

AN EXTENSION OF REDUCTION FORMULA FOR LITTLEWOOD-RICHARDSON COEFFICIENTS

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ABSTRACT. There is a well-known classical reduction formula by Griffiths and Harris for Littlewood-Richardson coefficients, which reduces one part from each partition. In this article, we consider an extension of the reduction formula reducing two parts from each partition. This extension is a special case of the factorization theorem of Littlewood-Richardson coefficients by King, Tollu, and Toumazet (the KTT theorem). This case of the KTT factorization theorem is of particular interest, because, in this case, the KTT theorem is simply a reduction formula reducing two parts from each partition. A bijective proof using tableaux of this reduction formula is given in this paper while the KTT theorem is proved using hives.

0. Introduction

Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ are important in many fields of mathematics. They count the number of column strict (skew) tableaux on the shape ν/λ of content μ that satisfy a certain condition on the word derived from each tableau. We call such a tableau a *Littlewood-Richardson tableau*. They explain the multiplication rule of Schur functions;

$$s_\lambda \cdot s_\mu = \sum_{\nu} c_{\lambda\mu}^\nu s_\nu$$

and the tensor product rule of irreducible polynomial representations of the general linear group $GL_n(\mathbb{C})$;

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} c_{\lambda\mu}^\nu V(\nu).$$

Received February 17, 2009.

2000 *Mathematics Subject Classification*. Primary 05E10; Secondary 14M15.

Key words and phrases. Littlewood-Richardson coefficients, Reduction formulae.

The first author was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD, Basic Research Promotion Fund) (KRF-2008-512-C00001).

The third author was supported by the Korea Science and Engineering Foundation(KOSEF) grant funded by the Korea government(MEST)(No. 2009-0076032).

They also appear in the Schubert calculus of Grassmannians as structural constants of the cohomology ring;

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu,$$

where σ_λ is the Schubert class in the cohomology ring of a Grassmannian, indexed by the partition λ . In [10], P. Griffiths and J. Harris used the inclusion map of the n -dimensional space into the $(n+1)$ -dimensional space to obtain a reduction formula for structural constants by reducing one part from each partition. A combinatorial bijective proof of this classical reduction formula using tableaux, the standard machinery to compute Littlewood-Richardson coefficients, is obtained in [3] (see also [5]).

Recently, inspired by the Saturation Theorem by A. Knutson and T. Tao [16], and the properties of *puzzles*, introduced by Knutson et al., which serve as another model of Littlewood-Richardson coefficients [17], King, Tollu, and Toumazet observed various properties of Littlewood-Richardson coefficients and their generating functions. They made several conjectures on them [14]. One of them is on the factorization of Littlewood-Richardson coefficients and it has been proved by themselves [15]. In this article, we call this the *KT* theorem for short. Roughly speaking, the theorem states that if $c_{\lambda\mu}^\nu > 0$ and any of Horn's inequalities is an equality, then $c_{\lambda\mu}^\nu$ can be written as a product of two Littlewood-Richardson coefficients indexed by certain subpartitions of λ, μ, ν , respectively. Note that Horn's inequalities give necessary and sufficient conditions for $c_{\lambda\mu}^\nu$ to be non-zero (see [8], for example).

In [6], we observed that if the given index sets in the theorem have cardinality $1, 2, (n-1)$ or $(n-2)$, then the KTT theorem gives a reduction formula of Littlewood-Richardson coefficients because there exists a unique Littlewood-Richardson tableau, if there is, when the length of partitions are at most 2. We also observed that it is the classical reduction formula when the size of index sets are $(n-1)$. The case when the cardinality of the index sets is $(n-2)$ can be realized as an extension of the classical reduction formula by Griffiths and Harris since we delete two parts from each partition and still get the same Littlewood-Richardson coefficient. This reduction formula ($(n-2)$ case of the KTT theorem) is not obtained by merely applying the classical reduction formula twice though.

In this article, we provide a combinatorial proof of this reduction formula similar to the one given by the authors in [3] for the classical reduction formula. The ingredients of the proof in this article are different from those of the classical reduction formula; Proposition 2.8 shows the difficulty to remove two parts while maintaining equality of Littlewood-Richardson coefficients. Moreover, we exploit a nice new equivalent condition, given in Lemma 3.13, for a reverse row word of a tableau to be a lattice word, and we complete our proof of the reduction formula.

The work presented in this paper began when the authors realized the classical reduction formula in [10] (and [3]) is a special case of the KTT conjecture in [14], which is the KTT theorem now. Our proof in [3] of the classical reduction formula is purely combinatorial using tableaux theory. That leads us to attempt to prove some special cases of the conjecture combinatorially using tableaux, even though the original conjecture by King, Tollu, and Toumazet was motivated by the hive and puzzle models of Littlewood-Richardson coefficients recently developed in [16], and eventually was proved using hives and puzzles ([15]). Our aim is achieved in the works presented here and [4], carrying out challenging and technical proofs. While preparing the manuscript of this paper, the authors acknowledged King, Tollu, and Toumazet for proving their conjecture using hives and puzzles [15].

We state the classical reduction formula and the KTT theorem and introduce necessary terminologies in Section 1. In Section 2, we provide an algorithm to obtain a combinatorial proof of the (extended) reduction formula. We prove that our algorithm is a well defined bijection in Section 3. In the appendix, we will see how our main idea can be translated into the language of hives.

1. Preliminaries, notations and terminologies

A *partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ is a nonincreasing sequence of nonnegative integers with finite number of positive numbers. The *size* of λ is defined as $|\lambda| = \sum_i \lambda_i$ and the *length* of λ , denoted by $\ell(\lambda)$, is defined as the number of positive numbers in λ .

The *Young diagram* of a partition λ is a left-justified array of boxes with λ_i boxes in its i th row. For a given partition λ , $\bar{\lambda}$ indicates the *conjugate* of λ , whose diagram is obtained by interchanging rows and columns of that of λ . A *tableau*, of shape λ , is a filling of a Young diagram of λ that is weakly increasing across each row and strictly increasing down each column. For two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\nu = (\nu_1, \nu_2, \dots)$, we write $\lambda \subseteq \nu$, if $\lambda_i \leq \nu_i$ for all i . For partitions λ and ν with $\lambda \subseteq \nu$, the *skew diagram* of shape ν/λ is the diagram consisting of boxes of ν which are not the boxes of λ . A *skew tableau* of shape ν/λ with *content* $\mu = (\mu_1, \mu_2, \dots)$ is a filling of boxes of a skew diagram ν/λ with μ_i i 's, where entries are weakly increasing in rows and strictly increasing in columns. The *reverse row word* of a skew tableau T , denoted by $w(T)$, is the word obtained by reading the entries of T from right to left and top to bottom. A word $w = x_1 \cdots x_r$ is called a *lattice word* if, for any $s \leq r$ and i , $x_1 \cdots x_s$ contains at least as many i 's as it contains $(i+1)$'s.

Definition 1.1. A skew tableau T is a Littlewood-Richardson skew tableau (LR-tableau) if its reverse row word $w(T)$ is a lattice word.

There are several ways to define Littlewood-Richardson coefficients, we however use the following combinatorial description for Littlewood-Richardson coefficients in this article.

Definition 1.2. Given three partitions λ, μ and ν , Littlewood-Richardson coefficient (LR-coefficient) $c_{\lambda\mu}^\nu$ is the number of LR-tableaux on the shape ν/λ of content μ .

When n is a positive integer, we let $[n] = \{1, 2, \dots, n\}$. For a pair of positive integers r, n with $r \leq n$, we let $I = \{i_1, i_2, \dots, i_r\}$, $i_1 < i_2 < \dots < i_r$, be an r -subset of $[n]$, and λ be a partition with $\ell(\lambda) \leq n$. We define $\lambda_I = (\lambda_{i_1}, \dots, \lambda_{i_r})$ as a subpartition of λ , which is associated to I . We denote by I^c the complement of I in $[n]$, i.e., $I^c = [n] - I$. The following is the classical reduction for LR-coefficients:

Theorem 1.3 (See [10], [3] and [4]). *Let λ, μ and ν be partitions whose lengths are at most n . Suppose there are $1 \leq i, j, k \leq n$ such that $i + j = k + n$ and $\nu_k = \lambda_i + \mu_j$. Then,*

$$c_{\lambda\mu}^\nu = c_{\lambda_{I^c}^c \mu_{J^c}^c}^{\nu_{K^c}^c},$$

where $I = \{i_1\}$, $J = \{j_1\}$, $K = \{k_1\}$.

We fix a positive integer n . For a given subset $I \subseteq [n]$, we let $\pi(I) = (i_r - r, i_{r-1} - (r - 1), \dots, i_1 - 1)$ be a partition associated to I .

Definition 1.4. For a pair of two positive integers $r \leq n$, define a set of triples of r -subsets of $[n]$ as follows:

$$R_r^n = \{(I, J, K) \mid I, J, K \text{ are } r\text{-subsets of } [n], c_{\pi(I)\pi(J)}^{\pi(K)} = 1\}.$$

Now we state the KTT theorem on the factorization of LR-coefficients.

Proposition 1.5 (KTT theorem [15]). *Let λ, μ and ν be partitions with at most n nonzero parts and $c_{\lambda\mu}^\nu > 0$. Suppose that there exists $(I, J, K) \in R_r^n$ for some $r < n$, which satisfies the equality $\sum_{k \in K} \nu_k = \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j$. Then*

$$(1.6) \quad c_{\lambda\mu}^\nu = c_{\lambda_I \mu_J}^{\nu_K} c_{\lambda_{I^c}^c \mu_{J^c}^c}^{\nu_{K^c}^c}.$$

Observe that if λ, μ, ν are partitions whose lengths are at most 2, then the corresponding LR-coefficient is 1 whenever it is non-zero. Hence, in the KTT theorem, if $r \leq 2$ or $r \geq n - 2$, then $c_{\lambda_I \mu_J}^{\nu_K} = 1$ or $c_{\lambda_{I^c}^c \mu_{J^c}^c}^{\nu_{K^c}^c} = 1$, respectively. Therefore, four cases $r = 1, 2, n - 1, n - 2$ of the KTT theorem will give us reduction formulae of LR-coefficients. More interestingly, the KTT theorem for the case $r = n - 1$ gives the classical reduction formula in [10], whose combinatorial proof using LR-tableaux is given in [3]. For more details of these observations, see [6]. Moreover, in [6], we interpret the condition that a triple (I, J, K) is in R_r^n in terms of triple (I^c, J^c, K^c) , and hence, we are able to state an *explicit and straightforward condition* (see equations (1.8)–(1.12) below) on the partitions, for that the reduction formula of $r = n - 2$ holds. These results including results on other reduction formulae are also presented in [4].

