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Fuzzy r-minimal Preopen Sets And Fuzzy r-*M* Precontinuous Mappings On Fuzzy Minimal Spaces

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요 약

fuzzy r-minimal preopen set, fuzzy r-M precontinuous와 fuzzy r-M preopen 함수의 개념을 소개하며 특성들을 조사한다.

Abstract

We introduce the concept of fuzzy r-minimal preopen set on a fuzzy minimal space. We also introduce the concept of fuzzy r-M precontinuous mapping which is a generalization of fuzzy r-M continuous mapping, and investigate characterization of fuzzy r-M precontinuity.

Key Words: fuzzy minimal structures, r-minimal open, r-minimal preopen, fuzzy r-M continuous, fuzzy r-M precontinuous

1. Intorduction

The concept of fuzzy set was introduced by Zadeh [5]. Chang [1] defined fuzzy topological spaces using fuzzy sets. In [2], Chattopadhyay, Hazra and Samanta introduced the smooth topological space which is a generalization of a fuzzy topological space. In [4], we introduced the concept of fuzzy r-minimal space which is an extension of the smooth topological space. The concepts of fuzzy r-open sets and fuzzy r-M continuous mappings are also introduced and studied.

In this paper, we introduce the concept of fuzzy r-minimal preopen sets and study some related properties. We also introduce the concept of fuzzy r-M precontinuous mappings, which is a generalization of fuzzy r-M continuous mapping. In particular, we investigate some characterization for the fuzzy r-M precontinuous mapping in terms of fuzzy r-minimal interior and fuzzy r-minimal closure operators.

2. Preliminaries

Let I be the unit interval $[0,1]$ of the real line. A member A of I^X is called a *fuzzy set* [5] of X. By $\tilde{0}$ and $\tilde{1}$, we denote constant maps on X with value 0 and 1, respectively.

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For any $A \in I^X$, A^c denotes the complement $\tilde{1}-A$. All other notations are standard notations of fuzzy set theory.

A *fuzzy point* x_α in X is a fuzzy set x_α is defined as follows

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

A fuzzy point x_α is said to belong to a fuzzy set A in X, denoted by $x_\alpha \in A$, if $\alpha \leq A(x)$ for $x \in X$.

A fuzzy set A in X is the union of all fuzzy points which belong to A.

Let $f: X \to Y$ be a mapping and $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} A(z)_{z \in f^{-1}(y)}, & \text{if } f^{-1}(y) \neq \varnothing, \\ 0, & \text{otherwise}, \end{cases}$$

for $y \in Y$ and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x))$, $x \in X$.

A *smooth topology* [2, 3] on X is a map $T: I^X \to I$ which satisfies the following properties:

(1) $T(\tilde{0}) = T(\tilde{1}) = 1$.

(2) $T(A_1 \wedge A_2) \geq T(A_1) \wedge T(A_2)$ for $A_1, A_2 \in I^X$.

(3) $T(\vee A_i) \geq \wedge T(A_i)$ for $A_i \in I^X$.

The pair (X, T) is alled a *smooth topological space*.

Let X be a nonempty set and $r\in(0,1]=I_0$. A fuzzy family $M:I^X\to I$ on X is said to have a *fuzzy r-minimal structure* [4] if the family

$$M_r=\{A\in I^X : M(A)\ge r\}$$

contains $\tilde{0}$ and $\tilde{1}$.

Then the (X,M) is called a *fuzzy r-minimal space* (simply r-FMS) [4]. Every member of M_r is called a *fuzzy r-minimal open set*. A fuzzy set A is called a *fuzzy r-minimal closed set* if the complement of A (simply, A^c) is a fuzzy r-minimal open set.

