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가격 추정이 계약 과정에 미치는 영향

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Price-Estimating and Contracting Process

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구매 계약과 구매에 직접적인 영향을 미치는 상품 가격에 반응하는 소비자의 행동은 꾸준한 주목을 받아왔다. 기존의 많은 연구에서 소비자는 상품을 사거나 계약을 할 때 과거 있었던 가격 또는 변할 수 있는 미래 가격을 고려하지 않고 현재 주어진 가격에만 반응을 보인다는 가정을 하였다. 또 다른 연구에서는 소비자는 미래 있을 가격이 어떠한 확률 분포를 따르는 지를 정확히 알고 반응을 보인다고 조금은 과장된 가정을 하였다. 하지만 최근의 연구에서는 소비자가 상품 판매자의 가격 결정 정책을 고려하면서 상품 구매 또는 구매 계약을 한다는 즉 가격에 민감한 반응을 보인다는 가정을 하게 되었다. 이 연구에서는 소비자가 상품 구매나 구매 계약 시 필요한 정보들 중 가장 중요한 상품의 가격을 어떻게 얻어서 이용하는지 분석해 보고자 한다. 즉 상품의 미래 가격 변화를 소비자가 어떻게 추정하여 구매 계약과 구매 시 이용 하는지를 분석하고자 한다. 이 연구에서는 소비자가 상품 구매 계약 시 가격을 추정하는 방법으로 과거의 경험에서 얻어진 가격들을 이용한다는 가정하의 수학적 모형을 세워 소비자의 가격 추정이 실제 상품 구매와 계약에 미치는 영향 그리고 판매자의 가격 결정에 미치는 영향을 분석해 보았다.

Keywords : Stochastic Approximation, Long-Term Contract, Price-Estimation, Strategic-Buyer

1. Introduction

In this paper, a long-term contract is modeled in a form of option : the buyer can obtain goods from the seller through two purchasing channels. The first channel is the long-tern contract of option. Given option price and contract price (called as strike price) provided by the seller, the buyer buys Q number of options and then on the day when the option is expired the buyer has right to purchase Q number of goods at the strike price even if the price in the market is higher that strike price. We call this market on the day as spot market and the price in the spot market as spot price. Applications where such a setting is typical are markets for freight transportation services, in which carriers (seller) and customers (buyers) can enter into long-term contracts that specify prices before the exact demand for the freight transportation is known, and in which customers (buyers) can also purchase transportation services when the exact demand for freight transportation is known. Typically, it is assumed that the transactions of option does not affect the price of the underlying products. However, in this paper, we consider the model where option contract do affect the price of the product. We call this as endogenous property and it can be explained in the following sense : the spot price on the day depends on the number of options sold to the buyer, that is, the pricing decision of the seller is affected by the purchasing decision

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of the buyer. Moreover, to decide how many options to purchase, the buyer needs to estimate the spot price on the expiration day of option. In most literatures, the price in the future market is assumed to follow a known exogenously given probability distribution and the buyer assumes to know its distribution. However, the former assumption is imaginary and the latter one overestimates the buyer. So, we need more reasonable and practical assumption for the buyer's price-estimating method for the spot price.

A classical model in economic literatures where prices might be subject to periodic fluctuations is Cobweb Model. [11] describes it as cyclical supply and demand in a market where the amount of production must be chosen before prices are observed. Producer's expectations about prices are assumed to be based on observations of previous prices. The main outcome from Cobweb model is that price converges to the equilibrium price if the slope of supply curve is greater than the slope of demand curve. It does not address any contract between buyer and seller before the market opens but gives a very simple model regarding price learning process from previous market.

To see what kind of information the buyer obtains and how the buyer processes the information in order to estimate the spot price, the following assumption is made on the buyer's estimating behavior : There are multiple periods in each of which there are two purchasing channels for buyers, the option type long-term contract and spot market. In each period, the buyer has seen and learned the spot prices in the previous spot market (information), and then forecasts the spot price in the next spot market with an estimate which is an average of the spot prices observed in the past spot market (process of the information). So, the buyer's purchasing decision is made based on an estimate averaged by the previous spot prices. Moreover, since the seller should decide the spot price on the expiration day after seeing the buyer's purchasing decision on the option, the spot price, which is the seller's decision, is affected by the buyer's option contracting decision and thus depends on th previous spot prices. This implies that the spot prices are not independent, and moreover that the spot price is not exogenously but endogenously decided. These findings raise interesting questions :

1. How would the seller's spot pricing decision be affected by the buyer's contracting decision?
2. Do the sequence of the buyer's contracting decisions and learning estimates have a finite limit?
3. If so, how can they be characterized and related?

The impact of the buyer's dynamically learning behavior on the seller's pricing decisions in multiple periods has not been well studied in the existing literature.