Our aim in this article is to prove the reduction formula (the KTT theorem when $r = n - 2$) using tableaux theory. The following is a precise statement of our reduction formula (the KTT theorem when $r = n - 2$):

Proposition 1.7 (reduction formula, KTT theorem for $r = n - 2$). *Let λ, μ and ν be partitions with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $c_{\lambda\mu}^\nu > 0$. Suppose that there are three subsets $I = \{i_1, i_2\}$, $J = \{j_1, j_2\}$, $K = \{k_1, k_2\}$ of $[n]$ such that*

$$(1.8) \quad i_1 + i_2 + j_1 + j_2 = k_1 + k_2 + 2n - 1,$$

$$(1.9) \quad k_2 + n \leq i_2 + j_2,$$

$$(1.10) \quad k_1 + n \leq i_2 + j_1,$$

$$(1.11) \quad k_1 + n \leq i_1 + j_2,$$

$$(1.12) \quad \nu_{k_1} + \nu_{k_2} = \lambda_{i_1} + \lambda_{i_2} + \mu_{j_1} + \mu_{j_2}.$$

Then, we have the following reduction formula on Littlewood-Richardson coefficients;

$$(1.13) \quad c_{\lambda\mu}^\nu = c_{\lambda_{I^c}, \mu_{J^c}}^{\nu_{K^c}}.$$

Note Proposition 1.7 is an extension of the classical reduction formula, in the sense that (1.13) reduces two parts from each partition. In the following sections, we give a combinatorial proof of Proposition 1.7 by constructing a bijection between two sets of LR-tableaux of corresponding shapes and contents.

2. Algorithms

In this section, we describe a bijection which will be used in Section 3 to prove Proposition 1.7. We start this section by providing some conventions. For a filling S of skew shape with n rows and integers $1 \leq p, \ell \leq n$, we let $\mathbf{n}_S^p(\ell)$ be the number of ℓ 's in the p th row of S . Also we let $S(p, q)$ denote the *entry* in the box at (p, q) -position of the filling S .

Assume λ, μ, ν and $I = \{i_1, i_2\}$, $J = \{j_1, j_2\}$, $K = \{k_1, k_2\}$ satisfy the conditions in Proposition 1.7 for a fixed integer $n \geq 2$. For the convenience, we let $a := i_1 - k_1$, $b := i_2 - k_2$ and $\alpha := (n - 1) - j_1$, $\beta := n - j_2$. It is obvious that $\alpha \geq \beta \geq 0$ from $j_1 < j_2$. Moreover, $a, b \geq 0$ because (1.9) and (1.11) imply $i_1 \geq k_1 + n - j_2 \geq k_1$ and $i_2 \geq k_2 + n - j_2 \geq k_2$. We rewrite the conditions in Equations (1.8)–(1.11) in terms of a, b, α and β ;

$$(2.1) \quad \alpha + \beta = a + b,$$

$$(2.2) \quad \beta \leq b,$$

$$(2.3) \quad k_1 \leq k_2 + b - \alpha - 1,$$

$$(2.4) \quad \beta \leq a.$$

Note that Equations (2.1), (2.2) and (2.4) imply that

$$(2.5) \quad \beta \leq a, b \leq \alpha.$$

We will describe our reduction algorithm for Proposition 1.7 by providing a bijective map Φ from the set of LR-tableaux on the shape ν/λ with content μ onto the set of LR-tableaux on the shape ν_{K^c}/λ_{I^c} with content μ_{J^c} . In this

section we only describe Φ and its inverse. We will prove that Φ is well-defined, and hence is a bijective map in Section 3.

The following lemmas are adopted from [3].

Lemma 2.6. *A skew tableau T with n rows is an LR-tableau, i.e., the reverse row word $w(T)$ is a lattice word, if and only if, for all $1 < p, s \leq n$,*

$$\sum_{h=1}^p \mathbf{n}_T^h(s) \leq \sum_{h=1}^{p-1} \mathbf{n}_T^h(s-1).$$

Lemma 2.7. *Suppose that s appears in the q th row of an LR-tableau T . Then $q \geq s$.*

Throughout this section, we assume that λ, μ, ν and $I = \{i_1, i_2\}, J = \{j_1, j_2\}, K = \{k_1, k_2\}$ satisfy the conditions in Proposition 1.7 for a given n , and T is an LR-tableau of the shape ν/λ with content μ . The following crucial observation leads us to establish an algorithm for Φ . Also the well-definedness of Φ is based on this.

Proposition 2.8. *We have*

$$\begin{aligned} & \mathbf{n}_T^{k_2}(k_2) + \cdots + \mathbf{n}_T^{k_2+\beta}(k_2) \\ &= \mathbf{n}_T^{k_2+1}(k_2+1) + \cdots + \mathbf{n}_T^{k_2+\beta+1}(k_2+1) \\ (2.9) \quad & \vdots \\ &= \mathbf{n}_T^{j_2}(j_2) + \cdots + \mathbf{n}_T^n(j_2) \\ &= \mu_{j_2}, \\ & \mathbf{n}_T^{k_2+b-\alpha-1}(k_2+b-\alpha-1) + \cdots + \mathbf{n}_T^{k_2+b}(k_2+b-\alpha-1) \\ &= \mathbf{n}_T^{k_2+b-\alpha}(k_2+b-\alpha) + \cdots + \mathbf{n}_T^{k_2+b+1}(k_2+b-\alpha) \\ (2.10) \quad & \vdots \\ &= \mathbf{n}_T^{j_1}(j_1) + \cdots + \mathbf{n}_T^n(j_1) \\ &= \mu_{j_1} \end{aligned}$$

and

$$\begin{aligned} & \mathbf{n}_T^{k_1}(k_1) + \cdots + \mathbf{n}_T^{k_1+a}(k_1) \\ (2.11) \quad &= \mathbf{n}_T^{k_1+1}(k_1+1) + \cdots + \mathbf{n}_T^{k_1+a+1}(k_1+1) \\ & \vdots \\ &= \mathbf{n}_T^{k_2+b-\alpha-1}(k_2+b-\alpha-1) + \cdots + \mathbf{n}_T^{k_2+\beta-1}(k_2+b-\alpha-1). \end{aligned}$$

Proof. First we note that $j_2 \geq k_2 + n - i_2 \geq k_2$ by Equation (1.9), $j_1 \geq k_1 + n - i_2 \geq k_1$ by Equation (1.10), and $k_2 + b - \alpha - 1 \geq k_1$ from Equation (2.3). Also note $j_1 = n - 1 - \alpha \geq i_2 - 1 - \alpha = k_2 + b - \alpha - 1$.

We use Lemma 2.6 and Lemma 2.7 repeatedly to obtain the following inequalities;

$$\begin{aligned}
& \mu_{j_2} + \mu_{j_1} \\
&= \left(\mathbf{n}_T^{j_2}(j_2) + \cdots + \mathbf{n}_T^n(j_2) \right) + \left(\mathbf{n}_T^{j_1}(j_1) + \cdots + \mathbf{n}_T^n(j_1) \right) \\
&\leq \left(\mathbf{n}_T^{j_2-1}(j_2-1) + \cdots + \mathbf{n}_T^{n-1}(j_2-1) \right) + \left(\mathbf{n}_T^{j_1}(j_1) + \cdots + \mathbf{n}_T^n(j_1) \right) \\
&\quad \vdots \\
&\leq \left(\mathbf{n}_T^{k_2}(k_2) + \cdots + \mathbf{n}_T^{k_2+\beta}(k_2) \right) + \left(\mathbf{n}_T^{j_1}(j_1) + \cdots + \mathbf{n}_T^n(j_1) \right) \\
&\leq \left(\mathbf{n}_T^{k_2}(k_2) + \cdots + \mathbf{n}_T^{k_2+\beta}(k_2) \right) + \left(\mathbf{n}_T^{j_1-1}(j_1-1) + \cdots + \mathbf{n}_T^{n-1}(j_1-1) \right) \\
&\quad \vdots \\
&\leq \left(\mathbf{n}_T^{k_2}(k_2) + \cdots + \mathbf{n}_T^{k_2+\beta}(k_2) \right) \\
&\quad + \left(\mathbf{n}_T^{k_2+b-\alpha-1}(k_2+b-\alpha-1) + \cdots + \mathbf{n}_T^{k_2+b}(k_2+b-\alpha-1) \right) \\
&= \left(\mathbf{n}_T^{k_2}(k_2) + \cdots + \mathbf{n}_T^{k_2+\beta}(k_2) \right) \\
&\quad + \left(\mathbf{n}_T^{k_2+b-\alpha-1}(k_2+b-\alpha-1) + \cdots + \mathbf{n}_T^{k_2+\beta-1}(k_2+b-\alpha-1) \right) \\
&\quad + \left(\mathbf{n}_T^{k_2+\beta}(k_2+b-\alpha-1) + \cdots + \mathbf{n}_T^{k_2+b}(k_2+b-\alpha-1) \right) \\
&\leq \left(\mathbf{n}_T^{k_2}(k_2) + \cdots + \mathbf{n}_T^{k_2+\beta}(k_2) \right) \\
&\quad + \left(\mathbf{n}_T^{k_2+b-\alpha-2}(k_2+b-\alpha-2) + \cdots + \mathbf{n}_T^{k_2+\beta-2}(k_2+b-\alpha-2) \right) \\
&\quad + \left(\mathbf{n}_T^{k_2+\beta}(k_2+b-\alpha-1) + \cdots + \mathbf{n}_T^{k_2+b}(k_2+b-\alpha-1) \right) \\
&\leq \\
&\quad \vdots \\
&\leq \underbrace{\left(\mathbf{n}_T^{k_2}(k_2) + \cdots + \mathbf{n}_T^{k_2+\beta}(k_2) \right)}_A + \underbrace{\left(\mathbf{n}_T^{k_1}(k_1) + \cdots + \mathbf{n}_T^{k_1+a}(k_1) \right)}_B \\
&\quad + \underbrace{\left(\mathbf{n}_T^{k_2+\beta}(k_2+b-\alpha-1) + \cdots + \mathbf{n}_T^{i_2}(k_2+b-\alpha-1) \right)}_C \\
&\leq (\nu_{k_1} - \lambda_{i_1}) + (\nu_{k_2} - \lambda_{i_2}) = \mu_{j_1} + \mu_{j_2}.
\end{aligned}$$

The last inequality is obtained from the following observations: Each column of T may contain at most one k_1 , and so $B \leq \nu_{k_1} - \lambda_{i_1}$ because $k_1 + a = i_1$. Note also $k_2 + b - \alpha - 1 < k_2$ because $b - \alpha - 1 < 0$ by Equation (2.5). Moreover, indices of the rows where k_2 's in A appear are always less than or equal to

indices of the rows where $(k_2 + b - \alpha - 1)$'s in C appear. We cannot have a $k_2 + b - \alpha - 1$ below a k_2 in a column of T since $b - \alpha \leq 0$ from (2.5). Thus, each of the columns from the $(\lambda_{i_2} + 1)$ st to the ν_{k_2} th may contribute at most once in counting $A + C$, and hence $A + C \leq \nu_{k_2} - \lambda_{i_2}$.

