International Journal of Reliability and Application
Vol. 11, No. 1, pp. 1-16, 2010

Parameters Estimation of Generalized Linear Failure Rate Semi-Markov Reliability Models

A. El-Gohary * and **A. Al-Khedhairi**
Department of Statistics and O. R., College of Science
King Saud University, P. O. Box 2455,
Riyadh, 11451 Saudi Arabia

Abstract. In this paper we will discuss the stochastic analysis of a three state semi-Markov reliability model. Maximum likelihood procedure will be used to obtain the estimators of the parameters included in this reliability model. Based on the assumption that the lifetime and repair time of the system units are generalized linear failure rate random variables, the reliability function of this system is obtained. Also, the distribution of the first passage time of this system will be derived. Some important special cases are discussed.

Key Words :*Linear failure rate distribution, Maximum likelihood estimators, Three state semi-Markov reliability model, System reliability, Operating unit, First passage.*

1. INTRODUCTION

A semi-Markov model has been used by Kao (1974) in some context of hospital administration in the study of the dynamics of movement of patients through care areas in a hospital. El-Gohary (2005) has used the semi-Markov process to describe a reliability system that consists of one active unit, an identical spare, a switch and a repair facility. In this paper it is assumed that the lifetimes of the active repair units are generalized exponentially distributed (El-Gohary(2004), Reinhard and Snoussi(2002) and Kulkarni(1995)).

The stochastic models have many applications in different fields such as reliability systems, social security policy analysis, health care services [3,4,6].

The evolution of many systems naturally ends as the first failure occurs, because external intervention is not practicable. These systems are non-repairable systems. For other systems, generally of high complexity, renewal possibilities exist, and their effectiveness therefore depends not only on their intrinsic reliability but also on the characteristics of maintenance and repair actions.

*Corresponding author.
E-mail address: elgohary0@yahoo.com
Permanent address: Department of Mathematics, College of Science, Mansoura University Mansoura, 35516, Egypt.

To discuss the stochastic analysis of our reliability model, we present some important definitions. A semi Markov process $\{X(t) : t \geq 0\}$ is a stochastic process in which changes of state occur according to a Markov chain with the time interval between two successive transitions is a random variable whose distribution depends on the state from which the transition takes place as well as the state to which the next transition takes place (Medhi(1982)). Generally a semi-Markov process with discrete state space can be defined as a Markov renewal process (Kao(1974)).

In this paper, in section *2*, we will display some important definitions and properties of a semi-Markov process and its kernel. In section *3* we use the stochastic analysis and semi-Markov model to estimate the parameters included in some reliability models. The maximum likelihood method is used to derive the point and confidence interval estimates of these parameters. Based on the assumption that the lifetime and repair time of the system units are generalized linear failure rate random variables the distribution of the first passage time of this system will be derived.

2. SEMI-MARKOV KERNEL

In this section, we consider some kind of generalization of a Markov process as well as of a renewal process. Also, we will discuss the semi-Markov kernel. Further, the properties of the semi-Markov kernel will be discussed.

Definition 2.1 Let the state of a stochastic process be denoted by the set of nonnegative integers, $S = \{0, 1, 2, \ldots\}$, and let the transitions of the process occur at time instants $t_0 = 0, t_1, t_2, \ldots (t_n < t_{n+1})$. Assume that X_n denote the transition occurring at time instant t_n. Then the twice $\{X_n, t_n\}$, $n = 0, 1, 2, \ldots$ is said to constitute a Markov renewal process if

$$\begin{aligned} P\{X_{n+1} = k, t_{n+1} - t_n \leq t | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n; t_0, t_1, \ldots, t_n\} &= \\ P\{X_{n+1} = k, t_{n+1} - t_n \leq t | X_n = i_n\}, \end{aligned} \tag{2.1}$$

Definition 2.2 The Markov renewal process $\{X_n, t_n\}$, $n = 0, 1, 2, \ldots$ is said to homogeneous if

$$P\{X_{n+1} = k, t_{n+1} - t_n \leq t | X_n = i\} = Q_{ik}(t) \tag{2.2}$$

does not depend on n

Lemma 2.3 Assume that $\{X_n, n = 0, 1, 2, \ldots\}$ constitutes a Markov chain with state space $S = \{0, 1, 2, \ldots\}$,and transition probability matrix $P = \{p_{ij}\}$. The continuous parameter process $Y(t)$ with state space $S = \{0, 1, 2, \ldots\}$, defined by

$$Y(t) = X_n, \ t_n \leq t < t_{n+1} \tag{2.3}$$

is called semi-Markov process.

The semi-Markov process is a stochastic process which changes its state occur according to a Markov chain and the time interval between two successive transitions is a random variable, whose distribution may be depend not only on the present state but also on the state of the next transition. Next, through this paper we assume the state space S of the semi-Markov process is finite number of renewal times on the time interval $[0, t]$.

Definition 2.4 A two-dimensional Markov process $\{\xi_n, \vartheta_n, n \in N\}$ with values in $S \times [0, \infty)$ is called a Markov renewal process if and only if

1. $Q_{ij} = P\{\xi_{n+1} = j, \vartheta_{n+1} \leq t | \xi_n = i, \vartheta_n = t_n, \ldots, \xi_0 = i_0, \vartheta_0 = t_0\}$

   $= P\{\xi_{n+1} = j, \vartheta_{n+1} \leq t | \xi_n = i\}$

2. $P\{\xi_0 = i, \vartheta_0 = 0\} = p_{i0}$

In the Markov renewal process, the non-negative random variables $\vartheta_n, n \geq 1$, define the interval between Markov renewal times:

$$\tau_n = \sum_{k=1}^{n} \vartheta_k, \quad n \geq 1, \tau_0 = 0$$

Now, let

$$\nu(t) := \sum_{n=1}^{\infty} I_{[0,t]}(\tau_n) \tag{2.4}$$

where

$$I_{[0,t]}(\tau_n) = \begin{cases} 1 & \text{if} \quad \tau_n \in [0, t] \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

The process $\nu(t)$ is called a counting process. It determines the number of renewal times on the segment $[0, t]$.