2. Literature Review

In the operations research area, the application of quantitative techniques or models have been used to improve profits by controlling the prices and availabilities of products. In almost every instance of published work, the starting point of the analysis for quantitative techniques and models is some set of assumptions regarding an underlying demand process based on an estimate for price.

In the traditional operations research area, demand process assumes that the demand at a particular point in time does not depend on the prices or availabilities of products at other points in time. Under this assumption, there are quite a large literature on the interaction between long-term contract and spot market. In [4], they present a static model where the selection problem of the long-term vs. short-term contract for a risk-averse buyer. Short-term contract in this model means the purchase of parts/goods in the (spot) market which should be immediately delivered to the buyer. The model is used to analyze the tradeoff between benefit of price certainty, cost improvement and fixed expense offered by the long-term contract and the flexibility and zero-fixed expense offered by the short-term contract. It shows that long-term contract may not always be optimal and introduce the conditions under which the short-term contract may be better off. [7] considers the impacts of a Internet-based secondary market in a two-period model where resellers can buy and sell excess inventories at the second period. However, the resellers can not buy any additional product from the manufacturer (supplier) at the second period but only order the products at the first period. Then it shows that the secondary market always improves allocative efficiency but total sales for the manufacturer at the first period may increase or decrease. [10] studies the role and value of B2B exchanges and their interaction with supply chain contracting. They consider two-level but three-stage supply chain with multiple sellers and buyers where forward-type contracting (first level) in the first stage, participants' reception of private information in the second stage and a B2B exchange (second level) as a spot market in the third stage are considered.

They derive the equilibrium on the clearing price, contracted quantities and the traded quantities of product, and investigates the effect of B2B exchanges on the four factors such as changes in industry structure, information effects in the second stage, volatility induced by the multiple manufacturers and price flexibility. [6] address the integration of option contract via spot market and survey the underlying theory and practices using the option in support of emerging business-to-business markets. It refers the option contract as long-term contracting and spot market as Business-to-Business exchanges and gives an excellent literature review on this option contract market. [13, 14] models contract between single seller and one or more buyers where the non-scalable goods or services are sold. By assuming that the price in the spot market is uncertain but its distribution function is common knowledge to both buyer and seller, it shows that the seller's optimal pricing policy is to set the strike price to the marginal cost and characterizes the condition for the existence of positive option contract. It is also shown that if the seller can sell all its residual goods in the spot market, then the seller will not make any option contract with the buyer. In its model, the price in the spot market assumed to follow the given probability distribution but not to be determined by the seller. This implies that the price in the spot market is independent of the option contract. [12] models the competition problem between procurement auction and long-term contract. It considers a tow-layer supply chain where multiple suppliers of a "part" transact with a set of manufacturers that utilize that part in the production of an end-product. Then, an auction-based e-market is used for the procurement of low quality parts and long-term relational contracts are employed for the procurement of high quality parts. (So, the manufacturer as a buyer purchases goods of different quality in each market.) It shows conditions under which each procurement mechanism will prevail and push the other out of the market, as well as conditions under which they coexist.

Models in above works somewhat underestimate the buyer's sophistication when making contracting and purchasing decision. However, recent works have developed models in which buyer's choice, that is demand process, depends on the prices or availabilities of products at multiple points in time. [1, 2, 5] consider the strategic buyer who takes into account the given future path of prices when making purchasing decisions and that the buyer make a purchase only if his/her current surplus is larger than the future expected surplus. They assume that the buyer knows exactly the future price path, which is the seller's decision. [9] considers the strategic buyer who is fully rational as commonly used assumption in game theory and the seller who even know that the buyer is strategic. They assume that the buyer knows how to use stochastic dynamic programming to evaluate his/her expected profit and to find optimal purchasing timing and even that the buyer knows and solves the seller's objective function to find seller's optimal pricing decision. Then, it shows the existence of a unique subgame-perfect equilibrium pricing policy.

However, the above recent works overestimate the buyer's sophistication when estimating the seller decision which is price. As shown above, much of the existing literature make an underestimated or overestimated assumption on the buyer's estimating strategy on the price, and those models are somewhat restrictive and unrealistic. So, in this paper, we want to make more practical and reasonable assumption on the buyer's price-estimating strategy.

3. Notations and Assumptions

The following notations are used in this paper

(1) $c \equiv$ production cost

(2) $K \equiv$ strike price

(3) $\pi \equiv$ option price

(4) $Q_n \equiv$ number of options in period n purchased by the buyers

(5) $q_{o,n} \equiv$ number of options to exercise in period $n (q_{o,n} \le Q_n)$

(6) $q_{s,n} \equiv$ number of goods purchased in period n through the spot market

(7) $q_n \equiv$ total number of goods purchased through both option contract and spot market, $(q_n = q_{o,n} + q_{s,n})$

(8) $p_n \equiv$ spot price in period n

Assumption 1 : The buyer's utility, denoted by $U(q)$, is a quadratic function of his/her total demand, denoted by q, and given by

$$U(q) \equiv -\frac{1}{2b}q^2 + \frac{a}{b}q$$

where a and b are constant with $a - bc \ge 0$.

