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CHARACTERIZATION OF CENTRAL UNITS OF $\mathbb{Z}A_n$

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Abstract. The structure of $V(\mathcal{Z}(\mathbb{Z}A_n))$ is known when $n \leq 6$. If $n = 5$ or 6, then a complete set of generators of $V(\mathcal{Z}(\mathbb{Z}A_n))$ has been determined. In this study, it was shown that $V(\mathcal{Z}(\mathbb{Z}A_n))$ is trivial when $n = 7, 8$ or 9 and it is generated by a single unit u when $n = 10$ or 11. This unit u is characterized explicitly for $n = 10$ or 11 by using irreducible characters of A_n.

1. Introduction

Let $V = V(\mathbb{Z}G)$ denote the group of normalized units of the integral group ring $\mathbb{Z}G$ of a finite group G. Let $\mathcal{Z}(V)$ denote the subgroup of the central units of V and $\mathcal{N}_V(G)$ denote the normalizer of G in the normalized units $V(\mathbb{Z}G)$. In [4], the following is considered as the normalizer property:

$$\mathcal{N}_V(G) = G\mathcal{Z}(V)?$$

Verification of the normalizer property reduces the problem to the construction of central units of normalized units of $\mathbb{Z}G$ [4, 5]. On the other hand, in order to find a counter example one must find a unit $u \in \mathcal{N}_V(G) \setminus \mathcal{Z}(V)$. In both cases, construction of $\mathcal{Z}(V)$ is important. Let $V(\mathcal{Z}(\mathbb{Z}G))$ denote the normalized units of the center of $\mathbb{Z}G$. Both of them are defined as follows [2]:

$$\mathcal{Z}(V) = \mathcal{Z}(V(\mathbb{Z}G)) = V(\mathbb{Z}G) \cap \mathcal{Z}(\mathbb{Z}G) = V(\mathcal{Z}(\mathbb{Z}G)).$$

The problem of finding the full structure of $V(\mathcal{Z}(\mathbb{Z}G))$, including a complete set of generators, has been determined for only a small number of special cases. When G is finite, Patay [8] proved the following theorem satisfying necessary and sufficient conditions for $U(\mathcal{Z}(\mathbb{Z}G))$ to be trivial.

Proposition 1.1. *Let G be a finite group. All central units of $\mathbb{Z}G$ are trivial if and only if for every $x \in G$ and for every $j \in \mathbb{N}$ relatively prime to $|G|$, x^j is conjugate to x or x^{-1}.*

Ritter and Sehgal constructed a finite set of generators for a subgroup of finite index in $\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$ for a finite group G [9]. On the other hand, Jespers,

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Parmenter, and Sehgal [6] found a different set of generators which works for finitely generated nilpotent groups and constructed these generators from Bass cyclic units in $\mathbb{Z}G$. The construction depended on the existence of a finite normal series in G. If we choose A_n $(n > 4)$ as a finite group, it is impossible to construct generators for $\mathcal{U}(\mathcal{Z}(\mathbb{Z}G))$ from Bass cyclic units since A_n is a simple group. However Aleev [1] constructed all central units of $\mathbb{Z}A_5$ and $\mathbb{Z}A_6$. Later, Li and Parmenter [7] independently constructed all central units of $\mathbb{Z}A_5$, too.

In this study we have given a method to construct generators for $V(\mathcal{Z}(\mathbb{Z}G))$, which works not only for A_n but also for any finite group G. We have mentioned this method in the second part. In the last part we have completed the construction of generators for $V(\mathcal{Z}(\mathbb{Z}A_n))$, where $n < 12$.

The main results are the following.

Theorem 1.2. *$V(\mathcal{Z}(\mathbb{Z}A_n))$ is generated by a single unit for $n = 5, 6, 10, 11$ as follows*:

i) $V(\mathcal{Z}(\mathbb{Z}A_5)) = \langle 49\upsilon_{1a} - 16\upsilon_{2a} + 26\upsilon_{5a} - 10\upsilon_{5b}\rangle$.

ii) $V(\mathcal{Z}(\mathbb{Z}A_6)) = \langle 18433\upsilon_{1a} - 2304\upsilon_{3a} - 2304\upsilon_{3b} + 3728\upsilon_{5a} - 1424\upsilon_{5b}\rangle$.

iii) $V(\mathcal{Z}(\mathbb{Z}A_{10})) = \langle \gamma^{180}\rangle$, *where*

$$\gamma = \frac{1}{4725}(5897\upsilon_{1a} - 72\upsilon_{3a} - 9\upsilon_{3c} + 12\upsilon_{5a} + 3\upsilon_{15a} - 3\upsilon_{5b} - 3\upsilon_{7a} + 7\upsilon_{21a} - 4\upsilon_{21b}).$$

iv) $V(\mathcal{Z}(\mathbb{Z}A_{11})) = \langle \gamma^{120}\rangle$, *where*

$$\begin{aligned}\gamma = \frac{1}{2800}&(8146\upsilon_{1a} + 54\upsilon_{2a} - 54\upsilon_{2b} - 405\upsilon_{3a} + 27\upsilon_{6a} + 27\upsilon_{6b} - 36\upsilon_{4a} - 9\upsilon_{12a} \\ &+ 18\upsilon_{4c} - 9\upsilon_{12b} + 81\upsilon_{5a} + 9\upsilon_{10a} + 9\upsilon_{20a} - 9\upsilon_{5b} - 9\upsilon_{7a} - 9\upsilon_{14a}) \\ &+ \frac{51}{5600}\upsilon_{21a} - \frac{33}{5600}\upsilon_{21b}.\end{aligned}$$

Let us first recall some basic concept about conjugacy classes and irreducible characters of A_n $(n < 12)$ prior to prove these theorems.

2. Motivation to construct generators for $V(\mathcal{Z}(\mathbb{Z}A_n))$

1 - Let us list all conjugacy classes of A_n by regarding of appearance first in A_n for $n < 12$:

$$\begin{aligned}1a &= \{(1)\},\\ 2a &= \{g(1,2)(3,4)g^{-1} : g \in A_n\},\\ 2b &= \{g(1,2)(3,4)(5,6)(7,8)g^{-1} : g \in A_n\},\\ 3a &= \{g(1,2,3)g^{-1} : g \in A_n\},\\ 3b &= \{g(1,2,3)(4,5,6)g^{-1} : g \in A_n\},\\ 3c &= \{g(1,2,3)(4,5,6)(7,8,9)g^{-1} : g \in A_n\},\\ 4a &= \{g(1,2)(3,4,5,6)g^{-1} : g \in A_n\},\end{aligned}$$