Definition 2.5 A stochastic process $\{X(t) : t \geq 0\}$ where $X(t) = \xi_{\nu(t)}$ is called a semi-Markov process that generated by the Markov renewal process with initial distribution $P_i^0 = p(\xi_0 = i)$ and the kernel $Q(t), t \geq 0$.

Since the counting process $\nu(t)$ keeps constant values on the half-interval $[t_n, t_{n+1})$ and is continuous from the right, then the semi-Markov process keeps also constant values on the half intervals $[\tau_n, \tau_{n+1})$: $X_n(t) = \xi_n$ for $t \in [\tau_n, \tau_{n+1})$. Moreover the sequence $\{X(\tau_n) : n \in N\}$ is a Markov chain with transition probability matrix $P = \{p_{ij} = Q_{ij}(\infty), i, j \in S\}$ that is called an embedded Markov chain. The concept of a Markov renewal process is a natural generalization of the concept of the ordinary renewal process given by a sequence of independent identically non-negative random variables $\theta_n, n \geq 1$. The random variables θ_n can be interpreted as lifetimes.

Definition 2.6 The stochastic matrix $\mathbf{Q}(t) = [\mathbf{Q}_{ij}(t); i, j \in S]$, $t \geq 0$ is said to be a renewal kernel if and only if the following conditions are satisfied:

1. The functions $\mathbf{Q}_{ij}(t)$ are nondecreasing functions in t.
2. $\sum_{j \in S} \mathbf{Q}_{ij} = G_i(t)$ are distribution functions in t.
3. $[\mathbf{Q}_{ij}(+\infty) = P_{ij}, i, j \in S] = P$ is a stochastic matrix.

Lemma 2.7 Assume that $\{X(t) : t \geq 0\}$ is a semi-Markov process with renewal kernel

$$\mathbf{Q}(t) = \mathbf{Q}_{ij}(t), i, j \in S,\ t \in [0, \infty) \tag{2.6}$$

then

$$P\{\xi_0 = i_0, \vartheta_0 = 0, \xi_1 = i_1, \vartheta_1 \leq s_1, \ldots, \xi_n = i_n, \vartheta_n \leq s_n\} = p_{i_0} \prod_{k=1}^{n} \mathbf{Q}_{i_{k-1} i_k}(s_k) \tag{2.7}$$

A main objective of this paper is to use a three state semi-Markov process to describe a reliability system which consists of operating unit, identical spare unit, a switch and repair facility. Also, use the maximum likelihood procedure to obtain the estimators of the unknown parameters included in this reliability system.

In what follows, we will use Lemma 2.7 and semi-Markov kernel to introduce the likelihood probability contribution function for the underlying reliability system.

2.1. Likelihood function

In this section, we use the semi-Markov realization to construct the likelihood function of the standby system with repair. In such study, we assume that the semi-Markov renewal kernel of the desired reliability system depends on a vector of unknown parameters $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$. Therefore,

$$\mathbf{Q}(t|\underline{\theta}) = \{\mathbf{Q}_{ij}(t|\underline{\theta}) : i, j \in S\}, \tag{2.8}$$

Let us assume that there is a sequence of random observations $(i_0, t_0), (i_1, t_1), \ldots, (i_n, t_n)$ of the random vector $(\xi_0, \vartheta_0), (\xi_1, \vartheta_1), \ldots, (\xi_n, \vartheta_n)$. Suppose z denotes the observation (i_0, t_0), $(i_1, t_1), \ldots, (i_n, t_n)$. We assume that there exist functions denoted by $q_{ij}(t|\theta), i, j \in S$ such that

$$\mathbf{Q}_{ij}(t|\underline{\theta}) = \int_0^t q_{ij}(u|\underline{\theta}) du \tag{2.9}$$

Using Lemma 2.7, the likelihood function for the given random observations of the semi-Markov process becomes

$$L(z; \underline{\theta}) = p_{i_0} \prod_{s=1}^{n} q_{i_{s-1} i_s}(t_s|\underline{\theta}) \tag{2.10}$$

Theorem 2.8 Suppose that $\{X(t) : t \in R_+\}$ be a semi-Markov process with a finite set of states $S\{1, 2, \ldots r\}$ and having semi-Markov kernel $\mathbf{Q}(.)$. Let $\{(i_0, t_0), (i_1, t_1), (i_2, t_2), \ldots, (i_n, t_n)\}$ be a given observation vector of two dimension random vector $\{(\xi_0, \vartheta_0), (\xi_1, \vartheta_1), (\xi_2, \vartheta_2), \ldots, (\xi_n, \vartheta_n)\}$, where $i_0, i_1, i_2, \ldots, i_n \in S$ and $t_0, t_1, t_2, \ldots, t_n \in R+$. Assume also that the continuous densities $q_{ij}(.)$ corresponding to the semi-Markov kernel exist such that

$$\mathbf{Q}_{ij}(t) = \int_0^t q_{ij}(s) ds,\ \forall i, j \in S,\ t \in R+ \tag{2.11}$$

Then the likelihood function of the given observations is given by:

$$L(i_0, i_1, i_2, \ldots, i_n; t_0, t_1, t_2, \ldots, t_n; \underline{\theta}) = p_{i_0} \prod_{s=1}^{n} q_{i_{s-1} i_s}(t_k, \underline{\theta}) \tag{2.12}$$

Proof.
The proof of this theorem can be reached by using Lemma 1.7. Therefore according to this lemma we have

$$P\{\xi_0 = i_0, \xi_s, t_s \leq t_s, s = 1, 2, \ldots, n\} = p_{i_0} \prod_{k=1}^{n} \mathbf{Q}_{i_{k-1}i_k}(t_k|\underline{\theta}) \tag{2.13}$$

Using (2.11)we get

$$\prod_{k=1}^{n} \mathbf{Q}_{i_{k-1}i_k}(t_k|\underline{\theta}) = \int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\ldots\int_0^{t_n} q_{i_0i_1}(s_1,\underline{\theta})q_{i_1i_2}(s_2,\underline{\theta})\ldots q_{i_{n-1}i_n}(s_n,\underline{\theta})ds_1ds_2\ldots ds_n \tag{2.14}$$

Combining (2.13) and (2.14) we can reach to (2.12) which completes the proof. Next, we proceed to apply maximum likelihood procedure to obtain estimators of the parameters included in a three-state semi Markov reliability model. In this study we will consider both of the life and repair times of the standby system with repair are generalized linear failure random variables.