Thus, all inequalities are actually equalities and we obtain desired relations. □

Remark. While an earlier version of this paper was circulating, it was pointed out that Proposition 2.8 could be naturally related with some conditions on hives under the bijection between LR-tableaux and hives in [14]. We discuss this in Appendix.

For convenience, we use the following convention:

Definition 2.12. We let $\mathcal{S}(T)$ be the union of the following sets:

- (S1) Set of boxes which contain k_1 and are in $(a + 1)$ consecutive rows from the k_1 th row to the i_1 th row.
- (S2) Set of boxes which contain $(k_2 + b - \alpha - 1)$ and are in $(b - \beta + 1)$ consecutive rows from the $(k_2 + \beta)$ th row to the i_2 th row.
- (S3) Set of boxes which contain k_2 and are in $(\beta + 1)$ consecutive rows from the k_2 th row to the $(k_2 + \beta)$ th row.

Now we let the number of boxes of (S1) be ω . Note that (2.9) implies the number of boxes in (S3) is μ_{j_2} , and (2.10) and (2.11) imply the number of boxes of (S2) is $\mu_{j_1} - \omega$. The following lemma is immediate from the proof of Proposition 2.8 and will be needed in Section 3.

Lemma 2.13. (1) *There is exactly one k_1 of (S1) in each column of T from the $(\lambda_{i_1} + 1)$ st column to the ν_{k_1} th column of T , and they are between the k_1 th row and the i_1 th row of T . Hence $\omega = \nu_{k_1} - \lambda_{i_1}$.*

(2) *There is exactly one $(k_2 + b - \alpha - 1)$ of (S2) in each column of T from the $(\lambda_{i_2} + 1)$ st column to the $(\nu_{k_2} - \mu_{j_2})$ th column, and they are between the $(k_2 + \beta)$ th row and the i_2 th row of T .*

(3) *There is exactly one k_2 of (S3) in each column of T from the $(\nu_{k_2} - \mu_{j_2} + 1)$ st column to the ν_{k_2} th column of T and they are between the k_2 th row and the $(k_2 + \beta)$ th row of T .*

Remark 2.14. Because of Lemma 2.7, there are μ_{j_2} j_2 's in the $(\beta + 1)$ consecutive rows from the j_2 th row to the n th row of T (note $\beta + 1 = n - j_2$). Then, Lemma 2.6 imply that there are at least μ_{j_2} k_2 's in β consecutive rows from the k_2 th row to the $(k_2 + \beta)$ th row (note $k_2 + \beta = k_2 + n - j_2$). Because $k_2 + n - j_2 \leq i_2$ and each column of T may contain at most one k_2 , we have $\nu_{k_2} - \lambda_{i_2} \geq \nu_{k_2} - \lambda_{k_2 + n - j_2} \geq \mu_{j_2}$. Hence, $\nu_{k_2} - \lambda_{i_2} - \mu_{j_2} = \lambda_{i_1} + \mu_{j_1} - \nu_{k_1} \geq 0$.

Corollary 2.15. *Each column of T may contain at most two boxes in $\mathcal{S}(T)$.*

Equations (2.9), (2.10), (2.11) and Lemma 2.13 imply that

Corollary 2.16. (1) For $\ell = k_1, k_1 + 1, \dots, k_2 + b - \alpha - 1$,

$$\sum_{q=\ell}^{\ell+\alpha} n_T^q(\ell) = \omega = \nu_{k_1} - \lambda_{i_1}.$$

(2) For $\ell = k_2 + b - \alpha - 1, \dots, j_1$,

$$\sum_{q=\ell}^{\ell+\alpha+1} n_T^q(\ell) = \mu_{j_1}.$$

(3) For $\ell = k_2, \dots, j_2$,

$$\sum_{q=\ell}^{\ell+\beta} n_T^q(\ell) = \mu_{j_2}.$$

For a skew shape ν/λ , we allow a *filling* S of shape ν/λ to contain boxes which are empty. As usual, the *reverse row word* $w(S)$ of a filling S is the word obtained by reading the entries of S from right to left and top to bottom, ignoring the empty boxes. Whenever we refer to the *order of a content* in a filling S , we mean the order in the corresponding reverse row word $w(S)$. We use the following operation on a filling of skew-shape to describe our reduction algorithm.

Definition 2.17. There are two types of operations: For a filling S of shape ν/λ ,

- (1) $\mathcal{R}_{p,q}^\ell(S)$ is the filling of shape ν/λ obtained by replacing the first ℓ p 's in S with q 's,
- (2) $\mathcal{R}_{p,\emptyset}^\ell(S)$ is the filling of shape ν/λ obtained by emptying the first ℓ boxes of S containing p .

Now we describe our reduction algorithm. Assume λ, μ, ν and I, J, K satisfy the conditions in Proposition 1.7 and let T be an LR-tableau on the shape ν/λ with content μ . Note that all j_1 's and j_2 's in T will be removed after we apply reduction procedure to T . During the reduction procedure, we empty boxes in $\mathcal{S}(T)$ and move the contents down until we remove all j_1 's and j_2 's. Equation (2.11) allows us to apply a sequence of \mathcal{R} operations to T as follows:

$$(2.18) \quad \mathcal{R}_{k_2+b-\alpha-1, k_2+b-\alpha-2}^\omega \circ \dots \circ \mathcal{R}_{k_1+2, k_1+1}^\omega \circ \mathcal{R}_{k_1+1, k_1}^\omega \circ \mathcal{R}_{k_1, \emptyset}^\omega.$$

Note (2.18) empties all boxes of (S1) in $\mathcal{S}(T)$ and replaces all $(k_2 + b - \alpha - 1)$'s from the $(k_2 + b - \alpha - 1)$ st row to the $(k_2 + \beta - 1)$ st row with $(k_2 + b - \alpha - 2)$'s.

Then we use Equation (2.10) and apply a sequence of \mathcal{R} operations to the filling we obtained in (2.18) as follows:

$$(2.19) \quad \mathcal{R}_{j_1, j_1-1}^{\mu_{j_1}} \circ \dots \circ \mathcal{R}_{k_2+b-\alpha, k_2+b-\alpha-1}^{\mu_{j_1}} \circ \mathcal{R}_{k_2+b-\alpha-1, \emptyset}^{\mu_{j_1}-\omega}.$$

Note $\mathcal{R}_{k_2+b-\alpha-1, \emptyset}^{\mu_{j_1}-\omega}$ empties boxes of (S2) in $\mathcal{S}(T)$. Then $\mathcal{R}_{k_2+b-\alpha, k_2+b-\alpha-1}^{\mu_{j_1}}$ will replace μ_{j_1} $(k_2 + b - \alpha)$'s in the next $(\alpha + 2)$ rows with $(k_2 + b - \alpha - 1)$'s. Remember

that $\omega (k_2 + b - \alpha - 1)$'s were replaced in (2.18) and $\mu_{j_1} - \omega (k_2 + b - \alpha - 1)$'s were in the boxes of (S2). After we apply (2.19), all j_1 's are removed.

Next step is to empty boxes of (S3) and remove all j_2 's. Again, Equation (2.9) enables us to apply one of the following operations to the filling we obtained in (2.19) depending on the given conditions:

If $k_2 > j_1$,

$$(2.20a) \quad \mathcal{R}_{j_2, j_2-1}^{\mu_{j_2}} \circ \cdots \circ \mathcal{R}_{k_2+1, k_2}^{\mu_{j_2}} \circ \mathcal{R}_{k_2, \emptyset}^{\mu_{j_2}}.$$

If $k_2 = j_1$,

$$(2.20b) \quad \mathcal{R}_{j_2, j_2-1}^{\mu_{j_2}} \circ \cdots \circ \mathcal{R}_{k_2+1, k_2-1}^{\mu_{j_2}} \circ \mathcal{R}_{k_2-1, \emptyset}^{\mu_{j_2}}.$$

If $k_2 < j_1$,

$$(2.20c) \quad \mathcal{R}_{j_2, j_2-1}^{\mu_{j_2}} \circ \cdots \circ \mathcal{R}_{j_1+2, j_1+1}^{\mu_{j_2}} \circ \mathcal{R}_{j_1+1, j_1-1}^{\mu_{j_2}} \circ \mathcal{R}_{j_1-1, j_1-2}^{\mu_{j_2}} \circ \cdots \\ \cdots \circ \mathcal{R}_{k_2, k_2-1}^{\mu_{j_2}} \circ \mathcal{R}_{k_2-1, \emptyset}^{\mu_{j_2}}.$$

After applying the above procedure to T , we obtain a filling with content $(\mu_1, \dots, \mu_{j_1-1}, 0, \mu_{j_1+1}, \dots, \mu_{j_2-1}, 0, \mu_{j_2+1}, \dots, \mu_n)$. Roughly speaking, an ℓ in the (p, q) -box of T will remain the same or be changed to either an $\ell - 1$ or an $\ell - 2$ depending on whether the procedures affect the (p, q) -box either once or twice. Note that the last case occurs only if $k_2 + 1 \leq \ell \leq j_1$ and the position of the box is in $(\beta + 1)$ consecutive rows from the ℓ th row to the $(\ell + \beta)$ th row of T . The illustration in Figure 1 helps us to understand what we discussed above. Note the numbers in boldface indicates that the content ℓ of a box in the area is decreased by 1 or 2.

We use the following definition to describe the whole process of applying (2.18), (2.19) and (2.20) in a single step.