$$
\begin{aligned}
4b &= \{g(1,2,3,4)(5,6,7,8)g^{-1} : g \in A_n\},\\
4c &= \{g(1,2)(3,4)(5,6)(7,8,9,10)g^{-1} : g \in A_n\},\\
5a &= \{g(1,2,3,4,5)g^{-1} : g \in A_n\},\\
5b &= \{g(1,2,3,4,5)(6,7,8,9,10)g^{-1} : g \in A_n\},\\
6a &= \{g(1,2)(3,4)(5,6,7)g^{-1} : g \in A_n\},\\
6b &= \{g(1,2)(3,4,5,6,7,8)g^{-1} : g \in A_n\},\\
6c &= \{g(1,2)(3,4)(5,6,7)(8,9,10)g^{-1} : g \in A_n\},\\
6d &= \{g(1,2)(3,4)(5,6)(7,8)(9,10,11)g^{-1} : g \in A_n\},\\
6e &= \{g(1,2)(3,4,5)(6,7,8,9,10,11)g^{-1} : g \in A_n\},\\
7a &= \{g(1,2,3,4,5,6,7)g^{-1} : g \in A_n\},\\
8a &= \{g(1,2)(3,4,5,6,7,8,9,10)g^{-1} : g \in A_n\},\\
9a &= \{g(1,2,3,4,5,6,7,8,9)g^{-1} : g \in A_n\},\\
10a &= \{g(1,2)(3,4)(5,6,7,8,9)g^{-1} : g \in A_n\},\\
11a &= \{g(1,2,3,4,5,6,7,8,9,10,11)g^{-1} : g \in A_n\},\\
12a &= \{g(1,2)(3,4,5)(6,7,8,9)g^{-1} : g \in A_n\},\\
12b &= \{g(1,2,3,4)(5,6,7,8,9,10)g^{-1} : g \in A_n\},\\
12c &= \{g(1,2,3)(4,5,6,7)(8,9,10,11)g^{-1} : g \in A_n\},\\
14a &= \{g(1,2)(3,4)(5,6,7,8,9,10,11)g^{-1} : g \in A_n\},\\
15a &= \{g(1,2,3)(4,5,6,7,8)g^{-1} : g \in A_n\},\\
15b &= \{g(1,2,3)(4,5,6)(7,8,9,10,11)g^{-1} : g \in A_n\},\\
20a &= \{g(1,2)(3,4,5,6)(7,8,9,10,11)g^{-1} : g \in A_n\},\\
21a &= \{g(1,2,3)(4,5,6,7,8,9,10)g^{-1} : g \in A_n\}.
\end{aligned}
$$

Using Gap 4.4 [10] it is easy to determine the size of conjugacy classes, the order of centralizers and the irreducible characters of A_n $(n < 12)$.

2- If we denote the class sum by υ_{kx}, then we write

$$\upsilon_{kx} = \sum_{g \in kx} g.$$

All class sums of A_n form a basis for the center of the rational group algebra $\mathbb{Q}A_n$. That is,

$$\mathcal{Z}(\mathbb{Q}A_n) = \langle \upsilon_{kx} : kx \in A_n \rangle_{\mathbb{Q}}.$$

Since $\mathcal{Z}(\mathbb{Q}A_n) \cong \oplus_i F_i$ and $F_i = \mathcal{Z}(D_i)$; the center of the division ring D_i, $\mathcal{Z}(\mathbb{Z}A_n)$ is an order of $\mathcal{Z}(\mathbb{Q}A_n)$ and also we say that $\mathcal{Z}(\mathbb{Z}A_n)$ is a suborder of maximal order

$$M \cong \oplus_i \mathcal{O}i, \tag{2.1}$$

where $\mathcal{O}i$ denotes the algebraic integers of F_i [3].

If we denote the central units of $\mathbb{Z}A_n$ by $U(\mathcal{Z}(\mathbb{Z}A_n))$ and the normalized central units of $\mathbb{Z}A_n$ by $V(\mathcal{Z}(\mathbb{Z}A_n))$, then we say that $V(\mathcal{Z}(\mathbb{Z}A_n))$ is trivial if and only if $V(\mathcal{Z}(\mathbb{Z}A_n))$=1. For $n < 12$ we can classify the central units of $\mathbb{Z}A_n$.

Proposition 2.1. *Let $n < 12$. $V(\mathcal{Z}(\mathbb{Z}A_n)) = 1$ if and only if $n \neq 5, 6, 10, 11$.*

Proof. By considering the equation (2.1) and the character tables for A_n one by one:

For $n = 2$, $\mathbb{Z}A_n \cong \mathbb{Z}$ so $V(\mathcal{Z}(\mathbb{Z}A_n))$ is trivial.

For $n = 3$ and 4, $\mathcal{O}_i = \mathbb{Z}$ or $\mathbb{Z}[\omega]$, where $\omega = \frac{-1+i\sqrt{3}}{2}$.

For $n = 5$ and 6, $\mathcal{O}_i = \mathbb{Z}$ or $\mathbb{Z}[\alpha]$, where $\alpha = \frac{-1+\sqrt{5}}{2}$.

For $n = 7$ and 8, $\mathcal{O}_i = \mathbb{Z}$ or $\mathbb{Z}[\omega]$, where $\omega = \frac{-1+i\sqrt{7}}{2}$.

For $n = 9$, $\mathcal{O}_i = \mathbb{Z}$ or $\mathbb{Z}[\omega]$, where $\omega = \frac{-1+i\sqrt{15}}{2}$.

For $n = 10$, $\mathcal{O}_i = \mathbb{Z}$ or $\mathbb{Z}[\alpha]$, where $\alpha = \frac{-1+\sqrt{21}}{2}$.

For $n = 11$, $\mathcal{O}_i = \mathbb{Z}$, $\mathbb{Z}[\omega]$ or $\mathbb{Z}[\alpha]$, where $\omega = \frac{-1+i\sqrt{11}}{2}$ and $\alpha = \frac{-1+\sqrt{21}}{2}$.

So by Proposition 1.1 for $n = 2, 3, 4, 7, 8, 9$ we get $V(\mathcal{Z}(\mathbb{Z}A_n)) = 1$, and for $n = 5, 6, 10, 11$ we get $V(\mathcal{Z}(\mathbb{Z}A_n)) \neq 1$. □

Proposition 2.2. *If χ_1 is a trivial character and $\gamma \in V(\mathcal{Z}(\mathbb{Z}A_n))$, then $\chi_1(\gamma) = 1$.*

Proof. Let us consider the augmentation map:

$$\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$$
$$\sum \gamma_g g \longmapsto \sum \gamma_g.$$

Let T_n be transversal for the alternating group A_n, l_g be the class size of g and $\overline{g}$ be the conjugacy class of g. Then we have

$$A_n = \bigcup_{g \in T_n} \overline{g}, \quad \upsilon_g = \sum_{g \in \overline{g}} g.$$