4. Model

In this paper, we consider the following model setting: Multiple periods are considered and at each period there are two stages of decisions between the seller and the buyer. Suppose that we are in period n. In the first stage, the seller offers an option price π and strike price K to the buyer. However, the buyer does not know the spot price p_n in the second stage yet. Thus, the buyer needs to forecast the spot price p_n and needs to use and estimate $\tilde{p}_n$. Based on this estimate, the buyer decides how many options to buy Q_n. In the second stage when the option is expired, the seller decides the spot price p_n for each good, and then the buyer decides how many options to exercise $q_{o,n} \in [0, Q_n]$ and how many additional non-contract goods to purchase through the spot market $q_{o,n} \in [0,\infty)$. Now, we need an assumption regarding what kind of estimating method a buyer would use to forecast the spot price. Since we consider multiple periods of two stage problems, we have a sequence of two stage problems so that a buyer can observe the past spot prices and keep learning one more spot price in each period. Thus, we assume that a buyer uses an average of past spot prices $p_1, p_2, \cdots, p_{n-1}$ to estimate the next spot price p_n, that is, $\tilde{p}_n = \frac{1}{n-1}\sum_{i=1}^{n-1} p_i$. As seen above, this model setting is a Stackelberg-type game continuously repeated between buyer and seller. Section 4.1 and 4.2 will introduce the buyer's and seller's problem in each stage and give the buyer's and Seller's optimal decision. In section 4.3, the convergence result is shown for the sequence of the solution from the two-stage problems which are the buyer's and seller's decision.

4.1 Buyer's Problem in Period n

In period n, a buyer should make two stage decisions the first of which is how many options to contract given option price π and strike price K based on the estimate for spot price, and the second of which is how many option to exercise and how many additional non-contract goods to purchase in the spot market based on the options contracted. To make the first decision, the buyer should solve his profit function given by;

$$\max_{Q \geq 0} f(\tilde{p}_n)(Q) \tag{1}$$

$$\equiv \max_{Q \geq 0} U(q_n) - \pi Q - Kq_{o,n} - \tilde{p}_n q_{s,n}$$

where $\tilde{p}_n$ is an buyer's estimate for the spot price p_n on the day when option is expired. As assumed above, an averaged estimate $\tilde{p}_n = \frac{1}{n-1}\sum_{i=1}^{n-1} p_i$ is used as an estimate for the next spot price which is a simple but very practical estimating method. Now, let Q_n be an optimal number of options to contract in period n. Thus, it is an optimal solution to (1).

Now, to make the second decision, the buyer should solve the following problem with given strike price K, spot price p_n and option contracted Q_n;

$$\begin{aligned} &\max_{q_{o,n}, q_{s,n}} R(q_{o,n}, q_{s,n}) \\ &\equiv \max_{q_{o,n}, q_{s,n}} U(q_n) - Kq_{o,n} - p_n q_{s,n} \\ &\equiv \max_{q_{o,n}, q_{s,n}} U(q_n) - Kq_n - (K - p_n) q_{s,n} \end{aligned}$$

where $q_n = q_{o,n} + q_{s,n}$ is total demand and $q_{o,n} \leq Q_n$. The following results show the buyer's optimal exercising and purchasing strategy in Theorem 1 and the buyer's optimal contracting strategy in Lemma 1. (See [8] for the proof.)

Theorem 1 : Suppose that buyer's utility function, $U(q_n)$, is given as Assumption 1, the spot price is p_n in the spot market and the number of options purchased by buyer is Q_n. Then, the optimal number of goods to buy in the spot market, $q_{s,n}$, and the optimal number of options to exercise, $q_{o,n}$, are given by :

$$q_n = q_{o,n} + q_{s,n} = \begin{cases} 0 + 0 & \text{if } \frac{a}{b} \leq p_n \text{ and } \frac{a}{b} \leq K_n \\ 0 + (a - bp_n) & \text{if } p_n \leq \frac{a}{b} \text{ and } p_n \leq K_n \\ (a - bK) + 0 & \text{if } \frac{a - Q_n}{b} \leq K_n \leq \frac{a}{b} \text{ and } K_n \leq p_n \\ Q_n + (a - bp_n - Q_n) & \\ & \text{if } K_n \leq p_n \leq \frac{a - Q_n}{b} \\ Q_n + 0 & \text{if } K_n \leq \frac{a - Q_n}{b} \leq p_n \end{cases}$$

where q_n is total optimal number of goods to purchase though both option contract and spot market.