Definition 2.21. We define h_T to be a filling of shape ν/λ with integers $0, -1, -2$ so that $h_T(p, q)$ is empty if (p, q) -box of T is in $\mathcal{S}(T)$; for $(p, q) \in \nu/\lambda - \mathcal{S}(T)$,

$$h_T(p, q) = \begin{cases} -1 & \text{if } k_1 + 1 \leq T(p, q) \leq k_2 + b - \alpha - 1 \text{ and } p \leq T(p, q) + a, \\ -1 & \text{if } k_2 + b - \alpha \leq T(p, q) \leq k_2 - 1 \text{ and } p \leq T(p, q) + \alpha + 1, \\ -2 & \text{if } k_2 + 1 \leq T(p, q) \leq j_1 + 1 \text{ and } p \leq T(p, q) + \beta, \\ -1 & \text{if } k_2 \leq T(p, q) \leq j_1 \text{ and } T(p, q) + \beta < p \leq T(p, q) + \alpha + 1, \\ -1 & \text{if } \max\{j_1 + 2, k_2 + 1\} \leq T(p, q) \leq j_2 \text{ and } p \leq T(p, q) + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Note if $k_2 \geq j_1$, then some cases are void in the above definition. The following is straightforward.

Lemma 2.22. Let F be the filling obtained by applying (2.18), (2.19) and (2.20) to T in that order. If the (p, q) -box of T is in $\mathcal{S}(T)$, then $F(p, q)$ is empty. Otherwise, $F(p, q) = T(p, q) + h_T(p, q)$.

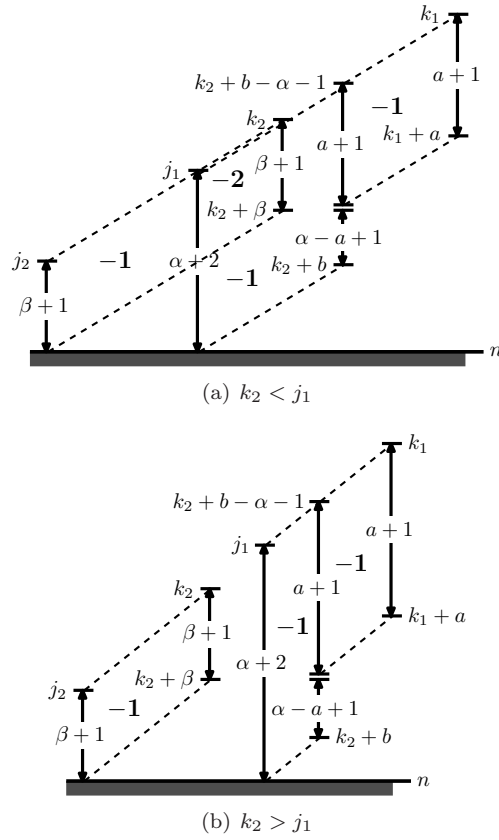


Figure 1: Illustration of reduction algorithm

Now our reduction algorithm is presented. Note, by *sliding*, we mean that empty box exchanges its position with a filled box in the same column.

Definition 2.23. The reduced LR-tableau $\Phi(T)$ on the shape ν_{K^c}/λ_{J^c} with content μ_{J^c} is obtained by applying the following algorithm:

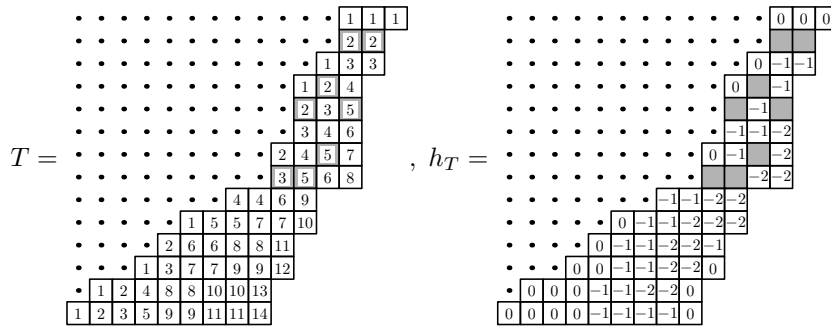
- Step 1: Empty all boxes in $\mathcal{S}(T)$.
- Step 2: Add $h_T(p, q)$ to the (p, q) -box which is not empty.
- Step 3: **for** $\ell = (j_1 + 1)$ **to** $(j_2 - 1)$ **do**
 Replace all ℓ 's with $(\ell - 1)$'s.
end for
- Step 4: **for** $\ell = (j_2 + 1)$ **to** n **do**
 Replace all ℓ 's with $(\ell - 2)$'s.
end for

- Step 5: Slide empty boxes of (S2) or (S3) in Definition 2.12 down to the i_2 th row or to the end of the column whichever occurs first. Also slide empty boxes of (S1) down to the i_1 th row or to the end of the column whichever occurs first.
- Step 6: Remove all the empty boxes which are in the i_1 th row, i_2 th row or at the end of columns.

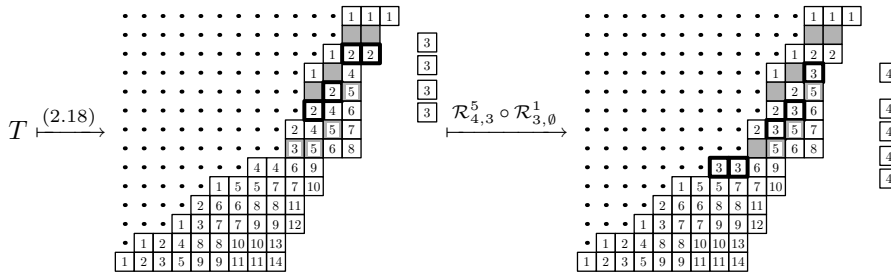
Note Proposition 2.8 guarantees that $\Phi(T)$ is a filling of ν_{K^c}/λ_{I^c} with content μ_{J^c} . We denote the filling obtained by applying Step 1 and Step 2 by $T + h_T$ and the filling obtained by applying Step 1 through Step 4 by T° . We will show that $\Phi(T)$ is an LR-tableau in Section 3.

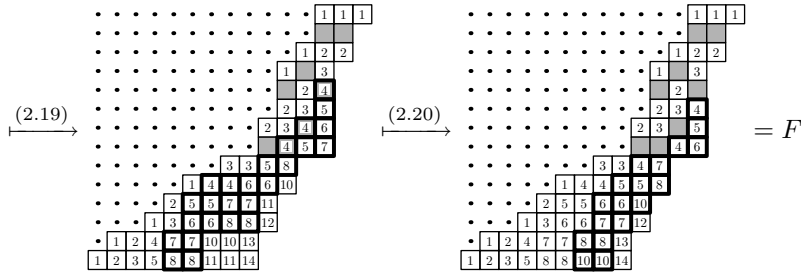
Now we give an example of our algorithm for $k_2 < j_1$.

Example 2.24. Let $n = 14$. For given partitions $\lambda = (12, 12, 11, 10, 10, 10, 9, 9, 7, 5, 4, 3, 1, 0)$, $\mu = (9, 8, 7, 6, 6, 5, 5, 5, 5, 3, 3, 1, 1, 1)$ and $\nu = (15, 14, 14, 13, 13, 13, 13, 11, 11, 10, 10, 9, 9)$, we choose $i_1 = 6, i_2 = 8, k_1 = 2, k_2 = 5, j_1 = 9, j_2 = 11$ so that $\lambda_6 + \lambda_8 + \mu_9 + \mu_{11} = 27 = \nu_2 + \nu_5$. In this case, $a = 4, b = 3, \alpha = 4, \beta = 3$ and $k_2 + b - \alpha - 1 = 3$. We are given the following LR-tableau T on the shape ν/λ with content μ .

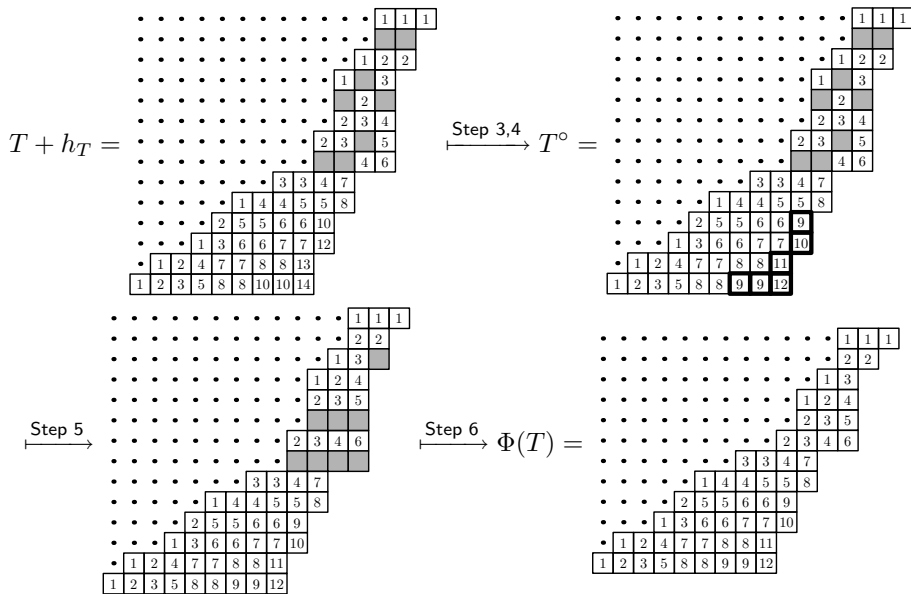


The following shows the process applying (2.18), (2.19) and (2.20).