Let (X,M) be an r-FMS and $r\in(0,1]=I_0$. The fuzzy r-minimal closure of A, denoted by mC(A,r), is defined as

$$\text{mC}(A,r)=\cap\{B\in I^X: B^c\in M_r \text{ and } A\subseteq B\}[4].$$

The fuzzy r-minimal interior of A, denoted by mI(A,r), is defined as

$$\text{mI}(A,r)=\cup\{B\in I^X: B\in M_r \text{ and } B\subseteq A\}[4].$$

Theorem 2.1 ([4]). Let (X,M) be an r-FMS and A, $B\in I^X$. Then the following properties hold:

(1) mI$(A,r)\subseteq A$ and if A is a fuzzy r-minimal open set, then mI$(A,r)=A$.
(2) $A\subseteq$ mC(A,r) and if A is a fuzzy r-minimal closed set, then mC$(A,r)=A$.
(3) If $A\subseteq B$, then mI$(A,r)\subseteq$ mI(B,r) and mC$(A,r)\subseteq$ mC(B,r).
(4) mI$(A\cap B,r)\subseteq$ mI$(A,r)\cap$ mI(B,r) and mC$(A,r)\cup$ mC$(B,r)\subseteq$ mC$(A\cup B,r)$.
(5) mI(mI$(A,r),r)$=mI(A,r) and mC(mC$(A,r),r)$= mC(A,r).
(6) $\tilde{1}$-mC(A,r)=mI$(\tilde{1}-A,r)$ and $\tilde{1}$-mI(A,r)=mC$(\tilde{1}-A,r)$.

Definition 2.2 ([4]). Let (X,M) and (Y,N) be r-FMS's. Then $f:X\to Y$ is said to be *fuzzy r-M continuous* function if for every $A\in N_r$, $f^{-1}(A)$ is in M_r.

Theorem 2.3 ([4]). Let $f:X\to Y$ be a function on r-FMS's (X,M) and (Y,N).
(1) f is fuzzy r-M continuous.
(2) $f^{-1}(B)$ is a fuzzy r-minimal closed set, for each fuzzy r-minimal closed set B in Y.
(3) f(mC$(A,r))\subseteq$ mC$(f(A),r)$ for $A\in I^X$.
(4) mC$(f^{-1}(B),r)\subseteq f^{-1}$(mC$(B,r))$ for $B\in I^Y$.
(5) f^{-1}(mI$(B,r))\subseteq$ mI$(f^{-1}(B),r)$ for $B\in I^Y$.

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

3. Fuzzy r-minmal preopen sets and fuzzy r-M precontinuity

Definition 3.1. Let (X,M) be an r-FMS and $A\in I^X$. Then a fuzzy set A is called a *fuzzy r-minimal preopen set* in X if $A\subseteq$ mI(mC$(A,r),r)$.

A fuzzy set A is called a *fuzzy r-minimal spre-closed* set if the complement of A is fuzzy r-minimal preopen.

Every fuzzy r-minimal open set is fuzzy r-minimal preopen but the converse may not be true in general.

Example 3.2 Let $X=[0,1]$, let A, B and C be fuzzy sets defined as follows

$$A(x)=\frac{1}{2}(x+1),\quad x\in I;$$

$$B(x)=-\frac{3}{4}(x-1),\quad x\in I;$$

$$C(x)=\frac{1}{5}(x+4),\quad x\in I.$$

Let us consider a fuzzy minimal structure M as follows

$$M(\mu)=\begin{cases}\frac{2}{3}, & \text{if } \mu=\tilde{0},\ \tilde{1},\ A,\ B,\\ 0, & otherwise\ .\end{cases}$$

Then the fuzzy set C is fuzzy $\frac{2}{3}$-minimal preopen but not fuzzy $\frac{2}{3}$-minimal open.

Lemma 3.3. Let (X,M) be an r-FMS and $A\in I^X$. Then a fuzzy set A is fuzzy r-minimal preclosed set if and only if mC(mI$(A,r),r)\subseteq A$.

Theorem 3.4. Let (X,M) be an r-FMS. Any union of fuzzy r-minimal preopen sets is fuzzy r-minimal preopen.

Proof. Let A_i be a fuzzy r-minimal preopen set for $i\in J$. Then from Theorem 2.1, $A_i\subseteq$ mI(mC$(A_i,r),r)\subseteq$ mI(mC$(\cup A_i,r),r)$. This implies $\cup A_i\subseteq$ mI(mC$(\cup A_i,r),r)$ and so $\cup A_i$ is fuzzy r-minimal preopen.

As shown in the next example, in general, the intersection of two fuzzy r-minimal preopen sets may not be fuzzy r-minimal preopen.

