore in sections below, we used the mathematical package MathCad to get the numerical solution.

4.1.2. Asymptotic confidence bounds

Since the MLE of the parameters cannot be derived in closed forms, we cannot get the exact confidence bounds of the parameters. The idea is to use the large sample approximation. The maximum likelihood estimators of θ can be treated as being approximately multi-normal with mean θ and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$\left(\hat{\theta}-\theta\right) \to N_4\left(0, \mathbf{I^{-1}}(\hat{\theta})\right), \tag{4.7}$$

where $\mathbf{I^{-1}}\left(\hat{\theta}\right)$ is the variance-covariance matrix of the unknown parameters θ. The element $I_{ij}(\hat{\theta})$, $i, j = 1, 2, 3, 4$, of the 4×4 matrix $\mathbf{I^{-1}}$ is given by

$$I_{ij}(\hat{\theta}) = -\left.\mathcal{L}_{\theta_i\theta_j}\right|_{\theta=\hat{\theta}}. \tag{4.8}$$

From expression (4.5), the second partial derivatives of the log-likelihood function are found to be

$$\mathcal{L}_{\theta_\ell\,\theta_r} = \sum_{i=1}^{k+1} n_i \frac{\mathcal{D}^{(i)}_{\theta_\ell\theta_r} D_i(\theta) - \mathcal{D}^{(i)}_{\theta_\ell}\mathcal{D}^{(i)}_{\theta_r}}{\left[D_i(\theta)\right]^2}, \quad \ell, r = 1, 2, 3, 4, \tag{4.9}$$

where

$$\mathcal{D}^{(i)}_{\alpha\alpha} = -\lambda \begin{cases} t_i^2 A_i(\alpha,\beta,\gamma)\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2}, & i = 1, \\ \\ t_i^2 A_i(\alpha,\beta,\gamma)\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2} \\ -t_{i-1}^2 A_{i-1}(\alpha,\beta,\gamma)\left[1-\lambda A_{i-1}(\alpha,\beta,\gamma)\right]\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-2}, & i = 2, \cdots, k+1, \end{cases}$$

$$\mathcal{D}^{(i)}_{\beta\beta} = -\lambda \begin{cases} t_i^{2\gamma} A_i(\alpha,\beta,\gamma)\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2}, & i = 1, \\ \\ t_i^{2\gamma} A_i(\alpha,\beta,\gamma)\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2} \\ -t_{i-1}^{2\gamma} A_{i-1}(\alpha,\beta,\gamma)\left[1-\lambda A_{i-1}(\alpha,\beta,\gamma)\right] \\ \times\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-2}, & i = 2, \cdots, k+1, \end{cases}$$

$$\mathcal{D}^{(i)}_{\gamma\gamma} = \lambda \begin{cases} \beta t_i^{\gamma} ln^2(t_i) A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2} \\ \quad\times\left[1-A_i(\alpha,\beta,\gamma)\right] - \beta t^{\gamma}\left[1-\lambda A_i(\alpha,\beta,\gamma)\right], & i = 1, \\ \\ \beta t_i^{\gamma} ln^2(t_i) A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2} \\ \quad\times\ \left[1-A_i(\alpha,\beta,\gamma)\right] - \beta t^{\gamma}\left[1-\lambda A_i(\alpha,\beta,\gamma)\right] \\ \quad-\ \beta t_{i-1}^{\gamma} ln^2(t_{i-1}) A_{i-1}(\alpha,\beta,\gamma)\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-2} \\ \quad\times\ \left[1-A_{i-1}(\alpha,\beta,\gamma)\right] - \beta t^{\gamma}\left[1-\lambda A_{i-1}(\alpha,\beta,\gamma)\right], & i = 2, \cdots, k+1, \end{cases}$$

$$\mathcal{D}^{(i)}_{\lambda\lambda} = \begin{cases} \left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda}\ln^2\left[1-A_i(\alpha,\beta,\gamma)\right] & i = 1, \\ \\ \left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda}\ln^2\left[1-A_i(\alpha,\beta,\gamma)\right] \\ \quad-\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda}\ln^2\left[1-A_{i-1}(\alpha,\beta,\gamma)\right], & i = 2, \cdots, k+1, \end{cases}$$