If χ_1 is a trivial character, then we write $\forall g \in G, \chi_1(g) = 1$. For any $\gamma \in V(\mathcal{Z}(\mathbb{Z}A_n))$ we have

$$\begin{aligned} \chi_1(\gamma) = \chi_1(\sum_{g \in T_n} \gamma_g \upsilon_g) &= \sum_{g \in T_n} \gamma_g \chi_1(\upsilon_g) \\ = \sum_{g \in T_n} \gamma_g \chi_1(\sum_{g \in \overline{g}} g) &= \sum_{g \in T_n} \gamma_g \sum_{g \in \overline{g}} \chi_1(g) \\ = \sum_{g \in T_n} \gamma_g l_g &= \sum_{g \in T_n} \sum_{g \in \overline{g}} \gamma_g \end{aligned}$$

$$= \sum_{g\in A_n} \gamma_g = \varepsilon(\gamma) = 1.$$

□

3. Construction of generators of central units of $\mathbb{Z}A_n$

Proof of Theorem 1.2. i) If $\gamma \in V(\mathcal{Z}(\mathbb{Z}A_5))$, then we can write

$$\gamma = \gamma_{1a}\upsilon_{1a} + \gamma_{2a}\upsilon_{2a} + \gamma_{3a}\upsilon_{3a} + \gamma_{5a}\upsilon_{5a} + \gamma_{5b}\upsilon_{5b} = \sum_{g\in T_5} \gamma_g \upsilon_g.$$

By using Gap 4.4 [10] we can get the character table of A_5. Therefore, the values of γ at the irreducible characters are

$$\chi_j(\gamma) = \sum_{g\in T_5} \gamma_g \chi_j(\upsilon_g) = \sum_{g\in T_5} \gamma_g l_g \chi_j(g).$$

Let us consider these values one by one. By Proposition 2.2, we have $\chi_1(\gamma) = 1$.

$$\gamma_{1a} + 15\gamma_{2a} + 20\gamma_{3a} + 12\gamma_{5a} + 12\gamma_{5b} = 1, \tag{3.1}$$

$$\chi_2(\gamma) = 3\gamma_{1a} - 15\gamma_{2a} + 12\alpha\gamma_{5a} + 12\overline{\alpha}\gamma_{5b} = 3\lambda_2,\ \lambda_2 \in \mathcal{U}(\mathbb{Z}[\sqrt{5}\,]),$$

$$\gamma_{1a} - 5\gamma_{2a} + 4\alpha\gamma_{5a} + 4\overline{\alpha}\gamma_{5b} = \lambda_2, \tag{3.2}$$

$$\chi_3(\gamma) = 3\gamma_{1a} - 15\gamma_{2a} + 12\overline{\alpha}\gamma_{5a} + 12\alpha\gamma_{5b} = 3\lambda_3,\ \lambda_3 \in \mathcal{U}(\mathbb{Z}[\sqrt{5}\,]),$$

$$\gamma_{1a} - 5\gamma_{2a} + 4\overline{\alpha}\gamma_{5a} + 4\alpha\gamma_{5b} = \lambda_3, \tag{3.3}$$

$$\chi_4(\gamma) = 4\gamma_{1a} + 20\gamma_{3a} - 12\gamma_{5a} - 12\gamma_{5b} = 4\lambda_4,\ \lambda_4 \in \mathcal{U}(\mathbb{Z}),$$

$$\gamma_{1a} + 5\gamma_{3a} - 3\gamma_{5a} - 3\gamma_{5b} = \pm 1, \tag{3.4}$$

$$\chi_5(\gamma) = 5\gamma_{1a} + 15\gamma_{2a} - 20\gamma_{3a} = 5\lambda_5,\ \lambda_5 \in \mathcal{U}(\mathbb{Z}),$$

$$\gamma_{1a} + 3\gamma_{2a} - 4\gamma_{3a} = \pm 1. \tag{3.5}$$

Since $d \mid (\chi_4(\gamma) - \chi_1(\gamma))$ and $d \mid (\chi_5(\gamma) - \chi_1(\gamma))$ for $d > 2$ we can write $\chi_4(\gamma) = \chi_5(\gamma) = 1$. So we can get the following system of linear equations

$$\begin{bmatrix} 1 & 15 & 20 & 12 & 12 \\ 1 & -5 & 0 & 4\alpha & 4\overline{\alpha} \\ 1 & -5 & 0 & 4\overline{\alpha} & 4\alpha \\ 1 & 0 & -5 & -3 & -3 \\ 1 & 3 & -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{1a} \\ \gamma_{2a} \\ \gamma_{3a} \\ \gamma_{5a} \\ \gamma_{5b} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_3 \\ 1 \\ 1 \end{bmatrix}.$$

If we consider the $1st$, $4th$ and $5th$ equations and the first three variables we get

$$\begin{bmatrix} 1 & 15 & 20 \\ 1 & 0 & -5 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} \gamma_{1a} \\ \gamma_{2a} \\ \gamma_{3a} \end{bmatrix} = \begin{bmatrix} 1 - 12(\gamma_{5a} + \gamma_{5b}) \\ 1 + 3(\gamma_{5a} + \gamma_{5b}) \\ 1 \end{bmatrix}.$$

So the parametric solution is

$$\gamma_{1a} = 1 + 3(\gamma_{5a} + \gamma_{5b}), \qquad \gamma_{2a} = -(\gamma_{5a} + \gamma_{5b}), \qquad \gamma_{3a} = 0.$$

By substituting this solution into (3.2), we get

$$(1+10(\gamma_{5a}+\gamma_{5b}))+2(\gamma_{5a}-\gamma_{5b})\sqrt{5}\in\mathcal{U}(\mathbb{Z}[\sqrt{5}\,])$$

or equivalently,

$$(1+10(\gamma_{5a}+\gamma_{5b}))+2(\gamma_{5a}-\gamma_{5b})\sqrt{5}=(2+\sqrt{5})^k$$

for some $k\in\mathbb{N}$. We get the smallest solution when $k=4$.

$$\gamma_{5a}=26,\ \gamma_{5b}=-10,\ \gamma_{3a}=-16,\ \gamma_{1a}=49,\ \gamma_{2a}=0.$$

Thus the generator is

$$\gamma=49\upsilon_{1a}-16\upsilon_{2a}+26\upsilon_{5a}-10\gamma_{5b}.$$

On the other hand the solution of the following equation

$$(1+10(\gamma_{5a}+\gamma_{5b}))+2(\gamma_{5a}-\gamma_{5b})\sqrt{5}=(2-\sqrt{5})^k$$

for the same $k\in\mathbb{N}$ gives us its inverse. Hence for $k=4$

$$\gamma_{5a}=-10,\ \gamma_{5b}=26,\ \gamma_{3a}=-16,\ \gamma_{1a}=49,\ \gamma_{2a}=0.$$

So the inverse of γ is

$$\gamma^{-1}=49\upsilon_{1a}-16\upsilon_{2a}-10\upsilon_{5a}+26\upsilon_{5b}.$$

ii) If $\gamma\in V(\mathcal{Z}(\mathbb{Z}A_6))$, then we can write

$$\begin{aligned}\gamma&=\gamma_{1a}\upsilon_{1a}+\gamma_{2a}\upsilon_{2a}+\gamma_{3a}\upsilon_{3a}+\gamma_{3b}\upsilon_{3b}+\gamma_{4a}\upsilon_{4a}+\gamma_{5a}\upsilon_{5a}+\gamma_{5b}\upsilon_{5b}\\&=\sum_{g\in T_6}\gamma_g\upsilon_g.\end{aligned}$$