Example 3.5. Let $X=[0,1]$, let C, D, E and F, be fuzzy sets defined as follows

$$C(x)=\frac{1}{3}x+\frac{1}{2},\quad \text{if } x\in I;$$

$$D(x)=-\frac{1}{3}(x-1)+\frac{1}{2}, \qquad \text{if } x\in I;$$

$$E(x)=\begin{cases} -x+\frac{1}{2}, & \text{if } 0\le x\le \frac{1}{4},\\ \frac{1}{3}(x-1)+\frac{1}{2}, & \text{if } \frac{1}{4}\le x\le 1;\end{cases}$$

$$F(x)=\begin{cases} \frac{1}{3}x+\frac{1}{2}, & \text{if } 0\le x\le \frac{3}{4},\\ -x+\frac{3}{2}, & \text{if } \frac{3}{4}\le x\le 1.\end{cases}$$

Let us consider a fuzzy minimal structure

$$N(\mu)=\begin{cases} \frac{2}{3}, & \text{if } \mu=\tilde{0},\ \tilde{1},\ E,F,\\ 0, & otherwise\ .\end{cases}$$

Then the fuzzy sets C and D are fuzzy $\frac{2}{3}$-minimal preopen. But $C\cap D$ is not fuzzy $\frac{2}{3}$-minimal preopen, because of mI(mC($C\cap D,\frac{2}{3}$),$\frac{2}{3}$)=$E\cup F\subseteq C,\ D$ but $E\cup F\neq C,\ D$.

Definition 3.6. Let (X,M) be an r-FMS and $A\in I^X$, mpC(A,r) and mpI(A,r), respectively, are defined as the following:

mpC$(A,r)=\cap\{F\in I^X$: $A\subseteq F$ and F is fuzzy r-minimal preclosed$\}$,

mpI$(A,r)=\cup\{U\in I^X$: $U\subseteq A$ and U is fuzzy r-minimal preopen$\}$.

Theorem 3.7. Let (X,M) be an r-FMS and $A\in I^X$. Then

(1) mpI$(A,r)\subseteq A$.
(2) If $A\subseteq B$, then mpI$(A,r)\subseteq$mpI(B,r).
(3) A is fuzzy r-minimal preopen if and only if mpI$(A,r)=A$.
(4) mpI(mpI$(A,r),r$)=mpI(A,r).
(5) mpC$(\tilde{1}-A,r)=\tilde{1}-mpI(A,r)$ and mpI$(\tilde{1}-A,r)=\tilde{1}-mpC(A,r)$.

Proof. (1), (2), (3) and (4) are obviously obtained from Theorem 3.4.

(5) For $A\in I^X$,

$\tilde{1}-$mpI(A,r)
$=\tilde{1}-\cup\{U\in I^X$: $U\subseteq A$ and U is fuzzy r-minimal preopen$\}$
$=\cap\{\tilde{1}-U$: $U\subseteq A$, U is fuzzy r-minimal preopen$\}$
$=\cap\{\tilde{1}-U$: $\tilde{1}-A\subseteq\tilde{1}-U$, U is fuzzy r-minimal preopen$\}$
$=$ mpC$(\tilde{1}-A,r)$.

Similarly, mpI$(\tilde{1}-A,r)=\tilde{1}-mpC(A,r)$.

Theorem 3.8. Let (X,M) be an r-FMS and $A\in I^X$. Then

(1) $A\subseteq$mpC(A,r).
(2) If $A\subseteq B$, then mpC$(A,r)\subseteq$mpC(B,r).
(3) F is fuzzy r-minimal preclosed if and only if mpC$(F)=F$.
(4) mpC(mpC$(A,r),r$)=mpC(A,r).

Proof. It is obvious from Theorem 3.7.

Lemma 3.9. Let (X,M) be an r-FMS and $A\in I^X$. Then

(1) $x_\alpha\in$mpC(A,r) if and only if $A\cap V\neq\varnothing$ for every fuzzy r-minimal preopen set V containing x_α.
(2) $x_\alpha\in$mpI(A,r) if and only if there exists a fuzzy r-minimal preopen set G such that $G\subseteq A$.

Proof. (1) If there is a fuzzy r-minimal preopen set V containing x_α such that $A\cap V=\varnothing$, then $\tilde{1}-V$ is a fuzzy r-minimal preclosed set such that $A\subseteq\tilde{1}-V$, $x_\alpha\not\in\tilde{1}-V$. So $x_\alpha\not\in$mpC(A,r).

The other relation is obvious.