$$\mathcal{D}^{(i)}_{\alpha\beta} = -\lambda \begin{cases} t_i^{\gamma+1} A_i(\alpha,\beta,\gamma)\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2}, & i=1, \\ t_i^{\gamma+1} A_i(\alpha,\beta,\gamma)\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2} \\ \quad -t_{i-1}^{\gamma+1} A_{i-1}(\alpha,\beta,\gamma)\left[1-\lambda A_{i-1}(\alpha,\beta,\gamma)\right] \\ \quad \times\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-2}, & i=2,\cdots,k+1, \end{cases}$$

$$\mathcal{D}^{(i)}_{\alpha\gamma} = -\lambda \begin{cases} t_i^{\gamma+1}\beta\, ln(t_i) A_i(\alpha,\beta,\gamma)\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2}, & i=1, \\ t_i^{\gamma+1}\beta\, ln(t_i) A_i(\alpha,\beta,\gamma)\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2} \\ \quad -t_{i-1}^{\gamma+1}\beta\, ln(t_{i-1}) A_{i-1}(\alpha,\beta,\gamma)\left[1-\lambda A_{i-1}(\alpha,\beta,\gamma)\right] \\ \quad \times\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-2}, & i=2,\cdots,k+1, \end{cases}$$

$$\mathcal{D}^{(i)}_{\alpha\lambda} = \begin{cases} t_i A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}\Big\{1+\lambda\ln\left[1-A_i(\alpha,\beta,\gamma)\right]\Big\}, & i=1, \\ t_i A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}\Big\{1+\lambda\ln\left[1-A_i(\alpha,\beta,\gamma)\right]\Big\} \\ \quad -t_{i-1} A_{i-1}(\alpha,\beta,\gamma)\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-1} \\ \quad \times\Big\{1+\lambda\ln\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]\Big\}, & i=2,\cdots,k+1, \end{cases}$$

$$\mathcal{D}^{(i)}_{\beta\gamma} = \begin{cases} \lambda\ln(t_i) {t_i}^{\gamma} A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2} \\ \Big\{(1-A_i(\alpha,\beta,\gamma)-\beta {t_i}^{\gamma}\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\Big\}, & i=1, \\ \lambda\ln(t_i) {t_i}^{\gamma} A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-2}\times \\ \Big\{(1-A_i(\alpha,\beta,\gamma)-\beta {t_i}^{\gamma}\left[1-\lambda A_i(\alpha,\beta,\gamma)\right]\Big\}+ \\ \quad \lambda\ln(t_{i-1}) {t_{i-1}}^{\gamma} A_{i-1}(\alpha,\beta,\gamma)\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-2}\times \\ \Big\{(1-A_{i-1}(\alpha,\beta,\gamma)-\beta {t_{i-1}}^{\gamma}\left[1-\lambda A_{i-1}(\alpha,\beta,\gamma)\right]\Big\}, & i=2,\cdots,k+1. \end{cases}$$

$$\mathcal{D}^{(i)}_{\beta\lambda} = \begin{cases} {t_i}^{\gamma} A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}\Big\{1+\lambda\ln\left[1-A_i(\alpha,\beta,\gamma)\right]\Big\}, & i=1, \\ {t_i}^{\gamma} A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}\Big\{1+\gamma\ln\left[1-A_i(\alpha,\beta,\gamma)\right]\Big\} \\ \quad -{t_{i-1}}^{\gamma} A_{i-1}(\alpha,\beta,\gamma)\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-1}\times \\ \quad \Big\{1+\lambda\ln\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]\Big\}, & i=2,\cdots,k+1. \end{cases}$$

$$\mathcal{D}^{(i)}_{\gamma\lambda} = \begin{cases} \beta {t_i}^{\gamma} ln(t_i) A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}\times \\ \Big\{1+\lambda\ln\left[1-A_i(\alpha,\beta,\gamma)\right]\Big\}, & i=1, \\ \beta {t_i}^{\gamma} ln(t_i) A_i(\alpha,\beta,\gamma)\left[1-A_i(\alpha,\beta,\gamma)\right]^{\lambda-1}\times \\ \Big\{1+\lambda\ln\left[1-A_i(\alpha,\beta,\gamma)\right]\Big\} - \\ \quad \beta {t_{i-1}}^{\gamma} ln(t_{i-1}) A_{i-1}(\alpha,\beta,\gamma)\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]^{\lambda-1}\times \\ \quad \Big\{1+\lambda\ln\left[1-A_{i-1}(\alpha,\beta,\gamma)\right]\Big\}, & i=2,\cdots,k+1. \end{cases}$$