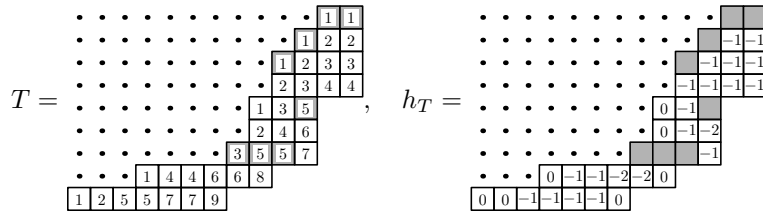


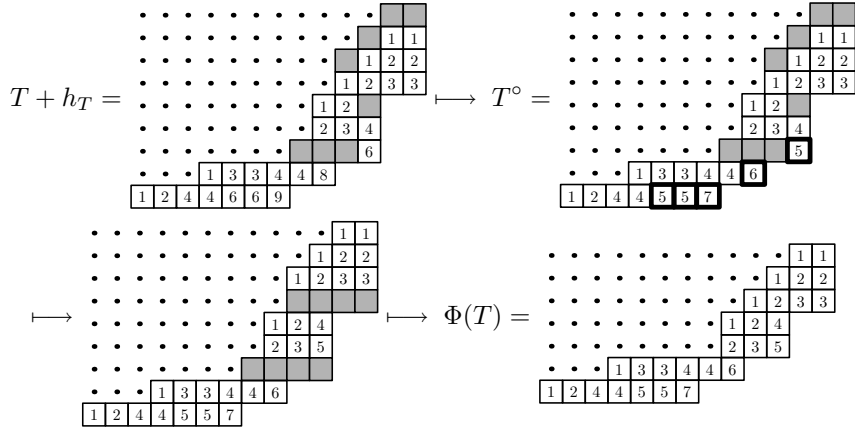
The following shows our reduction algorithm in Definition 2.23 to obtain $\Phi(T)$. We may verify Lemma 2.22 by comparing the filling obtained above and $T + h_T$ below.



Here is another example for $k_2 = j_1$.

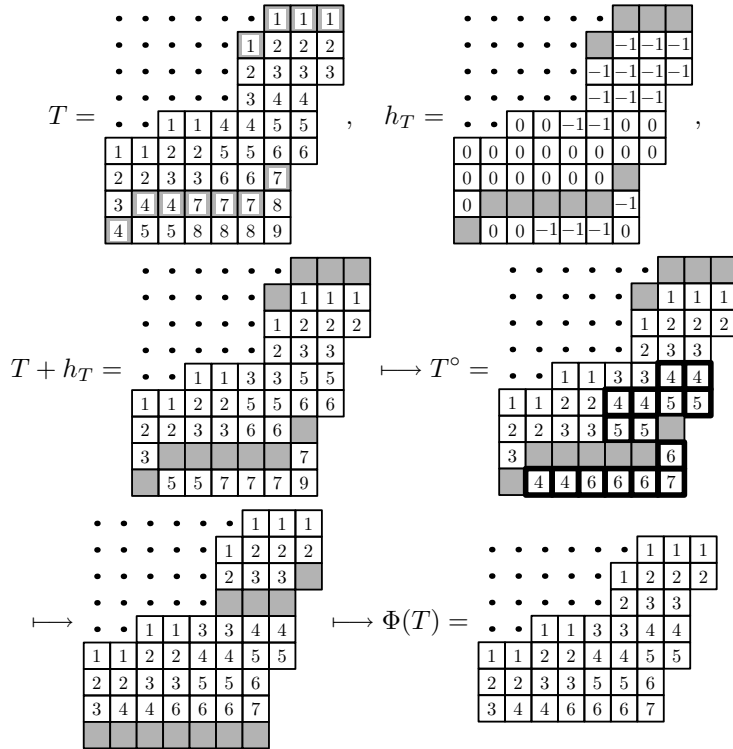
Example 2.25. Let $n = 9$. For given partitions $\lambda = (11, 10, 9, 9, 8, 8, 7, 3, 0)$, $\mu = (7, 6, 5, 5, 5, 3, 3, 1, 1)$ and $\nu = (13, 13, 13, 13, 11, 11, 11, 9, 7)$, we choose $i_1 = 4$, $i_2 = 7$, $k_1 = 1$, $k_2 = 5$, $j_1 = 5$, $j_2 = 7$ so that $\lambda_4 + \lambda_7 + \mu_5 + \mu_7 = 24 = \nu_1 + \nu_5$. In this case, $a = 3$, $b = 2$, $\alpha = 3$, $\beta = 2$ and $k_2 + b - \alpha - 1 = 3$.





The following is an example for $k_2 > j_1$.

Example 2.26. We let $n = 9$. For given partitions $\lambda = (6, 5, 5, 5, 2, 0, 0, 0, 0)$, $\mu = (8, 8, 7, 7, 6, 4, 4, 4, 1)$ and $\nu = (9, 9, 9, 8, 8, 8, 7, 7, 7)$, we choose $i_1 = 4$, $i_2 = 9$, $k_1 = 1$, $k_2 = 7$, $j_1 = 4$, $j_2 = 8$ so that $\lambda_4 + \lambda_9 + \mu_4 + \mu_8 = 16 = \nu_1 + \nu_7$. In this case, $a = 3$, $b = 2$, $\alpha = 4$, $\beta = 1$ and $k_2 + b - \alpha - 1 = 4$.



We now define a function Ψ that plays the role of the inverse of Φ . For λ, μ, ν and I, J, K which satisfy the conditions in Proposition 1.7, recall a, b, α, β are defined as $a = i_1 - k_1, b = i_2 - k_2, \alpha = n - 1 - j_1$ and $\beta = n - j_2$.

Definition 2.27. Let λ, μ, ν and I, J, K satisfy the conditions in Proposition 1.7. For a given LR-tableau U on the shape ν_{K^c}/λ_{I^c} with content μ_{J^c} , an LR-tableau $\Psi(U)$ on the shape ν/λ with content μ is obtained by applying the following algorithm:

- Step 1: Obtain a filling (with $(\mu_{j_1} + \mu_{j_2})$ empty boxes) of shape ν/λ by suitably inserting empty boxes between the $(i_1 - 1)$ st row and the i_1 th row of U , between the $(i_2 - 2)$ nd row and the $(i_2 - 1)$ st row of U and, if necessary, at the end of columns.
- Step 2: *Upper empty boxes* are leftmost $(\nu_{k_1} - \lambda_{i_1})$ empty boxes among empty boxes in the i_1 th row or above. Other $(\nu_{k_2} - \lambda_{i_2})$ empty boxes are *lower empty boxes*. Slide the empty boxes up so that columns are increasing when we put $(k_1 - 0.5)$ on upper empty boxes, $(k_2 + b - \alpha - 1.5)$ on leftmost $(\nu_{k_2} - \lambda_{i_2} - \mu_{j_2})$ lower empty boxes and $(k_2 - 1.5)$ on remaining lower empty boxes.
- Step 3: **for** $\ell = (j_2 - 1)$ **to** $(n - 2)$ **do**
 Replace ℓ 's with $(\ell + 2)$'s.
 end for
- Step 4: **for** $\ell = j_1$ **to** $(j_2 - 2)$ **do**
 Replace all ℓ 's with $(\ell + 1)$'s.
 end for

We denote the tableau obtained from U by applying the Step 1 to Step 4 by U' . Now we define h'_U to be a filling of shape ν/λ with integers $0, 1, 2$ so that $h'_U(p, q)$ is empty if (p, q) -box of U' is empty. Otherwise $h'_U(p, q)$ is defined as (2.28)

$$h'_U(p, q) = \begin{cases} 1 & \text{if } k_1 \leq U'(p, q) \leq k_2 + b - \alpha - 2 \text{ and } (p, q) \text{ is a position} \\ & \text{of the first } (\nu_{k_1} - \lambda_{i_1}) \text{ } U'(p, q)\text{'s in } w(U'), \\ 1 & \text{if } k_2 + b - \alpha - 1 \leq U'(p, q) \leq k_2 - 2 \text{ and } (p, q) \text{ is} \\ & \text{a position of the first } \mu_{j_1} \text{ } U'(p, q)\text{'s in } w(U'), \\ 2 & \text{if } k_2 - 1 \leq U'(p, q) \leq j_1 - 1 \text{ and } (p, q) \text{ is a position} \\ & \text{of the first } \mu_{j_2} \text{ } U'(p, q)\text{'s in } w(U'), \\ 1 & \text{if } k_2 - 1 \leq U'(p, q) \leq j_1 - 1 \text{ and } (p, q) \text{ is a position} \\ & \text{of } U'(p, q) \text{ which is between the } (\mu_{j_2} + 1)\text{st} \\ & \text{and the } \mu_{j_1}\text{th } U'(p, q)\text{'s in } w(U'), \\ 1 & \text{if } \max\{j_1 + 1, k_2\} \leq U'(p, q) \leq j_2 - 1 \text{ and } (p, q) \text{ is} \\ & \text{a position of the first } \mu_{j_2} \text{ } U'(p, q)\text{'s in } w(U'), \\ 0 & \text{otherwise.} \end{cases}$$

- Step 5: Add $h'_U(p, q)$ to the (p, q) -box of U' which is not empty.

Step 6: Place k_1 on upper empty boxes, $(k_2 + b - \alpha - 1)$ on leftmost $(\nu_{k_2} - \lambda_{i_2} - \mu_{j_2})$ lower empty boxes and k_2 on remaining lower empty boxes.

Remarks 2.29. (a) Again, note if $k_2 \geq j_1$, then some cases of (2.28) are void.

(b) The positions of empty boxes to be inserted in Step 1 are secured because λ, μ, ν and I, J, K are given and satisfy the conditions in Proposition 1.7.

(c) Note $c_{\lambda\mu}^\nu > 0$ and $k_2 + n - j_2 \leq i_2$ in Equation (1.9). Since $c_{\lambda\mu}^\nu > 0$, there is an LR-tableau T on the shape ν/λ with content μ . Because of Remark 2.14, $(\nu_{k_2} - \lambda_{i_2} - \mu_{j_2})$ in Step 2 is nonnegative and Step 2 is also secured.

(d) It is clear that (p, q) -box of $\Phi(T)'$ is empty if and only if that of T is in $\mathcal{S}(T)$. Moreover, the relation $h'_{\Phi(T)}(p, q) = -h_T(p, q)$ holds if (p, q) -box is not in $\mathcal{S}(T)$.

(e) For any LR-tableau T on the shape ν/λ with content μ , $\Phi(T)' = T + h_T$ holds.

(f) Finally, note that steps of Definition 2.27 reverse those of Definition 2.23. Thus, examples for Ψ may be obtained from Examples 2.24, 2.25 and 2.26 by reading the steps backwards.

3. Combinatorial proof

In this section, we show that Φ and Ψ are well-defined bijective maps between the set of LR-tableaux of shape ν/λ with content μ and the set of LR-tableaux of shape ν_{K^c}/λ_{J^c} with content μ_{J^c} . First, we show that the map Φ in Definition 2.23 is well defined, i.e., $\Phi(T)$ is an LR-tableau for an LR-tableau T of given condition. With similar arguments, we may show Ψ is well-defined. It is clear that $\Psi \circ \Phi$ and $\Phi \circ \Psi$ are the identity functions, and this will complete the combinatorial proof of the KTT theorem when $r = n - 2$.