Lemma 1 : Suppose that, in period n, $\widetilde{p_n}$ is an estimate used for the next spot price p_n and that the buyer does not know the seller's pricing policy. Then, the first-order derivative function of $f(\tilde{p}_n)(Q)$ with respect to Q denoted by $\nabla f(\tilde{p}_n)(Q)$ exists and is given as follows : *If* $a-bK \le 0$, then for all $Q \ge 0$

$$\nabla f(\tilde{p}_n)(Q) = -\pi$$

Otherwise

$$\nabla f(\tilde{p}_n)(Q)= \begin{cases} -\pi+(\tilde{p}_n-K)^+-(\tilde{p}_n-\frac{a-Q}{b})^+ & \text{for } Q\le a-bK \\ -\pi & \text{for } a-bK\le Q \end{cases}$$

Using Lemma 1, the buyer's first decision can be made. *If* $a-bK \le 0$, since, for all $Q \ge 0$, we have

$$\nabla f(\tilde{p}_n)(Q) = -\pi$$

, the contracted options in period n is 0. Otherwise, since we have

$$\nabla f(\tilde{p}_n)(Q)= \begin{cases} -\pi+(\tilde{p}_n-K)^+-(\tilde{p}_n-\frac{a-Q}{b})^+ & \text{for } Q\le a-bK \\ -\pi & \text{for } a-bK\le Q \end{cases}$$

, the contracted options in period n is Q_n such that

$$-\pi+(\tilde{p}_n-K)^+-(\tilde{p}_n-\frac{a-Q_n}{b})^+=0$$

4.2 Seller's Problem in Period n

Given the number of contracted option Q_n, the seller in the second stage should decide and offer the spot price in the spot market. Let p_n be the spot price in period n. Then, seller should solve her profit function,

$$\begin{aligned} &\max && pq(p) - cq(p) \\ &\text{subject to} && p \le K \\ &\max && p(q(p) - q_o) + Kq_o - cq(p) \\ &\text{subject to} && K \le p \\ &&& q_o \le Q_n \end{aligned}$$

where the first objective function does not have the revenue from the exercised options Kq_o, since the spot price, p, is less than strike price K and thus buyer does not have to exercise any option. Theorem 2 shows that the seller's pricing decision, which is the spot price, depends on the buyer's decision. (See [8] for the proof.)

Theorem 2 : In period n, given the number of contracted options Q_n, the seller decides the optimal spot price p_n as follows

$$p_n=\begin{cases} \frac{a}{2b}+\frac{c}{2} & if\ \frac{a}{2b}+\frac{c}{2}\le K \\ \max\left[\frac{a-Q_n}{2b}+\frac{c}{2}, K\right] & otherwise \end{cases}$$

The spot price, p_n, is a non-increasing function in the number of options purchased by buyer, Q_n.

4.3 Convergence Result for Estimated Spot Price $\tilde{p}_n$ and Actual Spot Price p_n

In previous sections, we introduced a seller's optimal pricing strategy p_n and a buyer's optimal contracting strategy Q_n which depends on an estimate $\tilde{p}_n$ for p_n. Since buyer makes a contracting decision Q_n based on $\tilde{p}_n$, Q_n is a function of p_n. Also, since seller makes a pricing decision based on buyer's contracting decision Q_n, actual spot price p_n is a function of Q_n. This observation implies that three sequences $(Q_n, \tilde{p}_n, p_n)$ are related. Thus, in this section, we will see if $(Q_n, \tilde{p}_n, p_n)$ converge. If so, we will see how the limits are related and how they are characterized, Lemma 2 shows that if one of Q_n, $\widetilde{p_n}$ and p_n converges, the others also converges. Theorem 3 gives a sufficient condition for which $\widetilde{p_n}$ converges and the characterize the relationship of its limit with others.

Lemma 2 : Suppose that $\tilde{p}_n$ converges to some finite limit p^*. Then, p_n converges to p^* and Q_n converges Q^* such that

$$\nabla f(p^*)(Q^*) = -\pi+(p^*-K)^+-(p^*-\frac{a-Q^*}{b})^+ = 0$$

Proof : If $a-bK \leq 0$, then $\nabla f(\tilde{p}_n)(Q) = -\pi$ for any $\tilde{p}_n \in [0, \infty)$ and all $n \geq 1$ by Lemma 1. Thus, for all $n \geq 1$, $Q_n = 0$ and $p_n = \frac{a}{2b} + \frac{c}{2}$ by Theorem 2. Moreover, $\tilde{p}_n = \sum_{i=1}^{n-1} p_i = \frac{a}{2b} + \frac{c}{2}$ for all $n \geq 1$. Thus, obviously, $\tilde{p}_n$, Q_n and p_n converge. Now, suppose $a-bK \geq 0$. There are three cases we can consider: $p^* > K$, $p^* = K$ and $p^* < K$. First, for each case, we show that $\nabla f(\tilde{p}_n)(Q)$ converges uniformly to $\nabla f(p^*)(Q)$ in $Q \geq 0$.