By using Gap 4.4 [10], we can get the character table of A_6. Thus, the values of γ at the irreducible characters are

$$\chi_j(\gamma)=\sum_{g\in T_6}\gamma_g\chi_j(\upsilon_g)=\sum_{g\in T_6}\gamma_g l_g\chi_j(g).$$

Let us consider these values one by one. By Proposition 2.2, we have $\chi_1(\gamma)=1$.

$$\gamma_{1a}+45\gamma_{2a}+40\gamma_{3a}+40\gamma_{3b}+90\gamma_{4a}+72\gamma_{5a}+72\gamma_{5b}=1, \tag{3.6}$$

$$\chi_2(\gamma)=5\gamma_{1a}+45\gamma_{2a}+80\gamma_{3a}-40\gamma_{3b}-90\gamma_{4a}=5\lambda_2,\ \lambda_2\in\mathcal{U}(\mathbb{Z}),$$

$$\gamma_{1a}+9\gamma_{2a}+16\gamma_{3a}-8\gamma_{3b}-18\gamma_{4a}=\pm1, \tag{3.7}$$

$$\chi_3(\gamma)=5\gamma_{1a}+45\gamma_{2a}-40\gamma_{3a}+80\gamma_{3b}-90\gamma_{4a}=5\lambda_3,\ \lambda_3\in\mathcal{U}(\mathbb{Z}),$$

$$\gamma_{1a}+9\gamma_{2a}-8\gamma_{3a}+16\gamma_{3b}-18\gamma_{4a}=\pm1, \tag{3.8}$$

$$\chi_4(\gamma)=8\gamma_{1a}-40\gamma_{3a}-40\gamma_{3b}+72\alpha\gamma_{5a}+72\overline{\alpha}\gamma_{5b}=8\lambda_4,\ \lambda_4\in\mathcal{U}(\mathbb{Z}[\alpha]),$$

$$\gamma_{1a}-5\gamma_{3a}-5\gamma_{3b}+9\alpha\gamma_{5a}+9\overline{\alpha}\gamma_{5b}=\lambda_4\in\mathcal{U}(\mathbb{Z}[\alpha]), \tag{3.9}$$

$$\chi_5(\gamma)=8\gamma_{1a}-40\gamma_{3a}-40\gamma_{3b}+72\overline{\alpha}\gamma_{5a}+72\alpha\gamma_{5b}=8\lambda_5,\ \lambda_5\in\mathcal{U}(\mathbb{Z}[\alpha]),$$

$$\gamma_{1a}-5\gamma_{3a}-5\gamma_{3b}+9\overline{\alpha}\gamma_{5a}+9\alpha\gamma_{5b}=\lambda_5\in\mathcal{U}(\mathbb{Z}[\alpha]), \tag{3.10}$$

$$\chi_6(\gamma) = 9\gamma_{1a} + 45\gamma_{2a} + 90\gamma_{4a} - 72\gamma_{5a} - 72\gamma_{5b} = 9\lambda_6,\ \lambda_6 \in \mathcal{U}(\mathbb{Z}),$$

$$\gamma_{1a} + 5\gamma_{2a} + 10\gamma_{4a} - 8\gamma_{5a} - 8\gamma_{5b} = \pm 1, \tag{3.11}$$

$$\chi_7(\gamma) = 10\gamma_{1a} - 90\gamma_{2a} + 40\gamma_{3a} + 40\gamma_{3b} = 10\lambda_7,\ \lambda_7 \in \mathcal{U}(\mathbb{Z}),$$

$$\gamma_{1a} - 9\gamma_{2a} + 4\gamma_{3a} + 4\gamma_{3b} = \pm 1. \tag{3.12}$$

Since $d \mid (\chi_j(\gamma) - \chi_1(\gamma))$ for $d > 2$ and $j = 2, 3, 6, 7$ we can write $\chi_j(\gamma) = 1$ $(j = 2, 3, 6, 7)$. If we consider the equations (3.6)-(3.12) we obtain the following system of linear equations

$$\begin{bmatrix} 1 & 45 & 40 & 40 & 90 \\ 1 & 9 & 16 & -8 & -18 \\ 1 & 9 & -8 & 16 & -18 \\ 1 & 5 & 0 & 0 & 10 \\ 1 & -9 & 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{1a} \\ \gamma_{2a} \\ \gamma_{3a} \\ \gamma_{3b} \\ \gamma_{4a} \end{bmatrix} = \begin{bmatrix} 1 - 72(\gamma_{5a} + \gamma_{5b}) \\ 1 \\ 1 \\ 1 + 8(\gamma_{5a} + \gamma_{5b}) \\ 1 \end{bmatrix}.$$

So the solution is

$$\gamma_{1a} = 1+8(\gamma_{5a}+\gamma_{5b}),\ \gamma_{2a} = 0,\ \gamma_{3a} = -(\gamma_{5a}+\gamma_{5b}),\ \gamma_{3b} = -(\gamma_{5a}+\gamma_{5b}),\ \gamma_{4a} = 0.$$

Therefore, by substituting these values in to the equation (3.9) we get

$$1 + 18(\gamma_{5a} + \gamma_{5b}) + 9(\frac{1+\sqrt{5}}{2})\gamma_{5a} + 9(\frac{1-\sqrt{5}}{2})\gamma_{5b} \in \mathcal{U}(\mathbb{Z}[\alpha]),$$

or simply we have

$$1 + \frac{45}{2}(\gamma_{5a} + \gamma_{5b}) + \frac{9}{2}\sqrt{5}(\gamma_{5a} - \gamma_{5b}) = (\frac{1+\sqrt{5}}{2})^k \text{ for some } k \in \mathbb{N}.$$

We can get the smallest solution when $k = 24$ as follows:

$$\gamma_{1a} = 18433, \gamma_{2a} = 0, \gamma_{3a} = -2304, \gamma_{3b} = -2304, \gamma_{4a} = 0, \gamma_{5a} = 3728, \gamma_{5b} = -1424.$$

Thus, the generator is

$$\gamma = 18433\upsilon_{1a} - 2304\upsilon_{3a} - 2304\upsilon_{3b} + 3728\upsilon_{5a} - 1424\upsilon_{5b}.$$

By considering the conjugate of the equation we get the inverse of γ

$$1 + \frac{45}{2}(\gamma_{5a} + \gamma_{5b}) + \frac{9}{2}\sqrt{5}(\gamma_{5a} - \gamma_{5b}) = (\frac{1-\sqrt{5}}{2})^k \text{ for the same } k.$$

So, we have

$$\gamma_{1a} = 18433, \gamma_{2a} = 0, \gamma_{3a} = -2304, \gamma_{3b} = -2304, \gamma_{4a} = 0, \gamma_{5a} = -1424, \gamma_{5b} = 3728.$$