We first show that Φ preserves the strict increasingness of columns.

Proposition 3.1. *Assume that λ, μ, ν and $I = \{i_1, i_2\}, J = \{j_1, j_2\}, K = \{k_1, k_2\}$ satisfy the given conditions in Proposition 1.7. Let T be an LR tableau of shape ν/λ and content μ . Then, $\Phi(T)$ is strictly increasing in columns.*

Proof. It is enough to show T° is strictly column increasing when we ignore empty boxes since the process to obtain $\Phi(T)$ from T° keeps each column unchanged when it is considered as an ordered set.

We show that column increasingness is invariant under each action of (2.18), (2.19) and (2.20). Let T' be the filling (with empty boxes) obtained after (2.18) is applied to T and T'' be the filling (with empty boxes) obtained after (2.19) is applied to T' .

We first consider the case when there is one or two empty boxes between two numbers in a column of T' or T'' . Suppose that $T'(p, q)$ is an empty box. We show that the column increasingness is unchanged after we apply (2.19) to T' by comparing $T''(p-1, q)$ and $T''(p+1, q)$. Note $T'(p-1, q) = T(p-1, q) < k_1 \leq T'(p+1, q)$. If $T'(p+1, q) = k_1$, then either $T''(p+1, q)$ is empty (when $k_1 = k_2 + b - \alpha - 1$) or $T''(p+1, q) = T'(p+1, q)$. If $T'(p+1, q) > k_1$, then

either $T''(p+1, q) = T'(p+1, q) - 1$ or $T''(p+1, q) = T'(p+1, q)$. In any case, $T''(p-1, q) = T(p-1, q) < k_1 \leq T''(p+1, q)$ or the $(p+1, q)$ -box of T'' is empty. Almost the same argument works for the case when $T''(p, q)$ is an empty box and we omit the proof of this case.

The second case we consider is when $T'(p, q) + 2 \leq T'(p+1, q)$ or $T''(p, q) + 2 \leq T''(p+1, q)$. The only action that reduces the content by two occurs when we apply (2.20) to T'' , and its action replaces $j_1 + 1$ with $j_1 - 1$. Therefore we only need care about the case $T''(p+1, q) = j_1 + 1$ and $T''(p+1, q)$ is among the first μ_{j_2} $(j_1 + 1)$'s in T'' . If $T''(p, q) < j_1 - 1$, then there is nothing to prove. So, we assume that $T''(p, q) = j_1 - 1$, then $T''(p, q)$ is among the first μ_{j_2} $(j_1 - 1)$'s because of Corollary 2.16 (3) and $T''(p, q) = j_1 - 1$ is replaced with $j_1 - 2$ by (2.20).

We now consider the effect of (2.18), (2.19) and (2.20) on two consecutive integers in adjacent boxes in a column.

Suppose that $s = T(p+1, q) = T(p, q) + 1$ for some p, q , and $T(p+1, q)$ is replaced by $s - 1$ after (2.18) is applied. Then $k_1 + 1 \leq T(p+1, q) \leq k_2 + b - \alpha - 1$ and $T(p+1, q) \leq p + 1 \leq T(p+1, q) + a$ by Corollary 2.16 (1). If $s - 1 = k_1$, then $T(p, q)$ must be replaced by an empty box. We, therefore, may assume that $s - 1 > k_1$. In this case, we can see that $k_1 + 1 \leq T(p, q) \leq k_2 + b - \alpha - 1$ and $T(p, q) \leq p \leq T(p, q) + a$, hence $T(p, q)$ must be replaced by $s - 2$ by Corollary 2.16. Therefore, $T'(p+1, q) > T'(p, q)$.

Suppose that $s = T'(p+1, q) = T'(p, q) + 1$ for some p, q , and $T'(p+1, q)$ is replaced by $s - 1$ after (2.19) is applied. Similar argument as above proves that (2.19) either empties the (p, q) -box of T' or replaces $T'(p, q)$ with $T'(p, q) - 1$.

There are three cases in (2.20). We only consider the third case of (2.20), which is less trivial than other two cases of (2.20). Suppose that $s = T''(p+1, q) = T''(p, q) + 1$ for some p, q , and $T''(p+1, q)$ is replaced by $s - 1$ after (2.20c) is applied. Then, Corollary 2.16 and the effects of the actions (2.18), (2.19) imply either $k_2 \leq T''(p+1, q) \leq j_1 - 1$ and $T''(p+1, q) + 1 \leq p + 1 \leq T''(p+1, q) + 1 + \beta$, or $j_1 + 2 \leq T''(p+1, q) \leq j_2$ and $T''(p+1, q) \leq p + 1 \leq T''(p+1, q) + \beta$. In either case, $T''(p, q) = s - 1$ must be replaced by $s - 2$ because of Corollary 2.16.

While we apply Step 3 and Step 4 of Definition 2.23 to get T° , we replace all s 's with $(s - 1)$'s for $j_1 < s < j_2$, and all t 's with $(t - 2)$'s for $j_2 < t$. Hence, Step 3 and Step 4 do not affect the column increasingness since T'' has neither j_1 nor j_2 . \square

Before we prove weakly increasingness in row of $\Phi(T)$ and the lattice word property of $w(\Phi(T))$, we prove some useful lemmas.

Let B and B' be boxes at the (p, q) -position and at the (p', q') -position, respectively, in a skew diagram. We say that B' is on the *South-west* of B if $p < p'$ and $q \geq q'$. Also B' is on the *south-West* of B if $p \leq p'$ and $q > q'$. Remember that whenever we refer to the *order of a content* in a filling S , we mean the order in the corresponding reverse row word $w(S)$.

Lemma 3.2. *Then, for any s and m , the followings hold:*

- (1) *For an LR-tableau T , the m th $(s + 1)$ is on the South-west of the m th s in T .*
- (2) *For a skew tableau T , the $(m + 1)$ st s is on the south-West of the m th s in T .*

Proof. (1) We use an induction on m for a fixed s . For $m = 1$, it is trivial because T is an LR-tableau.

Suppose that the lemma is true for m : The m th s is in the (p, q) -box of T and the m th $(s + 1)$ is in the (p', q') -box of T , where $p < p'$ and $q \geq q'$. Let the $(m + 1)$ st s be in the (r, l) -box and the $(m + 1)$ st $(s + 1)$ be in the (r', l') -box of T . Note that $p \leq r, q > l, p' \leq r'$ and $q' > l'$ since T is strictly increasing in columns and weakly increasing in rows.

Suppose that $p = r$. Then we have $l = q - 1$. Therefore, we have $l = q - 1 \geq q' - 1 > l' - 1$ and $l \geq l'$. We also have $r = p < p' \leq r'$, and the $(m + 1)$ st $s + 1$ is on the South-west of the $(m + 1)$ st s .

Suppose that $p < r$. Then we have $r < r'$ because T is an LR-tableau. Hence, we only need to show that $l \geq l'$. Assume that $l < l'$ on the contrary. Then $T(r, l') = s$ since $s = T(r, l) \leq T(r, l') < T(r', l') = s + 1$. This gives a contradiction since the s in the (r, l') -box is between two s 's in the (p, q) -box and the (r, l) -box, which are the m th and the $(m + 1)$ st s respectively. Therefore, we have $l \geq l'$.

(2) It is straightforward from weakly increasingness in rows and strictly increasingness in columns of skew tableaux. □

Definition 3.3. Let S be a filling of a skew shape. An **Sw**-route in S is, for some k , a collection of k boxes in S satisfying the following two conditions;

- (R1) there is exactly one box whose entry is s for each $s = 1, 2, \dots, k$,
- (R2) the box whose entry is $s + 1$ is on the South-west of the box whose entry is s for each $s = 1, 2, \dots, k - 1$.

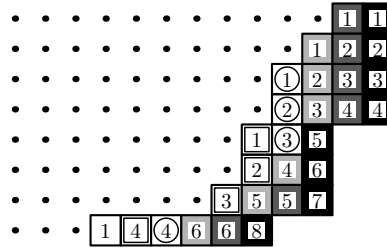
Definition 3.4. Assume T is an LR-tableau of shape ν/λ with content μ . For an integer $m, 1 \leq m \leq \mu_1$, we let $R_T(m)$ be a collection of $(\tilde{\mu})_m$ boxes in T whose entries are the m th s 's in T for $1 \leq s \leq (\tilde{\mu})_m$, i.e.,

$$R_T(m) = \{ ((p, q), T(p, q)) \mid T(p, q) \text{ is the } m\text{th } s \text{ with } 1 \leq s \leq (\tilde{\mu})_m \}.$$

From Lemma 3.2, we get the following lemma.

Lemma 3.5. *For an LR-tableau T of shape ν/λ with content μ , every $R_T(m)$ is an Sw-route in T , and T is a disjoint union of μ_1 Sw-routes.*

Example 3.6. In the following LR-tableau T , there are 6 $R_T(m)$ Sw-routes and T is a disjoint union of them.



For the rest of this section, we assume that λ, μ, ν and $I = \{i_1, i_2\}, J = \{j_1, j_2\}, K = \{k_1, k_2\}$ satisfy the conditions in Proposition 1.7 for a given n , and T is an LR-tableau of the shape ν/λ with content μ .

Because we obtain T° by applying (2.18), (2.19), (2.20), and Step 3 and Step 4 in Definition 2.23 to T in sequence, we have the following lemma.