Therefore, the approximate $100(1-\vartheta)\%$ two-sided confidence intervals for α, β, γ and

λ are, respectively, given by

$$\hat{\alpha} \pm Z_{\vartheta/2}\sqrt{\mathbf{I}_{11}^{-1}(\hat{\theta})}\,, \qquad \hat{\beta} \pm Z_{\vartheta/2}\sqrt{\mathbf{I}_{22}^{-1}(\hat{\theta})}\,, \qquad \hat{\gamma} \pm Z_{\vartheta/2}\sqrt{\mathbf{I}_{33}^{-1}(\hat{\theta})}\,, \qquad \hat{\lambda} \pm Z_{\vartheta/2}\sqrt{\mathbf{I}_{44}^{-1}(\hat{\theta})}\,.$$

Here, $Z_{\vartheta/2}$ is the upper $(\vartheta/2)th$ percentile of the standard normal distribution.

4.2. Bayes procedure

In this subsection, we use the Bayes procedure to derive the point and interval estimates of the parameters. To obtain the Bayes estimates, we need the following additional assumptions:

B.1) The parameters θ_i, $i = 1, 2, 3, 4$, behave as independent random variables.

B.2) The prior pdf of θ_i is symmetrical triangular on the interval $[a_i,\, b_i]$. Namely,

$$g_{\theta_i}(u) = \frac{1}{\varepsilon_i}\left(\varepsilon_i - |u - \mu_i|\right)\,, \quad u \in [a_i,\, b_i] \subset (0,\, \infty), \tag{4.10}$$

where $\varepsilon_i = \frac{b_i - a_i}{2}$ and $\mu_i = \frac{b_i + a_i}{2}$.

B.3) The loss function is

$$\ell(\theta, \hat{\theta}) = \sum_{i=1}^{4} \kappa_i \left(\theta_i - \hat{\theta}_i\right)^2\,, \quad \kappa_i > 0. \tag{4.11}$$

Based on the assumptions (B.1) and (B.2), the joint prior pdf of θ is

$$g(\theta) \;=\; \prod_{i=1}^{4} \frac{1}{\varepsilon_i}\left(\varepsilon_i - |\theta_i - \mu_i|\right), \quad \theta_i \in [a_i,\, b_i]. \tag{4.12}$$

When the joint prior pdf is (4.12), the joint posterior pdf of θ, given the available observations, is

$$\pi(\theta|\tau) \;=\; \frac{\psi(\theta)}{I_0}\,, \tag{4.13}$$

where

$$\begin{aligned}\psi(\theta) \;=\; & \left\{\prod_{i=1}^{4}\left(\varepsilon_i - |\theta_i - \mu_i|\right)\right\}\left\{1 - \left[1 - e^{-\left(\theta_1 t_k + \theta_2 t_k^{\theta_3}\right)}\right]^{\theta_4}\right\}^{n_{k+1}} \\ & \prod_{i=1}^{k}\left\{\left[1 - e^{-\left(\theta_1 t_i + \theta_2 t_i^{\theta_3}\right)}\right]^{\theta_4} - \left[1 - e^{-\left(\theta_1 t_{i-1} + \theta_2 t_{i-1}^{\theta_3}\right)}\right]^{\theta_4}\right\}^{n_i}\,, \end{aligned} \tag{4.14}$$

and

$$I_0 = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\int_{a_4}^{b_4} \psi(\theta_1,\theta_2,\theta_3,\theta_4)\,d\theta_4\,d\theta_3\,d\theta_2\,d\theta_1. \quad (4.15)$$

The marginal posterior pdfs of θ_i, $i = 1, 2, 3, 4$, are given by

$$\pi_1(\theta_1|\tau) = \frac{\theta_1}{I_0}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\int_{a_4}^{b_4} \psi(\theta_1,\theta_2,\theta_3,\theta_4)\,d\theta_4 d\theta_3\,d\theta_2, \quad (4.16)$$

$$\pi_2(\theta_2|\tau) = \frac{\theta_2}{I_0}\int_{a_1}^{b_1}\int_{a_3}^{b_3}\int_{a_4}^{b_4} \psi(\theta_1,\theta_2,\theta_3,\theta_4)\,d\theta_4\,d\theta_3\,d\theta_1, \quad (4.17)$$

$$\pi_3(\theta_3|\tau) = \frac{\theta_3}{I_0}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_4}^{b_4} \psi(\theta_1,\theta_2,\theta_3,\theta_4)\,d\theta_4 d\theta_2\,d\theta_1. \quad (4.18)$$

$$\pi_4(\theta_4|\tau) = \frac{\theta_3}{I_0}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} \psi(\theta_1,\theta_2,\theta_3,\theta_4)\,d\theta_3\,d\theta_2\,d\theta_1. \quad (4.19)$$