3. SEMI-MARKOV MODEL OF STANDBY SYSTEM

This section is devoted to introduce the assumptions of the underling reliability model. Also the semi-Markov kernel of the stochastic process that describe this reliability model will be introduced. Further, the densities corresponding to this kernel will be obtained.

The semi-Markov process is a nice tool to describe many reliability models. The model of this is a slight modification of well a known reliability model introduced by Barlow and Proschan (1965). In order to describe a reliability model of a standby system with a repair facility, the following assumptions are needed:

1. We consider a one reliability system which consists of one active unit, an identical spare, a switch and a repair facility.
2. As the active unit fails, the spare unit operates by the switch immediately.
3. The failed units can be repaired by the repair facility and the repair fully restore the units. Therefore, the repaired unit can be considered as new one.
4. The whole system fails as the active unit fails and repair unit has not been finished yet or as the active unit fails and the switch fails.
5. The lifetimes of the active units can be represented by independent and identical non-negative random variables T with probability density function $f(t), t \geq 0$.
6. The lengths of repair periods of the units can be represented by independent and identical non-negative random variable Θ with the distribution function $H(t) = P\{\Theta \leq t\}$.
7. The event E denotes the switch-over as the active unit fails. So the probability that the switch performs as required is represented by $P(E) = \theta_0$.
8. The whole system can also be repaired, and the failed system is replaced by a new identical one.

9. The replacing time is represented by a non-negative random variable K with distribution function $C(t) = P\{K \leq t\}$.

10. Finally, we assume that all the random variables that described above are independent.

The standby reliability system with repair facility can be described by a three state semi-Markov process with $S = \{0, 1, 2\}$.

Using the model assumptions that the states of the prescribed system can be considered as follows:

1. The system failure represents the first state of the semi-Markov describing the model and denoted by (0).

2. The failed unit is repaired and the standby unit is operating represents the second state of the semi-Markov describing the model and denoted by (1).

3. Both active and standby units are "Up" represents the third state of the semi-Markov describing the model and denoted by (2)

Let $\tau_0^*, \tau_1^*, \tau_2^*, \ldots$ denote the instants which the state of the system changes, where $\tau_0^* = 0$ and let $\{Y(t) : t \geq 0\}$ be a stochastic process with state space $S = \{0, 1, 2\}$. This process keeps constant values on the half intervals $[\tau_n^*, \tau_{n+1}^*)$ and is continuous from the right. Therefore, it is not a semi-Markov process.

Let us define a new stochastic process as follows:

Assuming that $\tau_0 = 0$ and $\tau_n, n = 1, 2, \ldots$ represent the instants when the components of the system failed or the whole system renewal. The stochastic process $\{X(t) : t \geq 0\}$ defined by

$$X(0) = 0, X(t) = Y(\tau_n),\ \tau_n \leq t < \tau_{n+1}, \tag{3.1}$$

is a semi-Markov process and its kernel density is given by the following square matrix

$$[q_{ij}] = \begin{bmatrix} 0 & 0 & q_{02} \\ q_{10} & q_{11} & 0 \\ q_{20} & q_{21} & 0 \end{bmatrix} \tag{3.2}$$

It is well-known that, the semi-Markov process $\{X(t), t \geq 0\}$ is completely specified by its semi-Markov kernel. Let us deduce the elements of the density semi-Markov kernel which describe the underlying reliability model as follows:

$$\begin{aligned}
q_{02}(t) &= \frac{d}{dt}P\{X(\tau_{n+1})=2,\vartheta_{n+1}\le t|\,X(\tau_n)=0\} = \frac{d}{dt}P\{k\le t\} = c(t) \\
q_{10}(t) &= \frac{d}{dt}P\{X(\tau_{n+1})=0,\vartheta_{n+1}\le t|\,X(\tau_n)=1\} \\
&= \frac{d}{dt}P\{T\le t,\Theta>T\} + P\{\bar{E},T\le t,\Theta<T\} \\
&= \frac{d}{dt}\left[\int_0^t[1-H(t)]dF(t) + (1-\theta_0)\int_0^t H(x)dF(x)\right] = (1-\theta_0)f(t), \\
q_{11}(t) &= \frac{d}{dt}P\{X(\tau_{n+1})=1,\vartheta_{n+1}\le t|X(\tau_n)=1\} \\
&= \frac{d}{dt}P\{E,T\le t,\Theta>T\} = \theta_0 H(t)f(t) \\
q_{21}(t) &= \frac{d}{dt}P\{X(\tau_{n+1})=1,\vartheta_{n+1}\le t|X(\tau_n)=2\} = \frac{d}{dt}P\{E,T\le t\} = \theta_0\, f(t) \\
q_{20}(t) &= \frac{d}{dt}P\{X(\tau_{n+1})=0,\vartheta_{n+1}\le t|X(\tau_n)=2\} = \frac{d}{dt}P\{\bar{E},\,T\le t\} = (1-\theta_0)\, f(t)
\end{aligned} \tag{3.3}$$

where $\bar{E}$ is the complementary event of E

It is well known that, some of the statistical distributions have a constant failure rate such as the exponential distribution, and other distributions have increasing failure rates such as linear failure rate distribution, and some others with decreasing failure rates such as Weibull distribution with shape parameter does not exceed one and other distributions with all of these types of failure rates on different periods of time such as those distributions having failure rate of the bath-tub curve shape see for example Jackson (1969), Lai, et al. (2001) and Lawless (2003). The generalized linear failure rate distribution having such these properties.