Lemma 3.7. *Assume the (p, q) -box of T is in $R_T(m)$ for some $m = 1, 2, \dots, \mu_1$. If the (p, q) -box of T° is not empty, then*

$$T(p, q) - T^\circ(p, q) = \begin{cases} 0 & \text{if } m \leq \omega = \nu_{k_1} - \lambda_{i_1} \text{ and } T(p, q) < k_1, \\ 0 & \text{if } \omega < m \leq \mu_{j_1} \text{ and } T(p, q) < k_2 + b - \alpha - 1, \\ 0 & \text{if } \mu_{j_1} < m, \\ 2 & \text{if } m \leq \mu_{j_2} \text{ and } k_2 + 1 \leq T(p, q), \\ 1 & \text{otherwise.} \end{cases}$$

Proof. The proof is straightforward from (2.18), (2.19), (2.20), and Definition 2.23. □

Lemma 3.8. *T° is weakly increasing in rows, if we ignore empty boxes.*

Proof. We compare $T^\circ(p, q)$ with $T^\circ(p, q + 1)$ or with the entry of the first nonempty box, say $T^\circ(p, q')$, on the p th row starting from the $(q + 1)$ st column in T° . Observe that $h_T(p, q)$ in Definition 2.21 depends on only p and $T(p, q)$, and it does not depend on q . Also the effects of Step 3 and Step 4 in Definition 2.23 on the (p, q) -box of F depend only on the value $F(p, q) = T(p, q) + h_T(p, q)$. Therefore, if $T(p, q) = T(p, q + \epsilon)$ with a nonnegative integer ϵ , then $T^\circ(p, q) = T^\circ(p, q + \epsilon)$. Thus, $T(p, q) - T^\circ(p, q) = 0$ and $T(p, q') - T^\circ(p, q') = 1$ imply $T(p, q) < T(p, q')$. Otherwise, $T(p, q) = T(p, q')$ implies $T^\circ(p, q) = T^\circ(p, q')$, and so we get $T(p, q) = T^\circ(p, q) = T^\circ(p, q') = T(p, q') - 1$, a contradiction. Therefore, we have $T^\circ(p, q) = T(p, q) \leq T(p, q') - 1 = T^\circ(p, q')$. Similarly, if $T(p, q) - T^\circ(p, q) = 1$ and $T(p, q') - T^\circ(p, q') = 2$, then $T^\circ(p, q) \leq T^\circ(p, q')$.

Thus, weakly increasingness in rows of T° , that is $T^\circ(p, q) \leq T^\circ(p, q')$, may be broken only if $T(p, q) - T^\circ(p, q) = 0$ and $T(p, q') - T^\circ(p, q') = 2$, and it is enough to consider this case. First assume that the $(p, q + 1)$ -box of T° is not empty, that is $q' = q + 1$. Since $T(p, q + 1) - T^\circ(p, q + 1) = 2$, we have $T(p, q + 1) \geq k_2 + 1$ and the $(p, q + 1)$ -box of T is in $R_T(m)$ for some $m \leq \mu_{j_2}$,

by Lemma 3.7. Moreover, from Proposition 2.8 we have

$$(3.9) \quad T(p, q + 1) \leq p \leq T(p, q + 1) + \beta.$$

Suppose $T(p, q) < k_2$. Then $T^\circ(p, q) = T(p, q) \leq k_2 - 1 \leq T(p, q + 1) - 2 = T^\circ(p, q + 1)$. So the weakly increasingness in the row holds.

Suppose $k_2 \leq T(p, q)$. Then, the (p, q) -box of T is in $R_T(m)$ with $\mu_{j_1} < m$ by Lemma 3.7. Note that μ_{j_1} th $(k_2 + b - \alpha - 1)$ is on the $(\lambda_{i_2} + 1)$ st column and $(\mu_{j_1} + 1)$ st $(k_2 + b - \alpha - 1)$ is below the i_2 th row by Lemma 2.13. Therefore m th $(k_2 + b - \alpha - 1)$ is also below the i_2 th row. Because $R_T(m)$ is an **Sw**-route in T by Lemma 3.5, the row index of m th $T(p, q)$ is larger than or equal to the row index of m th $(k_2 + b - \alpha - 1)$ plus $T(p, q) - (k_2 + b - \alpha - 1)$, i.e., we have

$$(3.10) \quad i_2 + (T(p, q) - (k_2 + b - \alpha - 1)) + 1 \leq p.$$

Equations (3.9) and (3.10) imply that $i_2 + \{T(p, q) - (k_2 + b - \alpha - 1)\} + 1 \leq p \leq T(p, q + 1) + \beta$ or equivalently $(\alpha - \beta) + 2 \leq T(p, q + 1) - T(p, q)$. Moreover $\beta \leq \alpha$ from Equation (2.5). Thus $T^\circ(p, q) = T(p, q) \leq T(p, q + 1) - 2 = T^\circ(p, q + 1)$ and the weakly increasingness in the row of T° holds.

Next assume that the $(p, q + 1)$ -box of T° is empty, that is $q' > q + 1$. We know that if $T(p, q) = T(p, q + 1)$ and the $(p, q + 1)$ -box of T is in $\mathcal{S}(T)$, then so is the (p, q) -box of T . This implies that $T(p, q) < T(p, q + 1) < T(p, q')$. Thus $T^\circ(p, q) = T(p, q) \leq T(p, q') - 2 = T^\circ(p, q')$. Therefore the weakly increasingness in the row of T° holds. \square

From Corollary 2.15, we know that each column of T may contain at most two boxes in $\mathcal{S}(T)$. So through Step 5 and Step 6, $T^\circ(p, q)$ moves up by at most 2 rows. The following lemma tells us how Step 5 and Step 6 move the contents of T° to get $\Phi(T)$.

Lemma 3.11. *If $i_2 < p + 2$, then $\Phi(T)(p, q) = T^\circ(p + 2, q)$. If $p + 2 \leq i_2$, then one of the following cases holds:*

$$\Phi(T)(p, q) = \begin{cases} T^\circ(p, q) & \text{if } T^\circ(p, q) < k_1 \text{ and } p < i_1, \\ T^\circ(p + 1, q) & \text{if } k_1 \leq T^\circ(p + 1, q) < k_2 - 1, \\ & k_1 \leq p + 1 \leq k_2 + \beta \text{ and } \nu_{k_2} - \mu_{j_2} < q \leq \nu_{k_1}, \\ T^\circ(p + 1, q) & \text{if } T^\circ(p + 1, q) < k_2 + b - \alpha - 1, i_1 \leq p + 1 \leq i_2, \\ & \text{and } \lambda_{i_2} < q \leq \nu_{k_2} - \mu_{j_2}, \\ T^\circ(p + 2, q) & \text{if } k_2 - 1 \leq T^\circ(p + 2, q), k_2 \leq p + 2 \leq i_2, \\ & \text{and } \nu_{k_2} - \mu_{j_2} < q \leq \nu_{k_2}, \\ T^\circ(p + 2, q) & \text{if } k_2 + b - \alpha - 1 \leq T^\circ(p + 2, q), \\ & k_2 + \beta \leq p + 2 \leq i_2, \text{ and } \lambda_{i_2} < q \leq \nu_{k_2} - \mu_{j_2}. \end{cases}$$

Proof. To get $\Phi(T)$ from T° , we first slide empty boxes of (S2) or (S3) in Definition 2.12 down to the i_2 th row or to the end of the column, and slide empty boxes of (S1) down to the i_1 th row. Then, we remove the empty boxes

in i_1 th, i_2 th rows and at the end of columns. Thus, boxes of T° that are above empty boxes of (S1) do not move, and by Lemma 2.13, these boxes are above the i_1 th row and their entries are less than k_1 . Therefore, if $T^\circ(p, q) < k_1$ and $p < i_1$, then $\Phi(T)(p, q) = T^\circ(p, q)$.

In each column of T° from the $(\lambda_{i_2} + 1)$ st column to the $(\nu_{k_2} - \mu_{j_2})$ th column, there is exactly one box of (S2) by Lemma 2.13. Note the ω th k_1 is on the $(\lambda_{i_1} + 1)$ st column and $(\omega + 1)$ st $k_2 + b - \alpha - 1$ is on the $(\nu_{k_2} - \mu_{j_2})$ th column by Lemma 2.8 and Lemma 2.13. Then, because of Lemma 3.2, $\nu_{k_2} - \mu_{j_2} < \lambda_{i_1} + 1$, and all boxes in the columns between the $(\lambda_{i_2} + 1)$ st and the $(\nu_{k_2} - \mu_{j_2})$ th are below the $(i_1 - 1)$ st row.

Note also from Lemma 2.13 that boxes of (S2) are placed between $(k_2 + \beta)$ th row and the i_2 th row. Remember that the entries of the boxes of (S2) are all $k_2 + b - \alpha - 1$ in T . Thus, if a box is above a box of (S2) in T° , then it moves up by one row, and if a box is below a box of (S2), then it moves up by two rows. Therefore, in a column between the $(\lambda_{i_2} + 1)$ st and the $(\nu_{k_2} - \mu_{j_2})$ th of T° , we have two cases: If $T^\circ(p + 1, q) < k_2 + b - \alpha - 1$ and $i_1 \leq p + 1 \leq i_2$, then $\Phi(T)(p, q) = T^\circ(p + 1, q)$. If $T^\circ(p + 2, q) \geq k_2 + b - \alpha - 1$ and $k_2 + \beta \leq p + 2 \leq i_2$, then $\Phi(T)(p, q) = T^\circ(p + 2, q)$.

We now consider columns of T° from the $(\nu_{k_2} - \mu_{j_2} + 1)$ st to the end. Similarly as above, a box in the column which is between a box of (S1) and a box of (S2) in the same column will move up by one row, and a box which is below a box of (S2) will move up by two rows. Thus, if $k_1 \leq T^\circ(p + 1, q) < k_2 - 1$ and $k_1 \leq p + 1 \leq k_2 + \beta$, then $\Phi(T)(p, q) = T^\circ(p + 1, q)$, and if $k_2 - 1 \leq T^\circ(p + 2, q)$ and $k_2 \leq p + 2 \leq i_2$, then $\Phi(T)(p, q) = T^\circ(p + 2, q)$. \square

We now prove the row increasingness of $\Phi(T)$ by means of Lemma 3.8 and Lemma 3.11.

Proposition 3.12. $\Phi(T)$ is weakly increasing in rows.

Proof. Note T° is weakly increasing in rows if we ignore empty boxes. Also Step 5 and Step 6 in Definition 2.23 only affect the row increasingness in the rows from the k_1 th row to i_2 th row of T° . Thus, it is enough to compare $\Phi(T)(p, q)$ and $\Phi(T)(p, q + 1)$ for $k_1 \leq p \leq i_2 - 2$. By Corollary 2.15, we consider the following nine cases:

- Case 1) $\Phi(T)(p, q) = T^\circ(p, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p, q + 1)$.
- Case 2) $\Phi(T)(p, q) = T^\circ(p + 1, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p + 1, q + 1)$.
- Case 3) $\Phi(T)(p, q) = T^\circ(p + 2, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p + 2, q + 1)$.
- Case 4) $\Phi(T)(p, q) = T^\circ(p, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p + 1, q + 1)$.
- Case 5) $\Phi(T)(p, q) = T^\circ(p + 1, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p + 2, q + 1)$.
- Case 6) $\Phi(T)(p, q) = T^\circ(p, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p + 2, q + 1)$.
- Case 7) $\Phi(T)(p, q) = T^\circ(p + 1, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p, q + 1)$.
- Case 8) $\Phi(T)(p, q) = T^\circ(p + 2, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p, q + 1)$.
- Case 9) $\Phi(T)(p, q) = T^\circ(p + 2, q)$ and $\Phi(T)(p, q + 1) = T^\circ(p + 1, q + 1)$.