Hence, the inverse of γ is

$$\gamma^{-1} = 18433\upsilon_{1a} - 2304\upsilon_{3a} - 2304\upsilon_{3b} - 1424\upsilon_{5a} + 3728\upsilon_{5b}.$$

iii) Let $\gamma \in V(\mathcal{Z}(\mathbb{Z}A_{10}))$. Then we can write γ as follows according to the appearance of the conjugacy classes in the software Gap 4.4 [10]:

$$\begin{aligned} \gamma \ = \ & \gamma_{1a}\upsilon_{1a} + \gamma_{2a}\upsilon_{2a} + \gamma_{2b}\upsilon_{2b} + \gamma_{3a}\upsilon_{3a} + \gamma_{6a}\upsilon_{6a} + \gamma_{3b}\upsilon_{3b} + \gamma_{6b}\upsilon_{6b} \\ & +\gamma_{3c}\upsilon_{3c} + \gamma_{4a} + \gamma_{4b}\upsilon_{4b} + \gamma_{4b}\upsilon_{4b} + \gamma_{4b}\upsilon_{4b} + \gamma_{12a}\upsilon_{12a} + \gamma_{4c}\upsilon_{4a} \end{aligned}$$

$$+\upsilon_{4c} + \gamma_{5a}\upsilon_{5a} + \gamma_{10a}\upsilon_{10a} + \gamma_{15a}\upsilon_{15a} + \gamma_{5b}\upsilon_{5b} + \gamma_{6c}\upsilon_{6c} + \gamma_{12b}\upsilon_{12b}$$
$$+\gamma_{7a}\upsilon_{7a} + \gamma_{21a}\upsilon_{21a} + \gamma_{21b}\upsilon_{21b} + \gamma_{8a}\upsilon_{8a} + \gamma_{9a}\upsilon_{9a} + \gamma_{9b}\upsilon_{9b}.$$

By using Gap 4.4 [10], we can get all irreducible characters. Only few values of the characters are irrational, the others are integers. We can find all values of irreducible characters of γ

$$\chi_j(\gamma) = \sum_{g\in T_{10}} \gamma_g \chi_j(\upsilon_g) = \sum_{g\in T_{10}} \gamma_g l_g \chi_j(g).$$

By Proposition 2.2, we have $\chi_1(\gamma) = 1$. That is,

$$\begin{aligned} 1 = {} & \upsilon_{1a} + 630\upsilon_{2a} + 4725\upsilon_{2b} + 240\upsilon_{3a} + 25200\upsilon_{6a} + 8400\upsilon_{3b} \\ & + 25200\upsilon_{6b} + 22400\upsilon_{3c} + 18900\upsilon_{4a} + 151200\upsilon_{4b} \\ & + 151200\upsilon_{12a} + 56700\upsilon_{4c} + 6048\upsilon_{5a} + 90720\upsilon_{10a} \\ & + 120960\upsilon_{15a} + 72576\upsilon_{15b} + 151200\upsilon_{6c} + 151200\upsilon_{12b} \\ & + 86400\upsilon_{7a} + 86400\upsilon_{21a} + 86400\upsilon_{21b} + 226800\upsilon_{8a} \\ & + 201600\upsilon_{9a} + 201600\upsilon_{9b}. \end{aligned} \tag{3.13}$$

For $j = 2, \ldots, 19, 22, 23, 24$ we get

$$\begin{aligned} \chi_j(\gamma) &= \sum_{g\in T_{10}} \gamma_g l_g \chi_j(g) = \chi_j(1)\lambda_j,\ \lambda_j \in \mathcal{U}(\mathbb{Z}), \\ &\Rightarrow \frac{\chi_j(\gamma)}{\chi_j(1)} = \pm 1. \end{aligned}$$

We can easily observe that $d \mid (\chi_j(\gamma) - \chi_1(\gamma))$ for $d > 2$ and $j = 2, \ldots, 19$, 22, 23, 24. So

$$\frac{\chi_j(\gamma)}{\chi_j(1)} = 1 \text{ for } j = 2, \ldots, 19, 22, 23, 24. \tag{3.14}$$

For $j = 20, 21$ we get

$$\chi_j(\gamma) = \sum_{g\in T_{10}} \gamma_g l_g \chi_j(g) = \chi_j(1)\lambda_j,\ \lambda_j \in \mathcal{U}(\mathbb{Z}[\alpha]), \tag{3.15}$$

$$\Rightarrow \frac{\chi_j(\gamma)}{\chi_j(1)} \in \mathcal{U}(\mathbb{Z}[\alpha]). \tag{3.16}$$

By transfering the $20th$ and $21st$ terms to the right hand side in the equations (3.13) and (3.14) we get the system of linear equations

$$AX^T = Y^T,$$

where A is the matrix of coefficients (See Appendix I),

$$X = [\gamma_{1a}, \ldots, \gamma_{7a}, \gamma_{8a}, \gamma_{9a}, \gamma_{9b}],$$

$$\begin{aligned} Y = [&1 - 86400(\gamma_{21a} + \gamma_{21b}), 1 + 9600(\gamma_{21a} + \gamma_{21b}), 1, 1 - 2400(\gamma_{21a} + \gamma_{21b}), \\ &1, 1 - 1152(\gamma_{21a} + \gamma_{21b}), 1, 1 + 960(\gamma_{21a} + \gamma_{21b}), 1, 1 + 540(\gamma_{21a} + \gamma_{21b}), \end{aligned}$$

$1,1,1,1-384(\gamma_{21a}+\gamma_{21b}),1,1-300(\gamma_{21a}+\gamma_{21b}),1+288(\gamma_{21a}+\gamma_{21b}),$
$1,1,1+192(\gamma_{21a}+\gamma_{21b}),1,1].$

So the solution of the system is

$\gamma_{1a}=1+384(\gamma_{21a}+\gamma_{21b})$	$\gamma_{4a}=0$	$\gamma_{6c}=0$
$\gamma_{2a}=0$	$\gamma_{4b}=0$	$\gamma_{12b}=0$
$\gamma_{2b}=0$	$\gamma_{12a}=0$	$\gamma_{7a}=-(\gamma_{21a}+\gamma_{21b})$
$\gamma_{3a}=-24(\gamma_{21a}+\gamma_{21b})$	$\gamma_{4c}=0$	$\gamma_{8a}=0$
$\gamma_{6a}=0$	$\gamma_{5a}=4(\gamma_{21a}+\gamma_{21b})$	$\gamma_{9a}=0$
$\gamma_{3b}=0$	$\gamma_{10a}=0$	$\gamma_{9b}=0.$
$\gamma_{6b}=0$	$\gamma_{15a}=(\gamma_{21a}+\gamma_{21b})$	
$\gamma_{3c}=-3(\gamma_{21a}+\gamma_{21b})$	$\gamma_{5b}=-(\gamma_{21a}+\gamma_{21b})$	