4.2.1. Point estimators

Under the assumption (B.3), the Bayes estimate of θ_i, say $\tilde{\theta}_i$, and the associated posterior risk, say $R(\tilde{\theta}_i)$, are

$$\tilde{\theta}_i = \frac{I^{(1)}_{\theta_i}}{I_0}, \quad (4.20)$$

and

$$R(\tilde{\theta}_i) = \frac{I^{(2)}_{\theta_i}}{I_0} - \left[\frac{I^{(1)}_{\theta_i}}{I_0}\right]^2, \quad (4.21)$$

where, for $j = 1, 2$,

$$I^{(j)}_{\theta_i} = \int_{a_4}^{b_4}\int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} \theta_i^j\,\psi(\theta_1,\theta_2,\theta_3,\theta_4)\,d\theta_1\,d\theta_2\,d\theta_3\,d\theta_4. \quad (4.22)$$

4.2.2. Interval estimates

The Bayesian approach to interval estimation is much more direct than the maximum likelihood approach. Once the marginal posterior pdf of θ_i has been obtained, a symmetric $100(1-\vartheta)\%$ two-sided Bayes probability interval [$100(1-\vartheta)\%$ TBPI] estimate of θ_i, $i = 1, 2, 3, 4$, denoted $[\ell_i,\, u_i]$, is obtained by solving the following two equations:

$$\begin{aligned}\int_0^{\ell_i} \pi_i(w|\tau)dw &= \frac{\vartheta}{2},\\ \int_{u_i}^{\infty} \pi_i(w|\tau)dw &= \frac{\vartheta}{2},\end{aligned} \quad (4.23)$$

for the limits ℓ_i and u_i. As it seems, the above two equations have no analytical solution in ℓ_i and u_i. We again use in Section 5 below, the MathCad package to get the numerical solution.

5. DATA ANALYSIS

In this section, we use the real data set from Nelson (1982), which reports a set of cracking data on 167 independent and identically parts in a machine. The test duration was 63.48 months and 8 unequally spaced inspections were conducted to obtain the number of cracking parts in each interval. The data were

$$(t_1, \cdots, t_8) = (6.12,\ 19.92,\ 29.64,\ 35.40,\ 39.72,\ 45.24,\ 52.32,\ 63.48)$$

and

$$(n_1, \cdots, n_9) = (5,\ 16,\ 12,\ 18,\ 18,\ 2,\ 6,\ 17,\ 73)$$

We assume that these data follow the following two distributions: (1) MWD(α, β, γ); and (2) NGWD$(\alpha, \beta, \gamma, \lambda)$. Then we compute the maximum likelihood estimates of the parameters included in each distribution and we compare these distributions based on two different criteria. The criteria used are: the log-likelihood function and the Kolmogorov-Smirnov (K-S) test statistic. Table 5.1 shows the MLE of the parameters of the distributions considered and the associated log-likelihood function values.

Table 5.1. The MLE of the parameters, the values of log-likelihood and K-S.

Distribution	parameters	$\mathcal{L}$	K-S
MWD	$\hat{\alpha} = 1,281 \times 10^{-3},\ \hat{\beta} = 1.134 \times 10^{-3},\ \hat{\gamma} = 1.566$	-309.651	0.138
NGWD	$\hat{\alpha} = 0.024,\ \hat{\beta} = 0.165,\ \hat{\gamma} = 0.002,\ \hat{\lambda} = 2.683$	-309.223	0.137

Based on the values of the log-likelihood function and the K-S statistic in Table 1, the NGWD$\big(0.024, 0.165, 0.002, 2.683\big)$ fits the above data slightly better than the MWD$\big(1.281 \times 10^{-3}, 1.134 \times 10^{-3}, 1.566\big)$.