Now, we assume that the lifetime of the active units have identically generalized linear failure rate distribution with the parameters θ_1, θ_2 and θ_3. Therefore, for $\theta_1>0, \theta_2>0$ and $\theta_3>0$, distribution function of the lifetime T of the active units is given by

$$F(t;\theta_1,\theta_2,\theta_3) = \left[1-e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\right]^{\theta_3},\ t\ge 0 \tag{3.4}$$

and the probability density and hazard rate functions of the lifetime T of the active units are given by

$$f(t;\theta_1,\theta_2,\theta_3) = \theta_3(\theta_1+\theta_2\,t)\,e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\left[1-e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\right]^{\theta_3-1},\ t\ge 0 \tag{3.5}$$

and

$$h(t;\theta_1,\theta_2,\theta_3) = \frac{\theta_3(\theta_1+\theta_2\,t)\,e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\left[1-e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\right]^{\theta_3-1}}{1-\left[1-e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\right]^{\theta_3}},\ t\ge 0 \tag{3.6}$$

respectively.

Substituting (3.5) into the densities (3.3) of the semi-Markov kernel density, we get

$$\left.\begin{aligned}
q_{10}(t\,|\,\underline{\theta}) &= \theta_3(1-\theta_0\,H(t))(\theta_1+\theta_2\,t)\,e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\left[1-e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\right]^{\theta_3-1},\\
q_{11}(t\,|\,\underline{\theta}) &= \theta_0\theta_3\;H(t)\,(\theta_1+\theta_2\,t)\,e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\left[1-e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\right]^{\theta_3-1}\\
q_{20}(t\,|\,\underline{\theta}) &= \theta_3(1-\theta_0)\,(\theta_1+\theta_2\,t)\,e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\left[1-e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\right]^{\theta_3-1},\\
q_{21}(t\,|\,\underline{\theta}) &= \theta_0\theta_3\,(\theta_1+\theta_2\,t)\,e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\left[1-e^{-\left(\theta_1 t+\frac{1}{2}\theta_2 t^2\right)}\right]^{\theta_3-1},
\end{aligned}\right\} \tag{3.7}$$

where θ_0, θ_1, θ_2, $\theta_3 > 0$, $t, \geq 0$.

4. PARAMETER ESTIMATION

In this section, we use the maximum likelihood procedure to derive point and interval estimates of the unknown vector parameters $\underline{\theta} = (\theta_0, \theta_1, \theta_2, \theta_3)$ included in the quadratic failure rate reliability model.

4.1 Maximum likelihood procedure

Suppose that z denotes the observations $\{(i_0, t_0),\ (i_1, t_1), \ldots, (i_n, t_n)\}$ of two dimensional random vector of variables, $\{(\xi_0, \vartheta_0), (\xi_1, \vartheta_1), \ldots, (\xi_n, \vartheta_n)\}$ where $i_0, i_1, \ldots, t_n$ and $t_0, t_1, \ldots, t_n \in [0, \infty)$. Further, we assume that this observation is classified as follows:

Let

$$A_{ij} = \{k : i_{k-1} = i, i_k = j, k = 1, 2, \ldots, n\} \tag{4.1}$$

be the set of numbers of direct observed transition from the state i to the state j and n_{ij} is the cardinal number of the set A_{ij} which represents the number of direct transitions from the state i to state j . In the present case we find that

$$n_{02} + n_{10} + n_{11} + n_{20} + n_{21} = n \tag{4.2}$$

Based on the above observation, the sample likelihood function $L(z; \underline{\theta})$ can be obtained as follows:

Substituting the semi-Markov densities from (3.7) into (2.12), the sample likelihood function $L(z; \underline{\theta})$ takes the form

$$L(z;\underline{\theta}) = C\,\theta_0^{n_{11}+n_{21}}\,(1-\theta_0)^{n_{20}}\,\theta_3^m\,W(\theta_0)\,\prod_{i\in\mathcal{B}}(\theta_1+\theta_2\,t_i)\,e^{-\left(\theta_1 t_i+\frac{1}{2}\theta_2 t_i^2\right)}\left[1-e^{-\left(\theta_1 t_i+\frac{1}{2}\theta_2 t_i^2\right)}\right]^{\theta_3-1} \tag{4.3}$$

where

$$\left.\begin{aligned}
&W(\theta_0) = \prod_{i\in A_{10}}\left[1-\theta_0 H(t_i)\right], \qquad C = \prod_{i\in A_{02}} c(t_i),\\
&\mathcal{B} = A_{10}\cup A_{11}\cup A_{20}\cup A_{21}, \quad m = n_{10}+n_{11}+n_{20}+n_{21}
\end{aligned}\right\} \tag{4.4}$$

Finally, the log of the sample likelihood function L can be written in the following form

$$\left.\begin{aligned}\mathcal{L} \;=\;& (n_{11}+n_{21})\ln\theta_0 + n_{20}\ln(1-\theta_0) + \ln W(\theta_0) + m\ln\theta_3 + \\ & \sum_{i\in\mathcal{B}}\ln\big(\theta_1+\theta_2 t_i\big) - \sum_{i\in\mathcal{B}}\big(\theta_1\, t_i + \frac{1}{2}\theta_2 t_i^2\big) + (\theta_3-1)\sum_{i\in\mathcal{B}}\ln\Big[1-Y_i(\theta_1,\theta_2)\Big]\end{aligned}\right\} \tag{4.5}$$

where $Y_i(\theta_1,\theta_2)=e^{-(\theta_1 t_i+\frac{1}{2}\theta_2 t_i^2)}$.

The maximum likelihood estimators $\hat{\theta}_0,\hat{\theta}_1,\hat{\theta}_2$ and $\hat{\theta}_3$ are the values of $\theta_0,\theta_1,\theta_2$ and θ_3, respectively that maximize the sample likelihood $\mathcal{L}$. Equivalently $\theta_0,\theta_1,\theta_2$ and θ_3 maximize the log sample likelihood since it is a monotone function of $L(z,\underline{\theta})$.