(i) Suppose that $p^* > K$. Then, take any $\epsilon > 0$ such that $p^* - \epsilon > K$. Since $\tilde{p}_n$ converges to some finite limit $p^* > K$, there exists a finite integer $N(\epsilon) < \infty$ such that for all $n \geq N(\epsilon)$

$$|\tilde{p}_n - p^*| < \epsilon$$

Now, by Lemma 1, for all $Q \in [a-bK, \infty)$,

$$\begin{aligned} |\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| &= |-\pi + \pi| \\ &= 0 \\ &< \epsilon \end{aligned}$$

For all $Q \in [\max[a-bp^*, a-b\tilde{p}_n], a-bK]$,

$$\begin{aligned} &|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| \\ &= |-\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+ \\ &\quad + \pi - (p^* - K)^+ + (p^* - \frac{a-Q}{b})^+| \\ &= |-\pi + (\tilde{p}_n - K) - (\tilde{p}_n - \frac{a-Q}{b}) \\ &\quad + \pi - (p^* - K) + (p^* - \frac{a-Q}{b})| \\ &< |0| \\ &< \epsilon \end{aligned}$$

For all $Q \in [\min[a-bp^*, a-b\tilde{p}_n], \max[a-bp^*, a-b\tilde{p}_n]]$, consider $\tilde{p}_n < p^*$ which is $a-bp^* < a-b\tilde{p}$.

$$\begin{aligned} &|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| \\ &= |-\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+ \\ &\quad + \pi - (p^* - K)^+ + (p^* - \frac{a-Q}{b})^+| \\ &= \left|-\pi + (\tilde{p}_n - K) + \pi - (p^* - K) + (p^* - \frac{a-Q}{b})\right| \\ &= \left|\tilde{p}_n - \frac{a-Q}{b}\right| \\ &< |\tilde{p}_n - p^*| \\ &< \epsilon \end{aligned}$$

Consider $\tilde{p}_n > p^*$ which is $a-bp^* > a-b\tilde{p}$.

$$\begin{aligned} &|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| \\ &= |-\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+ \\ &\quad + \pi - (p^* - K)^+ + (p^* - \frac{a-Q}{b})^+| \\ &= \left|-\pi + (\tilde{p}_n - K) - (\tilde{p}_n - \frac{a-Q}{b}) + \pi - (p^* - K)\right| \\ &= \left|p^* - \frac{a-Q}{b}\right| \\ &< |p^* - \tilde{p}_n| \\ &< \epsilon \end{aligned}$$

For all $Q \in [0, \min[a-bp^*, a-b\tilde{p}_n]]$,

$$\begin{aligned} &|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| \\ &= |-\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+ \\ &\quad + \pi - (p^* - K)^+ + (p^* - \frac{a-Q}{b})^+| \\ &= |-\pi + (\tilde{p}_n - K) + \pi - (p^* - K)| \\ &= |\tilde{p}_n - p^*| \\ &< \epsilon \end{aligned}$$

So, if $p^* > K$, $\nabla f(\tilde{p}_n)(Q)$ converges uniformly to $\nabla f(p^*)(Q)$ for all $Q \geq 0$.

(ii) Suppose that $p^* = K$. Then, take any $\epsilon > 0$. Since $\tilde{p}_n$ converges to $p^* = K$, there exists a finite integer $N(\epsilon) < \infty$ such that for all $n \geq N(\epsilon)$

$$|\tilde{p}_n - p^*| < \epsilon \text{ equivalently } |\tilde{p}_n - K| < \epsilon$$

First, consider $n \geq N(\epsilon)$ with $\tilde{p}_n \leq p^*$. Then, by Lemma 1, for all $Q \geq 0$, if $Q \geq a-bK$, then $\nabla f(\tilde{p}_n)(Q) = -\pi$. Otherwise $\nabla f(\tilde{p}_n)(Q) = -\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+ = -\pi$ since $\tilde{p}_n \leq p^* = K$. So, for any $n \geq 1$ with $\tilde{p}_n \leq$

p^*,

$$|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| = |-\pi + \pi| = 0 < \epsilon$$

Second, consider all $n \geq N(\epsilon)$ such that $p^* < \tilde{p}_n < p^* + \epsilon$. For all $Q \in [a - bK, \infty)$,

$$|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| = |-\pi + \pi| = 0 < \epsilon$$

For all $Q \in [a - b\tilde{p}_n, a - bK]$, where $a - bK = a - bp^*$,

$$\begin{aligned}
&|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| \\
&= |-\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+ \\
&\quad + \pi - (p^* - K)^+ + (p^* - \frac{a-Q}{b})^+| \\
&= \left|-\pi + (\tilde{p}_n - K) - (\tilde{p}_n - \frac{a-Q}{b}) + \pi\right| = \left|\frac{a-Q}{b} - K\right| \\
&= \left|\frac{a-Q}{b} - p^*\right| \quad (\text{since } p^* = K) \\
&\leq |\tilde{p}_n - p^*| \ (\text{since } Q \geq a - b\tilde{p}_n \text{ and } \tilde{p}_n > p^*) \\
&< \epsilon
\end{aligned}$$