The survival function of the data is estimated using non-parametric and parametric methods. In the case of non-parametric estimation, we used Kaplan-Meier (K-M) method. For the parametric estimations, we used the two models MWD and NGWD. Figure 5.1 shows the Kaplan-Meier estimate of the survival function and its two parametric estimates. It seems from this figure that the NGWD model is slightly closer to the K-M than the MWD model. This result agrees with the values of K-S test statistics and the values of the log-likelihood functions shown in Table 5.1.

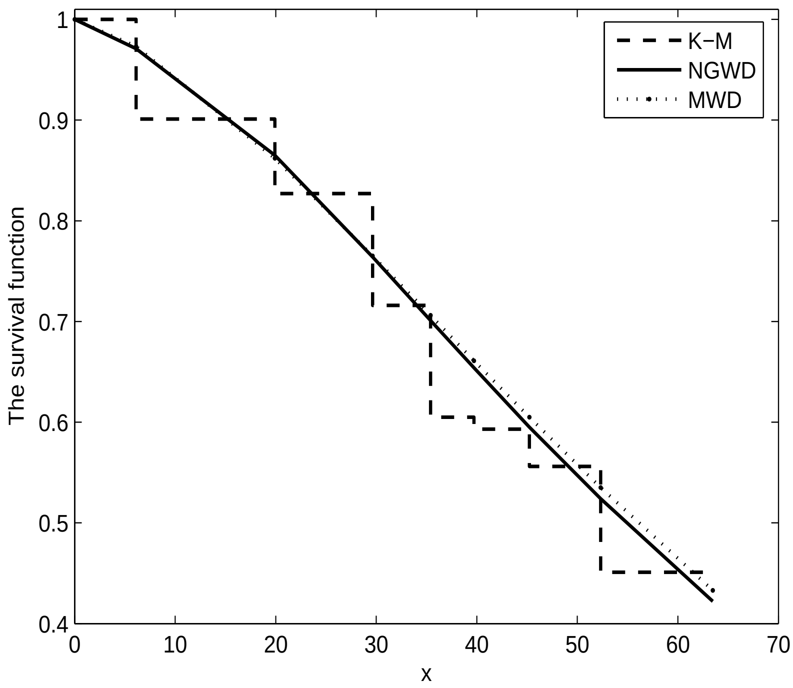

Figure 5.1. The empirical and fitted survival functions.

Figure 5.2 shows the variation of the hazard rate functions of the NGWD$\big(2.349\times10^{-5}, 0.078, 1.654, 9.313\big)$ and the MWD$\big(1.281\times10^{-3}, 1.134\times10^{-3}, 1.566\big)$.

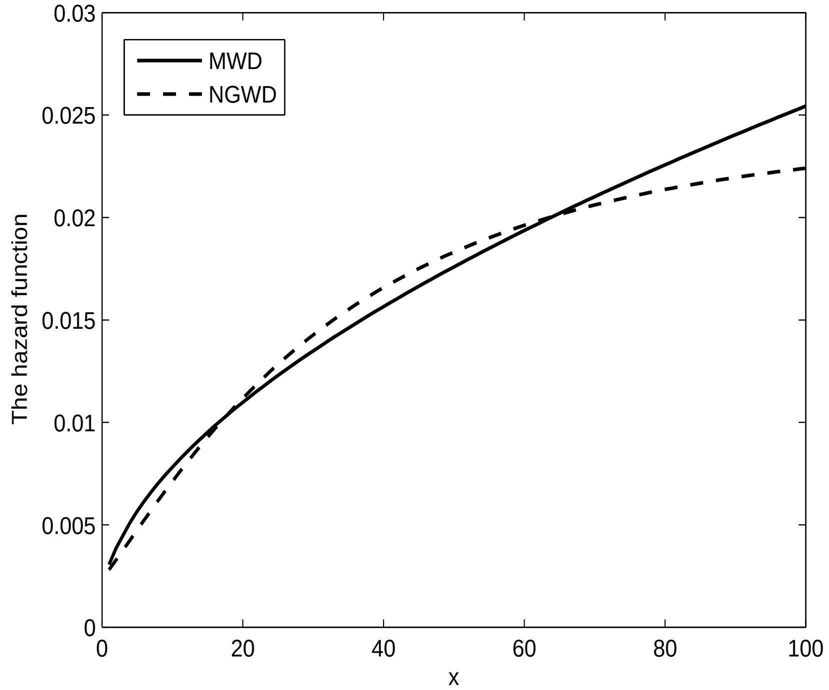

Figure 5.2. The estimated hazard rate functions.