The maximum likelihood equations are given by :

$$\frac{\partial\mathcal{L}}{\partial\theta_s}=0,\; s=0,\ldots,3. \tag{4.6}$$

Using (4.9) and (4.10) the maximum likelihood equations are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta_0} &= \frac{n_{11}+n_{21}}{\theta_0}-\frac{n_{20}}{1-\theta_0}+\frac{1}{W(\theta_0)}\frac{dW(\theta_0)}{d\theta_0}=0,\\ \frac{\partial\mathcal{L}}{\partial\theta_1} &= \sum_{i\in\mathcal{B}}\frac{1}{\theta_1+\theta_2 t_i}-\sum_{i\in\mathcal{B}}t_i+(\theta_3-1)\sum_{i\in\mathcal{B}}\left[\frac{t_iY_i(\theta_1,\theta_2)}{1-Y_i(\theta_1,\theta_2)}\right]=0,\\ \frac{\partial\mathcal{L}}{\partial\theta_2} &= \sum_{i\in\mathcal{B}}\frac{t_i}{\theta_1+\theta_2 t_i}-\frac{1}{2}\sum_{i\in\mathcal{B}}t_i^2+(\theta_3-1)\sum_{i\in\mathcal{B}}\left[\frac{t_i^2Y_i(\theta_1,\theta_2)}{2[1-Y_i(\theta_1,\theta_2)]}\right]=0,\\ \frac{\partial\mathcal{L}}{\partial\theta_3} &= \frac{m}{\theta_3}+\sum_{i\in\mathcal{B}}\ln\Big[1-Y_i(\theta_1,\theta_2)\Big]=0,\end{aligned} \tag{4.7}$$

4.2 Important special cases

This subsection is devoted to study some important special cases. Such cases occur when, both the time lengths of the repair periods of the units and the lifetimes of the active units are exponentially, linear failure rate and Rayleigh random variables.

In order to obtain the first special case, the following assumptions are needed:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1-\theta_0 H(t_i)=1-\theta_0$ for every $i\in A_{10}$.
2. The lifetimes of the active units can be represented by identically exponential random variables with parameter θ_1. That is, $\theta_2=0$, and $\theta_3=1$

   In this case, the maximum likelihood estimators are given by:

$$\hat{\theta}_0=\frac{n_{22}+n_{12}}{m},\;\;\hat{\theta}_1=\frac{m}{\tau},\;\tau=\sum_{i\in\mathcal{B}}t_i. \tag{4.8}$$

The second special case can be obtained by considering the following assumptions:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1-\theta_0 H(t_i) = 1-\theta_0$ for every $i \in A_{10}$.
2. The lifetimes of the active units can be represented by identically linear failure rate random variables with two parameters θ_1 and θ_2. That is, $\theta_3 = 1$

   In this case, the maximum likelihood estimators are given by:

$$\hat{\theta}_0 = \frac{n_{22}+n_{12}}{m}, \;\; \hat{\theta}_1 = \frac{2m - \hat{\theta}_2 \sum_{i \in \mathcal{B}} t_i^2}{2\tau}. \tag{4.9}$$

   where the estimator $\hat{\theta}_2$ is the solution of the nonlinear equation

$$2 \sum_{i \in \mathcal{B}} \left[\frac{1}{2m - \hat{\theta}_2 \sum_{s \in \mathcal{B}} t_s^2 + 2\hat{\theta}_2 \sum_{i \in \mathcal{B}} \sum_{s \in \mathcal{B}} t_i t_s} \right] = 1 \tag{4.10}$$

The third special case can be obtained by considering the following assumptions:

1. The distribution of the time lengths of the repair periods of the units satisfy the condition: $1-\theta_0 H(t_i) = 1-\theta_0$ for every $i \in A_{10}$.
2. The lifetimes of the active units can be represented by identically Rayleigh random variables with one parameter θ_2.

   In this case, the maximum likelihood estimators are given by:

$$\hat{\theta}_0 = \frac{n_{22}+n_{12}}{m}, \;\; \hat{\theta}_2 = \frac{2m}{\sum_{s \in \mathcal{B}} t_s^2}. \tag{4.11}$$

5. ASYPTOTIC CONFIDENCE BOUNDS

Since the maximum likelihood estimators $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ of the unknown parameters $(\theta_0, \theta_1, \theta_2, \theta_3)$ cannot be derived in closed forms, we cannot get the exact confidence bounds of the parameters. The idea is to use the large sample approximation. The maximum likelihood estimators of $\underline{\theta}$ can be treated as being approximately multi-normal with mean $\underline{\theta} = (\theta_0, \theta_1, \theta_2, \theta_3)$ and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$\left((\hat{\theta}_0 - \theta_0), (\hat{\theta}_1 - \theta_1), (\hat{\theta}_2 - \theta_2), (\hat{\theta}_3 - \theta_3) \right) \to N_4 \left(0, \mathbf{I^{-1}}(\underline{\hat{\theta}}) \right), \tag{5.1}$$

where $\mathbf{I^{-1}} \left(\underline{\hat{\theta}} \right)$ is the variance-covariance matrix of the unknown parameters θ. The element $I_{ij}(\hat{\theta})$, $i, j = 0, 1, 2, 3$, of the 4×4 matrix $\mathbf{I^{-1}}$ is given by

$$I_{ij}(\underline{\hat{\theta}}) = - \left. \mathcal{L}_{\theta_i \theta_j} \right|_{\theta = \hat{\theta}} . \tag{5.2}$$