For all $Q \in [0, a - b\tilde{p}_n]$,

$$\begin{aligned}
&|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| \\
&= |-\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+ \\
&\quad + \pi - (p^* - K)^+ + (p^* - \frac{a-Q}{b})^+| \\
&= |-\pi + (\tilde{p}_n - K) + \pi)| \\
&= |\tilde{p}_n - K| \\
&= |\tilde{p}_n - p^*| \ (\text{since } p^* = K) \\
&< \epsilon
\end{aligned}$$

So, if $p^* = K$, a function $\nabla f(\tilde{p}_n)(Q)$ converges uniformly to $\nabla f(p^*)(Q)$ for all $Q \geq 0$.

(iii) Suppose that $p^* < K$. Then, take any $\epsilon > 0$ such that $p^* + \epsilon < K$. Since $\tilde{p}_n$ converges to some finite limit $p^* < K$, there exists a finite integer $N(\epsilon) < \infty$ such that for all $n \geq N(\epsilon)$

$$|\tilde{p}_n - p^*| < \epsilon$$

which implies that, for all $n \geq N(\epsilon)$, $\tilde{p}_n < K$. Then, by Lemma 1, for all $Q \geq 0$ and all $n \geq N(\epsilon)$,

$$\begin{aligned}
&\nabla f(\tilde{p}_n)(Q) \\
&= \nabla f(p^*)(Q) \\
&= \begin{cases} -\pi & \text{if } Q \geq a - bK \\ -\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+ & \\ = -\pi & \textit{otherwise} \end{cases}
\end{aligned}$$

since $\tilde{p}_n \leq p^* = K$. Thus, for all $Q \geq 0$, trivially we have

$$|\nabla f(\tilde{p}_n)(Q) - \nabla f(p^*)(Q)| = |-\pi + \pi| = 0 < \epsilon$$

Therefor, for any $p^* \in R$, a function $\nabla f(\tilde{p}_n)(Q)$ converges uniformly to $\nabla f(\tilde{p}_n)(Q)$ for all $Q \geq 0$. Now, we need to show that Q_n, which is a solution to $\nabla f(\tilde{p}_n)(Q) = 0$, converges to some finite limit, Q^*, which is a solution to $\nabla f(p^*)(Q) = 0$. Due to the uniform convergence of $\nabla f(\tilde{p}_n)(Q)$, there exist ϵ_1 and a finite integer $N_1(\epsilon)$ such that for all $n \geq N_1(\epsilon)$

$$|\nabla f(p^*)(Q^*) - \nabla f(\tilde{p}_n)(Q^*)| < \epsilon_1$$

Equivalently,

$$|\nabla f(\tilde{p}_n)(Q^*)| < \epsilon_1$$

Moreover, for any $Q > 0$ such that $\nabla f(\tilde{p}_n)(Q)$ is strictly decreasing, its slope is $-\frac{a}{b}$. Now, for all $n \geq N_1(\epsilon)$, consider Q_n such that $\nabla f(\tilde{p}_n)(Q_n) = 0$ and $Q_n \neq Q^*$. Then

$$\left|\frac{a}{b}\right| = \left|\frac{\nabla f(\tilde{p}_n)(Q^*) - \nabla f(p^*)(Q^*)}{Q_n - Q^*}\right| = \frac{|\nabla f(\tilde{p}_n)(Q^*) - \nabla f(p^*)(Q^*)|}{|Q_n - Q^*|} < \frac{\epsilon_1}{|Q_n - Q^*|}$$

So, for all $n \geq N_1(\epsilon)$ and all Q_n such that $\nabla f(\tilde{p}_n)(Q_n) = 0$ and $Q_n \neq Q^*$,

$$|Q_n - Q^*| < \frac{b}{a}\epsilon_1$$

Therefore, Q_n converges to Q^*. Since

$$p_n = \begin{cases} \frac{a}{2b} + \frac{c}{2} & \text{if } \frac{a}{2b} + \frac{c}{2} \le K \\ \max\left[\frac{a-Q_n}{2b} + \frac{c}{2}, K\right] & otherwise \end{cases}$$

is a continuous function of Q, $\widetilde{p_n}$ converges to

$$p_* = \begin{cases} \frac{a}{2b} + \frac{c}{2} & \text{if } \frac{a}{2b} + \frac{c}{2} \le K \\ \max\left[\frac{a-Q^*}{2b} + \frac{c}{2}, K\right] & otherwise \end{cases}$$

Thus, $\tilde{p}_n = \Sigma_{n=1}^{\infty} p_n$ converges to p_* as p_n converges to p_*. Due to the assumption that $\tilde{p}_n$ converges to p^*, p_* should be equal to p^*. Thus the result holds: If $\tilde{p}_n$ converges to some finite limit, p^*, then Q_n and p_n converge.