(2) Obvious.

Definition 3.10. Let (X,M) and (Y,N) be r-FMS's. Then $f:X\to Y$ is said to be *fuzzy r-M preicontinuous* if for each fuzzy point x_α and each fuzzy r-minimal open set V containing $f(x_\alpha)$, there exists a fuzzy r-minimal preopen set U containing x_α such that $f(U)\subseteq V$.

Every fuzzy r-M continuous mapping is fuzzy r-M precontinuous but the converse is not true in general.

Example 3.11. Let X=[0,1]. Consider two fuzzy minimal structures (X,M) and (X,N) defined in Example 3.2 and Example 3.5, respectively. Then the identity mapping $f:(X,N)\to(X,M)$ is fuzzy $\frac{2}{3}$-M precontinuous but not fuzzy $\frac{2}{3}$-M continuous.

Theorem 3.12. Let $f:X\to Y$ be a mapping on fuzzy r-minimal spaces (X,M) and (Y,N). Then the following are equivalent:

(1) f is fuzzy r-M precontinuous.
(2) $f^{-1}(V)$ is a fuzzy r-minimal precopen set for each fuzzy r-minimal open set V in Y .

(3) $f^{-1}(B)$ is a fuzzy r-minimal precclosed set for each fuzzy r-minimal closed set B in Y.
(4) $f(\mathrm{mpC}(A,r)) \subseteq \mathrm{mC}(f(A),r)$ for $A \in I^X$.
(5) $\mathrm{mpC}(f^{-1}(B),r) \subseteq f^{-1}(\mathrm{mC}(B,r))$ for $B \in I^Y$.
(6) $f^{-1}(\mathrm{mI}(B,r)) \subseteq \mathrm{mpI}(f^{-1}(B),r)$ for $B \in I^Y$.

Proof. (1) ⇒ (2) Let V be any fuzzy r-minimal open set in Y and $x_\alpha \in f^{-1}(V)$. Then there exists a fuzzy r-minimal precopen set U containing x_α such that $f(U) \subseteq V$ and so $U \subseteq f^{-1}(V)$ for all $x_\alpha \in f^{-1}(V)$. Hence from Theorem 3.4, $f^{-1}(V)$ is fuzzy r-minimal precopen.

(2) ⇒ (3) Obvious.
(3) ⇒ (4) For $A \in I^X$,
$f^{-1}(\mathrm{mC}(f(A,r)))$
$=f^{-1}(\cap\{F \in I^Y\colon f(A) \subseteq F$ and F is a fuzzy r-minimal closed set$\})$
$=\cap\{f^{-1}(F) \in I^X\colon A \subseteq f^{-1}(F)$ and F is a fuzzy r-minimal precclosed set$\}$
$\supseteq \cap\{K \in I^X\colon A \subseteq K$ and K is a fuzzy r-minimal precclosed set$\}$
$=\mathrm{mpC}(A,r)$

Therefore, $f(\mathrm{mpC}(A,r)) \subseteq \mathrm{mC}(f(A),r)$.

(4) ⇒ (5) For $B \in I^Y$,
$f(\mathrm{mpC}(f^{-1}(B),r)) \subseteq \mathrm{mC}(f(f^{-1}(B)),r) \subseteq \mathrm{mC}(B,r)$.
Thus this implies $\mathrm{mpC}(f^{-1}(B),r) \subseteq f^{-1}(\mathrm{mC}(B,r))$.

(5) ⇒ (6) For $B \in I^Y$, from Theorem 3.7,
$f^{-1}(\mathrm{mI}(B,r))=f^{-1}(\tilde{1}-\mathrm{mC}(\tilde{1}-B,r))$
$=\tilde{1}-(f^{-1}(\mathrm{mC}(\tilde{1}-B),r))$
$\subseteq \tilde{1}-\mathrm{mpC}(f^{-1}(\tilde{1}-B))$
$=\mathrm{mpI}(f^{-1}(B),r)$.
Hence $f^{-1}(\mathrm{mI}(B,r)) \subseteq \mathrm{mpI}(f^{-1}(B),r)$.