From expression (4.11), the second partial derivatives of the log-likelihood function are found to be

$$
\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial \theta_0^2} &= -\frac{n_{11}+n_{21}}{\theta_0^2} - \frac{n_{20}}{(1-\theta_0)^2} - \frac{1}{W^2(\theta_0)}\left(\frac{\partial W(\theta_0)}{\partial \theta_0}\right)^2 + \frac{1}{W(\theta_0)}\frac{\partial^2 W(\theta_0)}{\partial \theta_0^2}, \\
\frac{\partial^2 W(\theta_0)}{\partial \theta_0 \theta_1} &= 0, \ \frac{\partial^2 W(\theta_0)}{\partial \theta_0 \theta_2} = 0, \ \frac{\partial^2 W(\theta_0)}{\partial \theta_0 \theta_3} = 0, \\
\frac{\partial^2 \mathcal{L}}{\partial \theta_1^2} &= -\sum_{i \in \mathcal{B}} \frac{1}{(\theta_1+\theta_2 t_i)^2} - (\theta_3 - 1)\sum_{i \in \mathcal{B}} \frac{t_i^2 Y_i}{(1-Y_i)^2}, \\
\frac{\partial^2 \mathcal{L}}{\partial \theta_1 \partial \theta_2} &= -\sum_{i \in \mathcal{B}} \frac{t_i}{(\theta_1+\theta_2 t_i)^2} - (\theta_3 - 1)\sum_{i \in \mathcal{B}} \frac{t_i^3 Y_i}{2(1-Y_i)^2}, \\
\frac{\partial^2 \mathcal{L}}{\partial \theta_1 \theta_3} &= \sum_{i \in \mathcal{B}} \frac{t_i Y_i}{2(1-Y_i)}, \ \frac{\partial^2 \mathcal{L}}{\partial \theta_2^2} = -\sum_{i \in \mathcal{B}} \frac{t_i^2}{(\theta_1+\theta_2 t_i)^2} + (\theta_3 - 1)\sum_{i \in \mathcal{B}} \frac{t_i^4 Y_i}{4(1-Y_i)^2}, \\
\frac{\partial^2 \mathcal{L}}{\partial \theta_2 \theta_3} &= -\sum_{i \in \mathcal{B}} \frac{t_i^2 Y_i}{(1-Y_i)}, \ \frac{\partial^2 \mathcal{L}}{\partial \theta_3^2} = -\sum_{i \in \mathcal{B}} \frac{t_i^2}{(\theta_1+\theta_2 t_i)^2} + (\theta_3 - 1)\sum_{i \in \mathcal{B}} \frac{t_i^4 Y_i}{4(1-Y_i)^2},
\end{aligned}
\tag{5.3}
$$

Therefore, the approximate $100(1-\alpha)\%$ two sided confidence intervals for $(\theta_0, \theta_1, \theta_2, \theta_3)$ are respectively, given by

$$
\hat{\theta}_s \pm Z_{\alpha/2}\sqrt{I_{ss}^{-1}(\hat{\theta}_s)}, \ s = 0, \ldots, 3 \tag{5.4}
$$

Here $Z_{\frac{\alpha}{2}}$ is the upper $100\frac{\alpha}{2}$-th percentile of the standard normal distribution.

From above results, we can deduce the following special cases:

Exponential case: setting $\theta_2 = 0$, $\theta_3 = 1$, from (5.3) and (5.4), we get the approximate $100(1-\alpha)\%$ two sided confidence intervals for θ_0 and θ_1 respectively

$$
\hat{\theta}_0 \pm \frac{\hat{\theta}_0(1-\hat{\theta}_0)\, Z_{\alpha/2}}{\sqrt{(n_{11}+n_{21})(1-\hat{\theta}_0)^2 + (n_{10}+n_{20})\hat{\theta}_0^2}}, \qquad \hat{\theta}_1 \pm \frac{\hat{\theta}_1\, Z_{\alpha/2}}{\sqrt{m}} \tag{5.5}
$$

Linear failure rate case: setting $\theta_3 = 1$, from (5.3) and (5.4), we get the approximate

$100(1-\alpha)\%$ two sided confidence intervals for θ_0, θ_1 and θ_2 respectively

$$\hat{\theta}_0 \pm \frac{\hat{\theta}_0(1-\hat{\theta}_0)}{\sqrt{(n_{11}+n_{21})(1-\hat{\theta}_0)^2+(n_{10}+n_{20})\hat{\theta}_0^2}}\ Z_{\alpha/2},$$

$$\hat{\theta}_1 \pm \left(\frac{\sum\limits_{i\in\mathcal{B}} \frac{t_i^2}{(\hat{\theta}_1+\hat{\theta}_2 t_i)^2}}{\sum\limits_{i\in\mathcal{B}} \frac{t_i}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2} \sum\limits_{i\in\mathcal{B}} \frac{t_s}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2} - \sum\limits_{s\in\mathcal{B}} \frac{1}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2} \sum\limits_{i\in\mathcal{B}} \frac{t_i^2}{(\hat{\theta}_1+\hat{\theta}_2 t_i)^2}} \right)^{\frac{1}{2}} Z_{\alpha/2},$$

$$\hat{\theta}_2 \pm \left(\frac{\sum\limits_{i\in\mathcal{B}} \frac{1}{(\hat{\theta}_1+\hat{\theta}_2 t_i)^2}}{\sum\limits_{s\in\mathcal{B}} \frac{1}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2} \sum\limits_{i\in\mathcal{B}} \frac{t_i^2}{(\hat{\theta}_1+\hat{\theta}_2 t_i)^2} - \sum\limits_{i\in\mathcal{B}} \frac{t_i}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2} \sum\limits_{i\in\mathcal{B}} \frac{t_s}{(\hat{\theta}_1+\hat{\theta}_2 t_s)^2}} \right)^{\frac{1}{2}} Z_{\alpha/2}. \tag{5.6}$$

Next, we discuss in details the reliability function of our standby system with repair that consists of one active unit, an identical spare, a switch, and a repair facility.

6. FIRST PASSAGE AND SYSTEM RELIABILITY

In this section, we will discuss the system reliability function of standby system with repair using semi-Markov procedure. The reliability function of the system will be derived. The distribution of the first passage time will be obtained.

6.1 The distribution of the first passage

Now, we will define the distribution first passage time. In order to define the first passage time, we should find an accurate answer for the question "how many transitions will the process take to reach state j for the first time if the system is in state i at time zero". The first passage time of the continuous-time semi-Markov process can be measured in time or in terms of the number of transitions. We will obtain the distribution $F_{iA}(t)$ of the first passage time from the state i to a state in a subset $A \subset S$ given that state i was entered at time zero and zeroth transition.