Q.E.D.

Now, Theorem 3 shows that there exists a sufficient condition for which the buyer's estimate $\widetilde{p_n}$ for the spot price converges. Also, Theorem 3 characterizes a relationship between limits of $\widetilde{p_n}$ and Q_n.

Theorem 3 : Suppose that for Q^* is the unique solution to following equation

$$-\pi + (p^* - K)^+ - \left(p^* - \frac{a-Q}{b}\right)^+ = 0$$

where

$$p^* = \begin{cases} \frac{a}{2b} + \frac{c}{2} & \text{if } \frac{a}{2b} + \frac{c}{2} \le K \\ \max\left[\frac{a-Q^*}{2b} + \frac{c}{2}, K\right] & otherwise \end{cases}$$

Then, as n goes to ∞, $\widetilde{p_n}$ converges to p^*.

Proof : Let $T_n = \tilde{p}_n - p^*$ and $Z_n = |T_n|^2$.

$$Z_{n+1} = |T_{n+1}|^2 = |\tilde{p}_{n+1} - p^*|^2 = \left|\tilde{p}_n + \frac{1}{n}(p_n - \widetilde{p_n}) - p^*\right|^2$$

$$= \left|\tilde{p}_n - p^* + \frac{1}{n}(p_n - \widetilde{p_n})\right|^2 = \left|T_n + \frac{1}{n}(p_n - \widetilde{p_n})\right|^2$$

$$= Z_n + 2\frac{1}{n}T_n(p_n - \widetilde{p_n}) + \frac{1}{n^2}|p_n - \widetilde{p_n}|$$

where the third equality holds since

$$\tilde{p}_{n+1} = \frac{1}{n}\sum_{i=1}^{n} p_i$$

$$= \tilde{p}_n + \frac{1}{n}(p_n - \widetilde{p_n})$$

Let F_n be the σ-field generated by $\{p_{n-1}, p_{n-2}, \cdots, p_1\}$. Then,

$$E[Z_{n+1}|F_n]$$
$$= E[Z_n + \frac{2}{n}T_n(p_n - \tilde{p}_n) + \frac{1}{n^2}|p_n - \tilde{p}_n|^2|F_n]$$
$$= E[Z_n|F_n] + E[\frac{2}{n}T_n(p_n - \tilde{p}_n)|F_n]$$
$$+ E[\frac{1}{n^2}|p_n - \tilde{p}_n|^2|F_n]$$
$$= Z_n + \frac{2}{n}T_n(E[p_n|F_n] - \tilde{p}_n) + \frac{1}{n^2}|E[p_n|F_n] - \tilde{p}_n|^2$$

Now, we need to show that the second term is strictly negative for all $n \ge 1$ if $\tilde{p}_n \ne p^*$. WLOG, suppose that $\tilde{p}_n < p^*$, (The other direction can be shown in the same way.) Then

$$\nabla f(\tilde{p}_n)(Q)$$
$$= -\pi + (\tilde{p}_n - K)^+ - (\tilde{p}_n - \frac{a-Q}{b})^+$$
$$\le -\pi + (p^* - K)^+ - (p^* - \frac{a-Q}{b})^+$$
$$= \nabla f(p^*)(Q)$$

Also, since $\nabla f(p^*)(Q) = 0$ has a unique solution by assumption, $Q_n \le Q^*$. Thus, we have

$$\begin{cases} E[p_n|F_n] \\ = \max[\frac{a}{2b} + \frac{c}{2}, K] \\ = \max[\frac{a}{2b} + \frac{c}{2}, K] \\ = p^* & \text{if } \frac{a}{2b} + \frac{c}{2} \le K \\ E[p_n|F_n] \\ = \max[\frac{a-Q_n}{2b} + \frac{c}{2}, K] \\ \ge \max[\frac{a-Q^*}{2b} + \frac{c}{2}, K] \\ = p^* & otherwise \end{cases}$$

and

$$T_n(E[p_n|F_n] - \tilde{p}_n)$$
$$= (\tilde{p}_n - p^*)(E[p_n|F_n] - \tilde{p}_n)$$
$$= (\tilde{p}_n - p^*)(E[p_n|F_n] - p^* + p^* - \tilde{p}_n)$$
$$= (\tilde{p}_n - p^*)(E[p_n|F_n] - p^*) + (\tilde{p}_n - p^*)(p^* - \tilde{p}_n)$$
$$\leq -(p^* - \tilde{p}_n)^2$$
$$< 0$$

where the first ineuqlaity holds since $\tilde{p}_n < p^*$ and $E[p_n|F_n] \geq p^*$. Now, take any $\epsilon > 0$ such that $-|T_n|^2 = (\tilde{p}_n - p^*)(p^* - \tilde{p}_n) \leq -\epsilon$. Then, we have

$$T_n(E[p_n|F_n] - \tilde{p}_n) < -(p^* - \tilde{p}_n)^2 \leq -\epsilon \quad (2)$$