In Case 1), Case 2), and Case 3), $\Phi(T)(p, q) \leq \Phi(T)(p, q + 1)$ holds because of Lemma 3.8.

In Case 4), by Lemma 3.11, $T^\circ(p, q) < k_1$ and $p < i_1$ because $\Phi(T)(p, q) = T^\circ(p, q)$. Then, by Lemma 3.11 again, $p < i_1$ imply $k_1 \leq T^\circ(p, q + 1) < k_2 - 1$ because $\Phi(T)(p, q + 1) = T^\circ(p + 1, q + 1)$. Thus, $\Phi(T)(p, q) = T^\circ(p, q) < k_1 \leq T^\circ(p, q + 1) < T^\circ(p + 1, q + 1) = \Phi(T)(p, q + 1)$ holds in Case 4).

Next, we consider Case 5). For $q < \nu_{k_2} - \mu_{j_2}$, by Lemma 3.11, $\Phi(T)(p, q) = T^\circ(p + 1, q) < k_2 + b - \alpha - 1 \leq T(p + 2, q + 1) = \Phi(T)(p, q + 1)$. For $q = \nu_{k_2} - \mu_{j_2}$, by Lemma 3.11 again, $\Phi(T)(p, q) = T^\circ(p + 1, q) < k_2 + b - \alpha - 1 < k_2 - 1 \leq T^\circ(p + 2, q + 1) = \Phi(T)(p, q + 1)$ because $b - \alpha \leq 0$. For $q > \nu_{k_2} - \mu_{j_2}$, $\Phi(T)(p, q) = T^\circ(p + 1, q) < k_2 - 1 \leq T^\circ(p + 2, q + 1) = \Phi(T)(p, q + 1)$.

Next, we consider Case 6). By Lemma 3.11, $T^\circ(p, q) < k_1$ and $k_2 - 1 \leq T^\circ(p + 2, q + 1)$ or $k_2 + b - \alpha - 1 \leq T^\circ(p + 2, q + 1)$. In any case, $\Phi(T)(p, q) = T^\circ(p, q) \leq T^\circ(p + 2, q + 1) = \Phi(T)(p, q + 1)$ holds because $k_1 \leq k_2 + b - \alpha - 1 \leq k_2$.

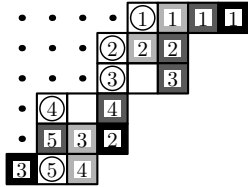
Next, we consider the three cases left. Lemma 2.13 and Lemma 3.2 imply that if (r, ℓ) -box and $(r', \ell + 1)$ -box of T are boxes of (S1) in Definition 2.12, then $r \geq r'$. Also, if (r, ℓ) -box and $(r', \ell + 1)$ -box of T are boxes of (S2) or (S3), then $r \geq r'$. Thus, the three cases left do not happen. \square

Finally, we show that $\Phi(T)$ is an LR-tableau, i.e., the reverse row word $w(\Phi(T))$ is a lattice word.

Lemma 3.13. *Let S be a filling of skew shape which is a disjoint union of Sw-routes. Then $w(S)$ is a lattice word.*

Proof. Take a Sw-route in S and consider $s + 1$ and s in the Sw-route. Because $s + 1$ is on the South-west of s , $s + 1$ will be read later than s in $w(S)$. That is true for all Sw-route in S , and $w(S)$ is a lattice word. \square

Example 3.14. The following filling of skew shape (with empty boxes) is a disjoint union of 4 Sw-routes and its reverse row word is a lattice word, even though it is not a tableau.



Proposition 3.15. *The reverse row word $w(\Phi(T))$ is a lattice word.*

Proof. Due to Corollary 3.5, we can decompose T as a disjoint union of $R_T(m)$ for $m = 1, \dots, \mu_1$. Note we apply Step 1 through Step 4 of Definition 2.23 on T to get T° . For each $m = 1, 2, \dots, \mu_{j_1}$, we let $R_T(m)^\circ$ be a filling (with empty boxes) obtained by applying a restriction of these actions on $R_T(m)$. Lemma 3.7 implies that $R_T(m)^\circ$ is an Sw-route if we ignore empty boxes. It is

clear that T° is a disjoint union of $R_T(m)^\circ$'s, and $w(T^\circ)$ is a lattice word by Lemma 3.13.

Next we apply Step 5 and Step 6 in the Definition 2.23 on T° . For each $m = 1, 2, \dots, \mu_{j_1}$, we let $R_T(m)'$ be a filling obtained by applying a restriction of these actions on $R_T(m)^\circ$. Note that these actions move up boxes in a column or delete empty boxes. Therefore, if a box at (p, q) -position has ℓ and a box at (p', q') -position has $\ell + 1$ in $R_T(m)'$, then $q \geq q'$. Hence, to show $R_T(m)'$ is an Sw-route, it is enough to show that $p < p'$.

However, $p \geq p'$ together with $q \geq q'$ will break the weakly increasingness in rows, which is contradict to Proposition 3.12. Therefore, each $R_T(m)'$ is an Sw-route in $\Phi(T)$, even though $R_T(m)'$ is not $R_{\Phi(T)}(m)$, a collection of the boxes in $\Phi(T)$ containing the m th ℓ 's in $w(\Phi(T))$. Now the proposition comes from Lemma 3.13. \square

Appendix

In this appendix, we use a known bijection between LR-tableaux and LR-hives (see [14]) to translate the main idea of our proof into the language of hives.

An n -hive is a graph of equilateral triangular shape with labeled edges as shown in Figure 2. For partitions λ, μ and ν of lengths at most n , an *LR-hive of type* (λ, μ, ν) is an n -hive with nonnegative integer (edge) labels and the boundary labels are determined by λ, μ, ν as shown in Figure 3, satisfying the following condition; for each rhombus in Figure 4, $\mathfrak{s} \geq \mathfrak{t}, \mathfrak{u} \geq \mathfrak{v}$, and in each triangle the sum of two values on oblique sides is same as the value on the horizontal side, and which imply that $\mathfrak{s} + \mathfrak{v} = \mathfrak{u} + \mathfrak{t}$. We use notations $\mathfrak{a}_{ij}(H)$, $\mathfrak{b}_{ij}(H)$ and $\mathfrak{c}_{ij}(H)$ for edge labels of a hive H as shown in Figure 2.

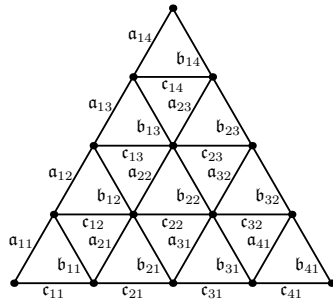


Figure 2: 4-hive

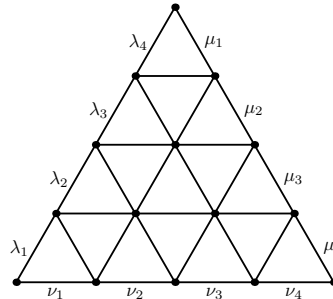


Figure 3: LR-hive

Proposition 4.1 (See [1]). *For given partitions λ, μ, ν , Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu$ is the number of LR-hives of type (λ, μ, ν) .*

A bijection between LR-tableaux and LR-hives is outlined in [14] (see also [1, Appendix A]). Under this bijection, for an LR-hive H corresponding to an

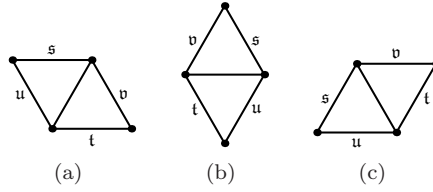


Figure 4: rhombus conditions

LR-tableau T on the shape ν/λ , we have

$$\begin{aligned} \mathbf{a}_{pq}(H) &= \sum_{\ell=0}^{p-1} \mathbf{n}_T^{p+q-1}(\ell), \\ \mathbf{b}_{pq}(H) &= \sum_{h=1}^{p+q-1} \mathbf{n}_T^h(p) = \sum_{h=p}^{p+q-1} \mathbf{n}_T^h(p), \\ \mathbf{c}_{pq}(H) &= \mathbf{a}_{pq}(H) + \mathbf{b}_{pq}(H), \end{aligned}$$

where $\mathbf{n}_T^h(0)$ is set to λ_h . With these observations, Equations (2.9), (2.10) and (2.11) of Proposition 2.8, which are crucial ingredients for our bijection, are translated into the following conditions:

$$(4.2) \quad \mathbf{b}_{\ell \beta+1} = \mu_{j_2} \quad \text{for } \ell = k_2, k_2 + 1, \dots, j_2,$$

$$(4.3) \quad \mathbf{b}_{\ell \alpha+2} = \mu_{j_1} \quad \text{for } \ell = k_2 + b - \alpha - 1, k_2 + b - \alpha, \dots, j_1,$$

$$(4.4) \quad \mathbf{b}_{\ell \alpha+1} = \omega = \nu_{k_1} - \lambda_{i_1} \quad \text{for } \ell = k_1, k_1 + 1, \dots, k_2 + b - \alpha - 1.$$

Note that these equalities say that some consecutive edge labels are constant across rhombi which are shown as in Figure 4 (a). In the proof of the KTT theorem [15], King, Tollu, and Toumazet show that, in some consecutive rhombi which are making corridors, opposite edge labels are constant along corridors; $\mathbf{s} = \mathbf{t}$ and $\mathbf{u} = \mathbf{v}$ hold. In fact, $\mathbf{b}_{\ell \beta+1}$'s in (4.2), $\mathbf{b}_{\ell \alpha+2}$'s in (4.3) and $\mathbf{b}_{\ell \alpha+1}$'s in (4.4) form a part of edges of these corridors, respectively. Basically the bijection in [15] is obtained by deleting these corridors and rearranging the rest part of the hive to create a smaller LR-hive (in the case of reduction formulae).

Acknowledgement. This research was done while authors were visiting Korea Institute for Advanced Study. The authors would like to thank Young-Tak Oh for introducing [14] when it was in preprint form. The authors are also grateful to R.C. King for sending the proof of the factorization theorem [15] prior to publication.

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