(6) ⇒ (1) Let V be any fuzzy r-minimal open set containing $f(x_\alpha)$ for a fuzzy point x_α. From (6), it follows

$$x_\alpha \in f^{-1}(V)=f^{-1}(\mathrm{mI}(V,r)) \subseteq \mathrm{mpI}(f^{-1}(V),r).$$

Since $x_\alpha \in \mathrm{mpI}(f^{-1}(V),r)$, there exists a fuzzy r-minimal precopen set U containing x_α such that $U \subseteq f^{-1}(V)$. Hence f is fuzzy r-M preccontinuous.

Lemma 3.13. Let (X,M) be an r-FMS and $A \in I^X$. Then
(1) $\mathrm{mC}(\mathrm{mI}(A,r),r) \subseteq \mathrm{mC}(\mathrm{mI}(\mathrm{mpC}(A,r),r),r) \subseteq \mathrm{mpC}(A,r)$.
(2) $\mathrm{mpI}(A,r) \subseteq \mathrm{mI}(\mathrm{mC}(\mathrm{mpI}(A,r),r),r) \subseteq \mathrm{mI}(\mathrm{mC}(A,r),r)$.

Proof. (1) Since $A \subseteq \mathrm{mpC}(A,r)$ and $\mathrm{mpC}(A,r)$ is fuzzy r-minimal preclosed, it is obtained from Lemma 3.3 and Theorem 3.8.

(2) Similarly, it follows from Theorem 3.7.

Theorem 3.14. Let $f: X \to Y$ be a mapping on r-FMS's (X,M) and (Y,N). Then the following statements are equivalent:
(1) f is fuzzy r-M precontinuous.
(2) $f^{-1}(V) \subseteq \mathrm{mI}(\mathrm{mC}(f^{-1}(V),r),r)$ for each fuzzy r-minimal open set V in Y.
(3) $\mathrm{mC}(\mathrm{mI}(f^{-1}(F),r),r) \subseteq f^{-1}(F)$ for each fuzzy r-minimal closed set F in Y.
(4) $f(\mathrm{mC}(\mathrm{mI}(A,r),r)) \subseteq \mathrm{mC}(f(A),r)$ for $A \in I^X$.
(5) $\mathrm{mC}(\mathrm{mI}(f^{-1}(B),r),r) \subseteq f^{-1}(\mathrm{mC}(B,\mathrm{r}))$ for $B \in I^Y$.
(6) $f^{-1}(\mathrm{mI}(B,r)) \subseteq \mathrm{mI}(\mathrm{mC}(f^{-1}(B),r),r)$ for $B \in I^Y$.

Proof. (1) ⇔ (2) It is obtained from definition of fuzzy r-minimal preopen sets and Theorem 3.12.

(1) ⇔ (3) It follows from Lemma 3.3 and Theorem 3.12.
(3) ⇒ (4) Let $A \in I^X$. Then from Lemma 3.13(1) and hypothesis, it follows $\mathrm{mC}(\mathrm{mI}(A,r),r) \subseteq \mathrm{mpC}(A,r) \subseteq f^{-1}(f(\mathrm{mpC}(A,r))) \subseteq f^{-1}(\mathrm{mC}(f(A),r))$.

Hence $f(\mathrm{mC}(\mathrm{mI}(A,r),r)) \subseteq \mathrm{mC}(f(A),r)$.

(4) ⇒ (5) Obvious.
(5) ⇒ (6) Let $B \in I^Y$. Then from Theorem 2.1,
$f^{-1}(\mathrm{mI}(B,r))=f^{-1}(\tilde{1}-\mathrm{mC}(\tilde{1}-B,r))$
$=\tilde{1}-(f^{-1}(\mathrm{mC}(\tilde{1}-B,r)))$
$\subseteq \tilde{1}-\mathrm{mC}(\mathrm{mI}(f^{-1}(\tilde{1}-B),r),r)$
$=\mathrm{mI}(\mathrm{mC}(f^{-1}(B),r),r)$.
Hence (6) is obtained.

(6) ⇒ (1) Let V be any fuzzy r-minimal open set. Then since $V=\mathrm{mI}(V,r)$, by (6), we have $f^{-1}(V)=f^{-1}(\mathrm{mI}(V,r)) \subseteq \mathrm{mI}(\mathrm{mC}(f^{-1}(V),r),r)$. By Theorem 3.7 (3), $f^{-1}(V)$ is fuzzy r-minimal preopen and so f is fuzzy r-M precontinuous.

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