Therefore,

$$E[Z_{n+1}|F_n]$$
$$= Z_n + \frac{2}{n} T_n(E[p_n|F_n] - \tilde{p}_n) + \frac{1}{n^2}|E[p_n|F_n] - \tilde{p}_n|^2$$
$$\leq (1 + \frac{2}{n})Z_n + \frac{2}{n} T_n(E[p_n|F_n] - \tilde{p}_n)$$
$$+ \frac{1}{n^2}|E[p_n|F_n] - \tilde{p}_n|^2$$

We know that $\sum_1^\infty \frac{1}{n} < \infty$ and $\sum_1^\infty \frac{1}{n^2} < \infty$. So, by Super-Martingale Type Lemma (See [3]),

$$Z_n \to Z < \infty \text{ and } -\sum_{n=1}^{\infty} \frac{2}{n} T_n(E[p_n|F_n] - \tilde{p}_n) < \infty$$

Suppose that $Z \neq 0$. Then, there exists $\epsilon > 0$ and $N < \infty$, such that for all $n > N$, we have $|T_n| \geq \epsilon$ and $\liminf -T_n(E[p_n|F_n] - \tilde{p}_n) \geq \epsilon > 0$ by (2). So, we have

$$-\sum_{n=1}^{\infty} \frac{2}{n} T_n(E[E[p_n|F_n] - \tilde{p}_n|F_n]) = \infty$$

But, this contradicts to

$$-\sum_{n=1}^{\infty} \frac{2}{n} T_n(E[E[p_n|F_n] - \tilde{p}_n|F_n]) < \infty$$

Therefore, $Z_n \to 0$ which is $|\tilde{p}_n - p^*|^2 \to 0$.
Therefore, $\tilde{p}_n \to p^*$. **Q.E.D.**

By Lemma 2 and Theorem 3, we find a sufficient condition for which buyer's estimate $\tilde{p}_n$, actual spot price p_n and the contracted option Q_n converge.

Corollary 1 : Suppose that for Q^* is the unique solution to following equation

$$-\pi + (p^* - K)^+ - (p^* - \frac{a - Q}{b})^+ = 0$$

where

$$p^* = \begin{cases} \frac{a}{2b} + \frac{c}{2} & \text{if } \frac{a}{2b} + \frac{c}{2} \leq K \\ \max\left[\frac{a - Q^*}{2b} + \frac{c}{2}, K\right] & \textit{otherwise} \end{cases}$$

Then, as n goes to ∞, $(\tilde{p}_n, p_n, Q_n)$, converge to (p^*, p^*, Q^*).

Proof : By Lemma 2 and Theorem 3, $(\tilde{p}_n, p_n, Q_n)$converge to (p^*, p^*, Q^*). **Q.E.D.**

5. Conclusion

A buyer's behavior responding to a price of good has been an important issue, which actually influences on the contracting or purchasing decision. In this paper, we consider a model using option-type contract in which there are two purchasing channels for buyer : The first channel is an option contract before knowing the actual spot price on the day when the contracted option will be expired. The second channel is the spot market which is open when the option contract is expired, the seller decides the spot price for the good and offer it to the buyer. Moreover, we consider a case where this two stage of purchasing channels repeats in many periods so that we will see the sequence of option contract and spot market. Thus, there is a chance the buyer can learn the spot price from the past spot markets. We assume that, as an estimate for the spot price in the future spot market, the buyer use average of the past observed spot prices. So, we analyze a model in which the buyer makes option-contracting decision based on an estimate for the spot price in the future market and the seller makes pricing decision based on the buyer's option-contracting decision. Consequently, we have answered the following questions :

1. Long-run behavior of buyer's estimate $\widetilde{p_n}=\sum_{i=1}^{n} p_i$
2. Long-run effect of the buyer's contracting decision Q_n on the seller's pricing decision p_n
3. Sufficient condition for which the sequence of buyer's price estimate to converge.

The second result above implies that more options are contracted by the buyer, the lower spot price will be made by the seller. More option contract will cause the seller to lower the spot price in the spot market so that the buyer would be enticed to buy additional product from the spot market. In the third result, the sufficient condition for convergence means that there is an unique equilibrium between the seller's pricing policy and buyer's contracting policy so that the buyer's estimate for price will finally approach the market equilibrium. Moreover, we have shown the long-run relationship among the buyer's estimate, buyer's contracting decision and the seller's pricing decision.

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