Thus, by substituting these values in to the equation (3.15) we get the following Pell-equation

$$1+\frac{4725}{2}(\gamma_{21a}+\gamma_{21b})+\frac{225}{2}(\gamma_{21a}-\gamma_{21b})\sqrt{21}=\left(\frac{5+\sqrt{21}}{2}\right)^k. \tag{3.17}$$

This can be obtained when $k=180$ by using Maple 7.0 [11]. But it is not useful to express the other coefficients in terms of γ_{21a} and γ_{21b}. Therefore, we will solve this for $\frac{5+\sqrt{21}}{2}$ instead of $(\frac{5+\sqrt{21}}{2})^{180}$. Hence, the solution gives us the $180th$ root of the solution.

$$\gamma=\frac{1}{4725}(5897\gamma_{1a}-72\gamma_{3a}-9\gamma_{3c}+12\gamma_{5a}+3\gamma_{15a}-3\gamma_{5b}-3\gamma_{7a}+7\gamma_{21a}-4\gamma_{21b}).$$

By regarding rational conjugate of the right hand side of the equation (3.16) we get the inverse of γ as follows:

$$\gamma^{-1}=\frac{1}{4725}(5897\gamma_{1a}-72\gamma_{3a}-9\gamma_{3c}+12\gamma_{5a}+3\gamma_{15a}-3\gamma_{5b}-3\gamma_{7a}+7\gamma_{21a}-4\gamma_{21b}).$$

Hence,

$$V(\mathcal{Z}(\mathbb{Z}A_{10}))=\left\langle\gamma^{180}\right\rangle.$$

iv) For $\gamma\in V(\mathcal{Z}(\mathbb{Z}A_{11}))$, we can write γ as follows according to the appearance of conjugacy classes in the software Gap 4.4 [10]:

$$\begin{aligned}\gamma \;=\;& \gamma_{1a}\upsilon_{1a}+\gamma_{2a}\upsilon_{2a}+\gamma_{2b}\upsilon_{2b}+\gamma_{3a}\upsilon_{3a}+\gamma_{6a}\upsilon_{6a}+\gamma_{6b}\upsilon_{6b}+\gamma_{3b}\upsilon_{3b}\\ &\gamma_{6c}\upsilon_{6c}+\gamma_{3c}\upsilon_{3c}+\gamma_{4a}\upsilon_{4a}+\gamma_{4b}\upsilon_{4b}+\gamma_{12a}\upsilon_{12a}+\gamma_{4c}\upsilon_{4c}+\gamma_{12b}\upsilon_{12b}\\ &+\gamma_{5a}\upsilon_{5a}+\gamma_{10a}\upsilon_{10a}+\gamma_{15a}\upsilon_{15a}+\gamma_{15b}\upsilon_{15b}+\gamma_{20a}\upsilon_{20a}+\gamma_{5b}\upsilon_{5b}\\ &+\gamma_{6d}\upsilon_{6d}+\gamma_{6e}\upsilon_{6e}+\gamma_{12c}\upsilon_{12c}+\gamma_{7a}\upsilon_{7a}+\gamma_{14a}\upsilon_{14a}+\gamma_{21a}\upsilon_{21a}\\ &+\gamma_{21b}\upsilon_{21b}+\gamma_{8a}\upsilon_{8a}+\gamma_{9a}\upsilon_{9a}+\gamma_{11a}\upsilon_{11a}+\gamma_{11b}\upsilon_{11b}.\end{aligned}$$

We can determine all irreducible characters by the same software. There are 4 irreducible characters without integer values, two of which have irrational values while the other two have complex values. We can find all values of γ at irreducible characters

$$\chi_j(\gamma) = \sum_{g\in T_{11}} \gamma_g \chi_j(\upsilon_g) = \sum_{g\in T_{11}} \gamma_g l_g \chi_j(g).$$

Let us consider these values one by one. By Proposition 2.2, we have

$$\chi_1(\gamma) = 1. \tag{3.18}$$

On the other hand, for $j = 2, \ldots, 17, 20, \ldots, 31$ we get

$$\begin{aligned} \chi_j(\gamma) &= \sum_{g\in T_{11}} \gamma_g l_g \chi_j(g) = \chi_j(1)\lambda_j,\ \lambda_j \in \mathcal{U}(\mathbb{Z}), \\ &\Rightarrow \frac{\chi_j(\gamma)}{\chi_j(1)} = \pm 1. \end{aligned}$$

But we can easily guess that $d \mid (\chi_j(\gamma) - \chi_1(\gamma))$ for $d > 2$ and $j = 2, \ldots, 17, 20, \ldots, 31$. Thus we write

$$\frac{\chi_j(\gamma)}{\chi_j(1)} = 1 \text{ for } j = 2, \ldots, 17, 20, \ldots, 31. \tag{3.19}$$

For $j = 18, 19$ we get

$$\chi_j(\gamma) = \sum_{g\in T_{11}} \gamma_g l_g \chi_j(g) = \chi_j(1)\lambda_j,\ \lambda_j \in \mathcal{U}(\mathbb{Z}[\alpha]), \tag{3.20}$$

$$\Rightarrow \frac{\chi_j(\gamma)}{\chi_j(1)} \in \mathcal{U}(\mathbb{Z}[\alpha]). \tag{3.21}$$

By transfering the $18th$ and $19th$ terms to the right hand side in the equations (3.17) and (3.18) we get the system of linear equations

$$AX^T = Y^T,$$

where A is the matrix of coefficients (See Appendix II),

$$X = [\gamma_{1a}, \ldots, \gamma_{14a}, \gamma_{8a}, \ldots, \gamma_{11b}],$$

Y=$[1 - 950400(\gamma_{21a} + \gamma_{21b}), 1, 1 + 21600(\gamma_{21a} + \gamma_{21b}), 1, 1 - 8640(\gamma_{21a} + \gamma_{21b}),$
$1 - 7920(\gamma_{21a} + \gamma_{21b}), 1, 1, 1 + 7200(\gamma_{21a} + \gamma_{21b}), 1, 1, 1, 1 - 2880(\gamma_{21a} + \gamma_{21b}),$
$1, 1, 1, 1 + 1600(\gamma_{21a} + \gamma_{21b}), 1 + 1440(\gamma_{21a} + \gamma_{21b}), 1, 1 + 1152(\gamma_{21a} + \gamma_{21b}), 1,$
$1, 1, 1 - 864(\gamma_{21a} + \gamma_{21b}), 1, 1, 1, 1, 1].$