Assuming that $A \subset S = \{0,1,2\}$ and $\bar{A} = S - A$, we introduce the following notations

$$\Delta_A = \inf\{n \in N : X(\tau_n) \in A \subset S\}, \tag{6.1}$$

and

$$f_{iA}(n) = P\{\Delta_A = n | X(0) = i\}, T_A = \tau_{\Delta A}, \tag{6.2}$$

Therefore, the distribution function of the first passage time $F_{iA}(t)$ is given by

$$F_{iA}(t) = P\{T_A \le t | X(0) = i\},\ i \in \bar{A}, \tag{6.3}$$

The function $F_{iA}(t)$ represents the distribution of the first passage time of the semi-Markov process $\{X(t) : t \geq 0\}$, from the state $i \in \bar{A}$ to state in the subset A.

6.2 The system reliability function

Now, we will define, the first and the second moments of the first passage time distribution as follows

$$\bar{\Phi}_{iA} = \int_0^\infty t dF_{iA}(t), \text{ and}, \qquad \bar{\Phi}^2_{iA} = \int_0^\infty t^2 dF_{iA}(t), \tag{6.4}$$

If we assumed that, A denotes the subset of the failed states of the system and $i \in \bar{A}$ is an initial operating state such that $P\{X(0) = i\} = 1$, then the random variable T_A represents the lifetime or the time to failure of the reliability system. That is, the reliability of this system is

$$R(t) = 1 - F_{iA}(t), \ t \geq 0, \tag{6.5}$$

Next, we define the first and second moments of the semi-Markov kernel as reliability characteristics of the system as follows:

$$\bar{q}_{ik} = \int_0^\infty t q_{ik}(t)dt, \ \bar{q}^2_{ik} = \int_0^\infty t^2 q_{ik}(t)dt, \ \ \bar{g}_i = \int_0^\infty t dG_i(t)dt, \ \bar{g}^2_i = \int_0^\infty t^2 dG_i(t)dt \tag{6.6}$$

Following theorem can be used to get an integral equations and two linear algebraic systems of equation for the first passage distribution function $F_{iA}(t)$, the first moment $\bar{\Phi}_{iA}$ and the second moment $\bar{\Phi}^2_{iA}$, $i \in \bar{A}$.

Theorem 6.1 If the following three conditions

$$\left.\begin{array}{l} 1. \ f_{iA} = 1 \ \ \forall \ i \in \bar{A} \subset S; \\ \\ 2. \ \exists \ d > 0 \ s.t. \ \bar{q}^2_{ij} < d, \forall \ i, j \in S, \\ \\ 3. \ \sum_{k=1}^\infty k^2 f_{iA}(k) < \infty \ \forall, \ \ i \in \bar{A} \end{array}\right\} \tag{6.7}$$

are fulfilled

Then the distribution function $F_{iA}(t)$, the mean $\bar{\Phi}_{iA}$ and the second moments $\bar{\Phi}^2_{iA}$, $i \in \bar{A}$ are only the solution of the following system:

$$\left.\begin{array}{lcl} F_{iA}(t) & = & \sum_{j \in A} Q^{(k)}_{ij}(t) + \sum_{k \in \bar{A}} \int_0^t F_{kA}(t-u) dQ_{ik}(u), \ i \in \bar{A} \\ \\ \bar{g}_i & = & \bar{\Phi}_{iA} - \sum_{k \in \bar{A}} p_{ik} \bar{\Phi}_{ik}, \ \ i \in \bar{A} \\ \\ \bar{g}^2_i & = & \bar{\Phi}^2_{iA} - \sum_{k \in \bar{A}} p_{ik} \bar{\Phi}^2_{ik} - 2 \sum_{k \in \bar{A}} \bar{q}_{ik} \bar{\Phi}_{kA}, \ \ i \in \bar{A}, \end{array}\right\} \tag{6.8}$$

which consist of a system of integral equations and two linear algebraic systems of equations.

The system of integral equations is equivalent to its Laplace-Stieltjes system

$$\tilde{\varphi}_{iA}(s) = \sum_{j \in A} \tilde{q}_{ij}(s) + \sum_{k \in \bar{A}} \tilde{q}_{ik}(s) \tilde{\varphi}_{kA}(s), \ i \in \bar{A} \tag{6.9}$$

where

$$\tilde{\varphi}_{iA}(s) = \int_0^\infty e^{-st} dF_{iA}(t), \qquad \tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t) \tag{6.10}$$

For underling system, we find that $A = \{0\}$ and $\bar{A} = \{1, 2\}$. From the solution of the system (6.9), we have

$$\tilde{\varphi}_{10}(s) = \frac{\tilde{q}_{10}(s)}{1 - \tilde{q}_{11}(s)}, \quad \tilde{\varphi}_{20}(s) = \tilde{q}_{20}(s) + \frac{\tilde{q}_{21}\tilde{q}_{10}}{1 - \tilde{q}_{11}(s)} \tag{6.11}$$

Using the Laplace transformation, the system reliability function (6.5) of the underling reliability system is given by

$$\tilde{R}(s) = \frac{1 - \tilde{\varphi}_{20}(s)}{s}, \tag{6.12}$$

From the system of equations (6.8), we can get

$$\bar{\Phi}_{20} = \bar{g}_2 + \frac{p_{21}\bar{g}_1}{1 - p_{11}}, \tag{6.13}$$

For the present system $\bar{g}_1$ and $\bar{g}_2$ are the average of the lifetimes of the active units. That is

$$\bar{g}_1 = \bar{g}_2 = E(T) = \int_0^\infty t f(t)\, dt \tag{6.14}$$

6.3 Standby system with exponential lifetime

In this subsection, we will obtained the Laplace-Stieljes of the reliability function. Some semi-Markov reliability characteristics such as the first and second moments of the kernel density will be obtained.