Figure 5.3 depicts the forms of the estimated pdf of the data considering the two models NGWD$\big(2.349 \times 10^{-5}, 0.078, 1.654, 9.313\big)$ and MWD$\big(1.281 \times 10^{-3}, 1.134 \times 10^{-3}, 1.566\big)$.

For the Bayes technique, it is assumed that the prior pdf of α, β , γ and λ have the supports $[0, 0.1]$, $[0, 0.1]$, $[0, 0.1]$,and $[0.5, 1.7]$, respectively. These supports were chosen with the help of the MLE of the parameters obtained above. The Bayes point estimates of the parameters were found as

$$\tilde{\alpha} = 0.017, \quad \tilde{\beta} = 0.02, \quad \tilde{\gamma} = 0.399, \quad \tilde{\lambda} = 1.653$$

with corresponding minimum posterior risk:

$$R(\tilde{\alpha}) = 1.39 \times 10^{-5}, \quad R(\tilde{\beta}) = 3.279 \times 10^{-4}, \quad R(\tilde{\gamma}) = 0.088, \quad R(\tilde{\lambda}) = 0.014.$$

The value of the K-S test statistic when we use the Bayes estimate of the parameters is 0.159

which is greater than the K-S value using the MLE of the parameters. This means that the maximum likelihood procedure provides better estimates than the Bayes procedure in this sense. Also, both MathCad and Matlab packages were unable to converge to a solution in the set of equations (4.23) which give the two-sided Bayes probability interval estimates of the parameters.

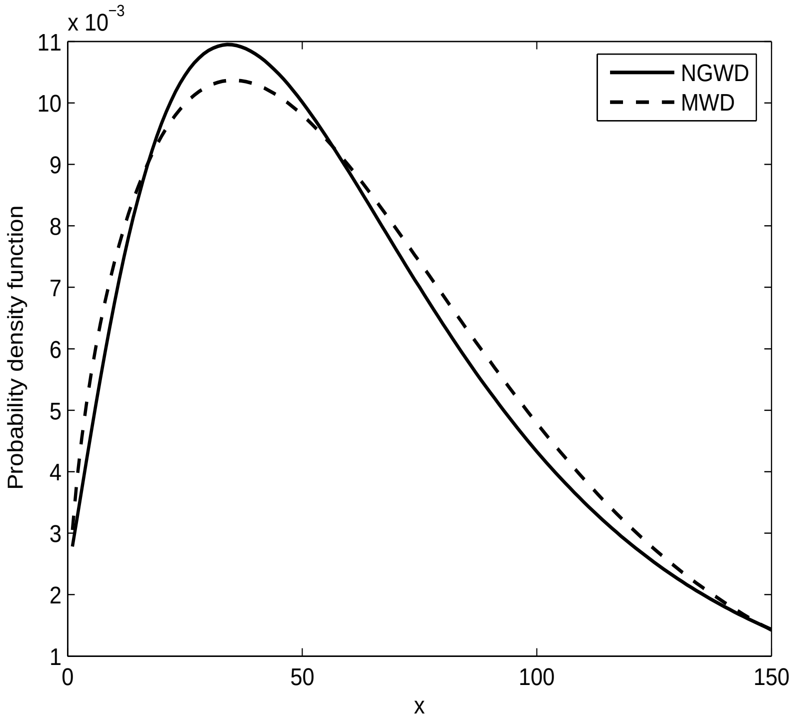

Figure 5.3. The estimated probability density functions.

6. CONCLUSION

In this paper, we discussed the parameter estimation of the NGWD$(\alpha, \beta, \gamma, \lambda)$. The maximum likelihood and Bayes techniques have been used. The MWD(α, β) is tested against NGWD$(\alpha, \beta, \gamma, \lambda)$ using a set of real data. Based on the two criteria (the values of the log-likelihood function and K-S test statistic), we found that the NGWD$(\alpha, \beta, \gamma, \lambda)$ fits the data better than the MWD(α, β, γ). Further, we used the MLE of the parameters of NGWD($\alpha, \beta, \gamma, \lambda$) to construct suitable prior supports for α, β, γ, and λ. In spite of this, the maximum likelihood procedure provides better estimates than the Bayes procedure in the sense of having smaller K-S. Also, we failed to obtain numerically the TBPI of the parameters. Finally, we conclude that the maximum likelihood procedure outperforms the Bayes procedure in this situation.

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