So the solution is

$$\begin{array}{lll}
\gamma_{1a}=1+594(\gamma_{21a}+\gamma_{21b}) & \gamma_{4b}=0 & \gamma_{6d}=0\\
\gamma_{2a}=6(\gamma_{21a}+\gamma_{21b}) & \gamma_{12a}=-(\gamma_{21a}+\gamma_{21b}) & \gamma_{6e}=0\\
\gamma_{2b}=-6(\gamma_{21a}+\gamma_{21b}) & \gamma_{4c}=2(\gamma_{21a}+\gamma_{21b}) & \gamma_{12c}=0\\
\gamma_{3a}=-45(\gamma_{21a}+\gamma_{21b}) & \gamma_{12b}=-(\gamma_{21a}+\gamma_{21b}) & \gamma_{7a}=-(\gamma_{21a}+\gamma_{21b})\\
\gamma_{6a}=3(\gamma_{21a}+\gamma_{21b}) & \gamma_{5a}=9(\gamma_{21a}+\gamma_{21b}) & \gamma_{14a}=-(\gamma_{21a}+\gamma_{21b})\\
\gamma_{6b}=3(\gamma_{21a}+\gamma_{21b}) & \gamma_{10a}=(\gamma_{21a}+\gamma_{21b}) & \gamma_{8a}=0\\
\gamma_{3b}=0 & \gamma_{15a}=0 & \gamma_{9a}=0\\
\gamma_{6c}=0 & \gamma_{15b}=0 & \gamma_{11a}=0\\
\gamma_{3c}=0 & \gamma_{20a}=(\gamma_{21a}+\gamma_{21b}) & \gamma_{11b}=0.\\
\gamma_{4a}=-4(\gamma_{21a}+\gamma_{21b}) & \gamma_{5b}=-(\gamma_{21a}+\gamma_{21b}) &
\end{array}$$

Therefore, by substituting these values in to the equation (3.19) we get the following Pell-equation

$$1+16800(\gamma_{21a}+\gamma_{21b})+800(\gamma_{21a}-\gamma_{21b})\sqrt{21}=\left(55+12\sqrt{21}\right)^k. \tag{3.22}$$

Using Maple 7.0 [11] we get the solution when $k=120$. But it is not useful to express the other coefficients in terms of γ_{21a} and γ_{21b}. Therefore, we will solve this for $55+12\sqrt{21}$ instead of $(55+12\sqrt{21})^{120}$. Hence, we obtain the $120th$ root of the solution as follows:

$$\begin{aligned}\gamma=&\frac{1}{2800}(8146\upsilon_{1a}+54\upsilon_{2a}-54\upsilon_{2b}-405\upsilon_{3a}+27\upsilon_{6a}+27\upsilon_{6b}-36\upsilon_{4a}\\&-9\upsilon_{12a}+18\upsilon_{4c}-9\upsilon_{12b}+81\upsilon_{5a}+9\upsilon_{10a}+9\upsilon_{20a}-9\upsilon_{5b}-9\upsilon_{7a}-9\upsilon_{14a})\\&+\frac{51}{5600}\upsilon_{21a}-\frac{33}{5600}\upsilon_{21b}.\end{aligned}$$

If we consider the rational conjugate of the right hand side of the equation (3.20) we obtain the inverse of γ as follows:

$$\begin{aligned}\gamma^{-1}=&\frac{1}{2800}(8146\upsilon_{1a}+54\upsilon_{2a}-54\upsilon_{2b}-405\upsilon_{3a}+27\upsilon_{6a}+27\upsilon_{6b}-36\upsilon_{4a}\\&-9\upsilon_{12a}+18\upsilon_{4c}-9\upsilon_{12b}+81\upsilon_{5a}+9\upsilon_{10a}+9\upsilon_{20a}-9\upsilon_{5b}-9\upsilon_{7a}-9\upsilon_{14a})\\&-\frac{33}{5600}\upsilon_{21a}+\frac{51}{5600}\upsilon_{21b}.\end{aligned}$$

As a result

$$V(\mathcal{Z}(\mathbb{Z}A_{11}))=\left\langle\gamma^{120}\right\rangle.$$

□

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Appendix I. The coefficient matrix for the characterization of $V(\mathcal{Z}(\mathbb{Z}A_{10}))$

$A =$

1	630	4725	240	25200	8400	25200	22400	18900	18900	151200	56700	6048	90720	120960	72576	151200	151200	86400	226800	201600	201600
1	350	525	160	5600	2800	-2800	0	6300	-2100	0	6300	2688	0	13440	-8064	16800	-16800	19200	-25200	0	0
1	198	405	96	1440	480	1440	-640	1620	1620	0	-1620	864	2592	-3456	0	0	0	0	6480	-5760	-5760
1	140	-525	100	-700	700	-700	0	1050	-1050	-4200	0	1008	-5040	0	2016	-4200	4200	2400	0	0	0
1	90	225	0	0	600	1800	-1600	0	-1800	0	2700	-432	2160	0	3456	-3600	-3600	0	0	0	0
1	126	189	48	1008	0	0	896	252	-756	2016	-756	0	0	0	0	0	0	-2304	-3024	0	0
1	0	-225	60	-900	300	900	800	-450	450	1800	0	288	0	1440	-864	-1800	-1800	0	0	0	0
1	98	105	16	560	280	-280	0	0	840	0	1260	-336	-1008	1344	0	-1680	1680	-960	0	0	0
1	-70	225	40	200	400	-400	0	-600	0	-1200	-900	48	720	960	576	0	0	0	0	0	0
1	63	0	51	-315	-105	-315	-280	0	0	0	0	189	567	-756	0	0	0	-540	0	1260	1260
1	18	-135	-24	360	0	0	320	-360	0	-720	540	144	432	-576	0	0	0	0	0	0	0
1	-45	0	15	225	75	225	-100	0	0	0	0	-27	-405	-540	-324	0	0	0	0	-900	1800
1	-45	0	15	225	75	225	-100	0	0	0	0	-27	-405	-540	-324	0	0	0	0	1800	-900
1	14	189	16	-112	-224	224	0	-84	252	-672	252	0	0	0	0	0	0	384	-1008	0	0
1	20	75	-20	-100	100	-100	0	-150	150	600	0	48	-720	-480	576	600	-600	0	0	0	0
1	35	0	-5	-175	175	-175	0	0	0	0	0	-147	315	-420	-504	0	0	300	0	0	0
1	0	63	-12	-252	84	252	224	126	-126	-504	0	0	0	0	0	504	504	-288	0	0	0
1	38	-75	16	80	-80	80	0	-60	-60	-480	-180	-96	-288	384	0	480	-480	0	720	0	0
1	-18	-27	24	-72	-24	-72	-64	-108	-108	432	324	0	0	0	0	432	432	0	0	-576	-576
1	14	21	-8	56	-56	56	0	-84	-84	336	-252	0	0	0	0	-336	336	384	0	0	0
1	-18	45	0	0	-48	-144	128	108	-36	0	108	0	0	0	0	-288	-288	0	432	0	0
1	-10	-75	0	0	0	0	0	100	100	0	-100	-32	160	0	256	0	0	0	-400	0	0

Appendix II. The coefficient matrix for the characterization of $V(\mathcal{Z}(\mathbb{Z}A_{11}))$