For the exponential distribution the lifetimes of the active units , we find that:

$$\bar{g}_1 = \bar{g}_2 = E(T) = \frac{1}{\theta_1}, \;\; p_{21} = \theta_0 \tag{6.15}$$

Substituting from (6.15) into (6.13) we obtain a simple form of the mean lifetime of the underling reliability system

$$E(T_A|X(0) = 2) = \Phi_{20} = \frac{1}{\theta_1} + \frac{\theta_0}{\theta_1(1 - p_{11})} \tag{6.16}$$

where

$$p_{11} = \theta_0\theta_1 \int_0^\infty H(u)\, e^{-\theta_1 u}\, du. \tag{6.17}$$

After observation of a piece of the considered semi-Markov process realization, we can substitute $\theta_0 = \hat{\theta}_0$ and $\theta_1 = \hat{\theta}_1$.

6.4 Numerical simulation

In this subsection, we discuss some numerical example for the maximum likelihood estimators of the unknown parameters $\theta_0, \theta_1, \theta_2$ and θ_3. The following table displays the mean square errors (MSE) of the parameters against the different values of the sample size.

n	$MSE(\theta_1)$	$MSE(\theta_2)$	$MSE(\theta_3)$	n	$MSE(\theta_1)$	$MSE(\theta_2)$	$MSE(\theta_3)$
40	2.50	1.50	3.34	350	0.35	0.21	0.32
80	1.26	0.81	1.54	400	0.32	0.20	0.31
130	0.82	0.56	0.98	450	0.30	0.19	0.29
150	0.75	0.55	0.95	500	0.25	0.14	0.25
200	0.52	0.37	0.79	550	0.23	0.13	0.23
250	0.39	0.34	0.58	600	0.21	0.12	0.22
300	0.45	0.33	0.45	700	0.17	0.11	0.21

where the assumed values of the parameters are $\theta_0 = 0.5, \theta_1 = 1.5, \theta_2 = 3.0$ and the partial of the sample size are such that $n_{11} = n_{21} = n_{10} = n_{20}$. Further the distribution of the length of the repair time is such that $H(t_l) = 1, \forall l \in A_{10}$. Note that the mean square error of the parameter θ_0 is zero for all different values of the sample size.

7. CONCLUSION

In this paper, we have used the stochastic analysis to discuss an important semi-Markov reliability system. Also the likelihood procedure is employed to derive estimators of the unknown parameters include in this semi-Markov reliability system. The distribution of the first passage time is discussed. The reliability function of this model is derived. Many important special cases are derived.

REFERENCES

Ahmad, I. A. (2001). Moment inequality of aging families of distributions with hypotheses testing applications, *J. Statist. Plann. Inf.*, **92**, 121–132.

A-Hameed, M. S. and Proschan, F. (1973). Nonstationary shock models, *Stoch. Proc. Appl.*, **1**, 333–404.

Balakrishnan, N. and Basu, A. P. (Eds.) (1995). *The Exponential Distribution: Theory, Methods and Applications*, Gordon and Breach, Newark, New Jersey.

Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing, Probability Models*, To Begin With, Silver Spring, Maryland.

El-Gohary, A. (2004) *Bayesian estimations of parameters in a three state reliability semi-Markov models*, *Applied Mathematics and Computation*, **154**(1), 53-68.

El-Gohary, A. (2005) *Estimation of Parameters in a Generalized Exponential Semi-Markov Reliability Models*, *International Journal of Reliability and Applications*, **6**(1), 13-30.

El-Gohary, A. and Sarhan, A. (2005). Estimations of the parameters in a three non-independent component series system subjected to sources of shocks, *Applied Mathematics and Computation*, **160**(1), 29-40.

Grabski, F. and Sarhan, A. (1996). Bayesian Estimation of two nonindependent components exponential series system, *Zagadnienia Eksploatacji Maszyn Zeszyt*, **2**(102), 175–187.

Jackson, O.A.Y. (1969). Fitting a gamma or log-normal distribution to fibre-diameter measurements of wool tops, *Appl. Statist.*, **18**, 70-75.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions – Vol. 1*, Second edition, John Wiley & Sons, New York.

Kotz, S., Balakrishnan, N. and Johnson, N. L. (2000). *Continuous Multivariate Distributions – Vol. 1*, Second edition, John Wiley & Sons, New York.

Kulkarni,V.G. (1995). *Modeling and Analysis of Stochastic Systems*, *Chapman & Hall*

Lai, C. D. Xie, M. and Murthy, D. N. P. (2001). Bathtub shaped failure rate distributions," in Handbook in Reliability, N. Balakrishnan and C. R. Rao, Eds., **20**, 69-104.

Lawless, J. F. (2003). *Statistical Models and Methods for Lifetime Data*, John Wiley and Sons, New York.

Marshall, A. W. and Olkin, I. A. (1967a). A multivariate exponential distribution, *J. Amer. Statist. Assoc.*, **62**, 30–44.

Marshall, A. W. and Olkin, I. A. (1967b). A generalized bivariate exponential distribution, *J. Appl. Prob.*, **4**, 291–302.

Medhi, J. (1982). *Stochastic Processes*, Wiley Eastern Limited.

Pandey, A. Singh, A. and Zimmer, W. (1993). Bayes estimation of the linear hazard rate model, *IEEE Trans. on Reliab.*, **42**, 636–640.

Reinhard, J. M. and Snoussi, M. (2002). The severity of run in a discrete semi Markov risk model, Stochastic Models **18**, 85–107.

Rojo, J. and Ghebremichael, M. (2006). Estimation of two ordered bivariate mean residual life functions, *Journal of Multivariate Analysis,* **97**, 431–454.

Sarhan, A. (1996). Bayesian Estimation in reliability Theory, *Ph. D. Dissertation*, Institute of Mathematics, Gdansk University, Gdansk, Poland.

Sarhan, A. and Abouammoh, A. (2001). Reliability of $k-$out$-$of$-n$ nonrepairable systems with nonindependent and non-identical components subjected to common shocks, *Microelectrons & Reliability*, **41**, 617–621.

Zacks, S. (1991). *Introduction to Reliability Analysis*, Springer-Verlag, New York.