$A =$

1	990	17325	330	69300	34650	18480	277200	123200	41580	207900	831600	207900	415800	11088	498960	443520	443520	997920	798336	554400	1108800	1663200	237600	712800	2494800	2217600	1814400	1814400
1	594	3465	231	20790	-3465	7392	0	12320	16632	0	83160	41580	-41580	5544	49896	88704	-44352	-99792	0	110880	-110880	0	71280	-71280	0	221760	-181440	-181440
1	360	1575	150	6300	3150	2100	6300	-2800	5670	9450	0	0	0	2268	11340	0	0	22680	-18144	12600	25200	-37800	10800	32400	0	-50400	0	0
1	288	-1155	154	1540	-2310	2464	-12320	0	4620	-13860	-18480	4620	9240	2464	-22176	9856	9856	0	0	0	0	0	15840	-15840	-55440	0	40320	40320
1	234	945	87	3150	-945	336	5040	2240	1512	0	7560	-3780	3780	504	4536	-4032	8064	-9072	0	0	0	0	-4320	-12960	0	-20160	0	0
1	88	-1155	99	-2310	1155	924	4620	3080	0	0	0	0	0	924	-8316	3696	3696	0	0	-9240	9240	0	1980	5940	0	0	-15120	-15120
1	110	825	55	550	-825	880	-4400	0	-1320	0	-6600	-3300	3300	88	3960	3520	3520	7920	6336	0	0	0	0	0	0	0	14400*C	14400*D
1	110	825	55	550	-825	880	-4400	0	-1320	0	-6600	-3300	3300	88	3960	3520	3520	7920	6336	0	0	0	0	0	0	0	14400*D	14400*C
1	150	525	15	1050	-525	840	4200	-2800	0	0	0	6300	-6300	-672	0	3360	3360	0	12096	-8400	8400	0	-1800	-5400	0	0	0	0
1	174	525	42	2100	1050	336	-1680	2240	252	1260	5040	1260	2520	-336	-3024	2688	-5376	6048	0	-3360	-6720	10080	-4320	4320	-15120	0	0	0
1	-88	165	66	-660	330	792	1320	1760	-1188	1980	0	-1980	-3960	264	2376	4224	-2112	-4752	0	-2640	-5280	-7920	0	0	0	0	8640	8640
1	150	-75	90	-300	-150	240	-1200	-1600	900	900	-3600	-900	-1800	768	0	-3840	-3840	0	3456	-2400	-4800	7200	0	0	10800	0	0	0
1	88	105	-6	-420	210	504	840	-1120	0	-2520	0	1260	2520	-336	3024	-1344	-1344	0	0	-1680	-3360	-5040	720	2160	0	0	0	0
1	54	405	42	-540	810	-384	0	-840	-108	1620	-2160	540	1080	144	1296	-1152	2304	-2592	0	0	0	0	0	0	-8640	5760	0	0
1	30	-75	-30	300	-150	120	-600	600	-540	900	0	900	1800	168	-1080	-1920	-1920	-2160	3456	1200	2400	-3600	0	0	0	0	0	0
1	90	-63	42	252	-126	-168	-504	224	0	-1512	0	-756	-1512	0	0	0	0	0	0	1008	2016	-3024	-1296	1296	0	4032	0	0
1	10	-175	50	-700	-350	0	0	0	-280	0	2800	700	1400	168	840	0	0	1680	-1344	0	0	0	-400	-1200	0	0	0	0
1	24	-105	-18	420	-210	-84	420	560	-378	-630	0	0	0	84	756	-672	1344	-1512	0	-840	-1680	2520	720	2160	0	0	0	0
1	70	-75	10	100	-150	160	-600	0	-60	900	-1200	300	600	-272	-720	640	640	-1440	-2304	0	0	0	0	0	3600	0	0	0
1	-18	189	-6	-252	378	0	0	896	252	-756	-1008	252	504	0	0	0	0	0	0	0	0	0	-288	-864	3024	0	0	0
1	-60	75	30	300	150	180	300	-400	-270	-450	0	0	0	-12	-540	-480	-480	-1080	-864	600	1200	1800	0	0	0	0	0	0
1	34	105	7	70	-105	-224	1120	0	-168	0	-840	-420	420	-56	-504	448	-896	1008	0	0	0	0	720	-720	0	0	0	0
1	34	105	-14	-140	210	112	-560	0	-168	0	1680	-420	-840	-56	-504	-896	448	1008	0	0	0	0	720	-720	0	0	0	0
1	-18	189	15	-126	-189	-168	-504	224	0	0	0	756	-756	0	0	0	0	0	0	0	0	0	216	648	0	-2016	0	0
1	6	-75	-6	-300	-150	144	240	320	180	180	-720	-180	-360	-96	864	-384	-384	0	0	480	960	1440	0	0	-2160	0	0	0
1	0	-225	15	0	225	-60	0	-100	0	0	0	0	0	-72	0	360	360	0	1296	900	-900	0	0	0	0	-1800	0	0
1	6	105	-21	-210	-105	84	420	-280	0	0	0	0	0	84	-756	336	336	0	0	840	-840	0	-540	540	0	0	0	0
1	-36	-45	6	180	-90	12	180	80	162	270	0	0	0	-36	-324	-576	288	648	0	-360	-720	-1080	0	0	0	1440	0	0
1	-6	-15	-9	30	15	-48	-240	-160	72	0	360	-180	180	24	216	384	-192	-432	0	-480	480	0	0	0	0	0	0	0

References

[1] R. Z. Aleev, *Higman's central unit theory, units of integral group rings of finite cyclic groups and Fibonacci numbers*, Internat. J. Algebra Comput. **4** (1994), no. 3, 309–358.

[2] ———, *Central elements of integral group rings*, Algebra Log. **39** (2000), no. 5, 513–525, 630; translation in Algebra and Logic **39** (2000), no. 5, 293–300.

[3] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, a division of John Wiley & Sons, New York-London 1962.

[4] M. Hertweck and E. Jespers, *Class-preserving automorphisms and the normalizer property for Blackburn groups*, J. Group Theory **12** (2009), no. 1, 157–169.

[5] M. Hertweck and W. Kimmerle, *Coleman automorphisms of finite groups*, Math. Z. **242** (2002), no. 2, 203–215.

[6] E. Jespers, M. M. Parmenter, and S. K. Sehgal, *Central units of integral group rings of nilpotent groups*, Proc. Amer. Math. Soc. **124** (1996), no. 4, 1007–1012.

[7] Y. Li and M. M. Parmenter, *Central units of the integral group ring ZA_5*, Proc. Amer. Math. Soc. **125** (1997), no. 1, 61–65.

[8] Z. F. Patay, *Structure of the center of the multiplicative group of a group ring*, Visnik Kiiv. Univ. Ser. Mat. Mekh. No. **27** (1985), 90–91, 127.

[9] J. Ritter and S. K. Sehgal, *Integral group rings with trivial central units*, Proc. Amer. Math. Soc. **108** (1990), no. 2, 327–329.

[10] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12, 2008. (http://www.gap-system.org)

[11] Waterloo Maple Inc., Maple, Version 7.00, 2008. (http://www.maplesoft.com)

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