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Some Notes on L_p-metric Space of Fuzzy Sets

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Abstract

It is well-known that the space E^n of fuzzy numbers (i.e., normal, upper-semicontinuous, compact-supported and convex fuzzy subsets) in the n-dimensional Euclidean space R^n is separable but not complete with respect to the L_p-metric.

In this paper, we introduce the space $F_p(R^n)$ that is separable and complete with respect to the L_p-metric. This will be accomplished by assuming p-th mean bounded condition instead of compact-supported condition and by removing convex condition.

Key Words: Fuzzy numbers, Compact sets, L_p-metric.

1. Introduction

The metric in a space of fuzzy sets plays an important role both in the theory and in its applications. There are various useful metrics defined on the fuzzy number space E^n of normal, upper-semicontinuous, compact-supported and convex fuzzy subsets of n-dimensional Euclidean space R^n. The readers may refer to [2] for supremum metric, sendograph metric and L_p-metric, and refer to [6] for Skorohod metric.

It is well-known that E^n is complete and separable if it is equipped with the metric except L_p-metric. Characterizations of compact subsets of E^n equipped supremum metric, sendograph metric and the Skorohod metric were given by Greco [4], Greco and Moschen [5], Greco [3], Zhao and Wu [10], Joo and Kim [6], respectively.

However, it is known that E^n is separable but not complete with respect to the L_p-metric. This problem arises from the fact that compact-supported condition is inadequate for the L_p-metric.

Related to this problem, Kraschmer [7] dealt with completion of E^n w.r.t. the L_p-metric by introducing the notion of support function for noncompact fuzzy number and Degang et al. [1] proposed the completion of E^1 w.r.t. the L_1-metric by using representation theorem of noncompact fuzzy number in E^1. But these approaches cannot be valid any more if we drop the convexity condition.

In this paper, we introduce the space $F_p(R^n)$ without convexity that is complete and separable with respect to the L_p-metric.

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2. Preliminaries

Let $K(R^n)$ denote the family of all non-empty compact subsets of the n-dimensional Euclidean space R^n with the usual norm $|\cdot|$. Then the space $K(R^n)$ is metrizable by the Hausdorff metric h defined by

$$h(A,B) \;=\; \max[\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|].$$

The norm of $A\in K(R^n)$ is defined by

$$\|A\| = h(A,\{0\}) = \sup_{a\in A}|a|.$$

It is well-known that $K(R^n)$ is complete and separable with respect to the Hausdorff metric h. Also, if we denote by $K_c(R^n)$ the family of all $A\in K(R^n)$ which is convex, then $K_c(R^n)$ is a closed subspace of $(K(R^n),h)$.

Let $F(R^n)$ denote the family of all fuzzy sets $u : R^n \to [0,1]$ with the following properties;

(i) u is normal, i.e., there exists $x\in R^n$ such that $u(x) = 1$.

(ii) $L_\alpha u = \{x\in R^n : u(x)\geq\alpha\}$ is a compact subset of R^n for each $0<\alpha\leq 1$.

$L_\alpha u$ is called the α-level set of u. We denote by $F_c(R^n)$ the family of all $u\in F(R^n)$ which is convex, i.e., $u(\lambda x+(1-\lambda)y)\geq\min(u(x),u(y))$ for all $x,y\in R^n$ and $0\leq\lambda\leq 1$. Then $u\in F_c(R^n)$ if and only if $L_\alpha u\in K_c(R^n)$ for each $0<\alpha\leq 1$.

Also, we denote by $F_\infty(R^n)$ (resp. $F_{c\infty}(R^n)$) the family of all $u \in F(R^n)$ (resp. $F_c(R^n)$) with compact support, i.e., $L_0u = \overline{\{x \in R^n : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A w.r.t. the usual norm in R^n. Briefly, $F_{c,\infty}(R^n)$ is denoted by E^n and a member of E^n is called a fuzzy number.

Joo and Kim [6] showed that $u \in F_\infty(R^n)$ can be characterized by a function f_u defined as $f_u : [0,1] \to K(R^n)$, $f_u(\alpha) = L_\alpha u$, which is non-increasing, left-continuous on $(0,1]$, right-continuous at 0 and right-limits on $[0,1)$. By very similar arguments, we can obtain the following lemma.

Lemma 2.1. *For $u \in F(R^n)$, we define*

$$f_u : (0,1] \longrightarrow (K(R^n),h),\ f_u(\alpha) = L_\alpha u.$$

Then the followings hold;

(i) f_u is non-increasing, i.e., $\alpha \le \beta$ implies $f_u(\alpha) \supset f_u(\beta)$,

(ii) f_u is left continuous on $(0,1]$,

(iii) f_u has right-limits on $(0,1)$.

Conversely, if $g : [0,1] \to K(R^n)$ is a function satisfying the above conditions $(i)-(iii)$, then there exists a unique $v \in F(R^n)$ such that $g(\alpha) = L_\alpha v$ for all $\alpha \in (0,1]$.

If we denote by $L_{\alpha^+}u$ the right-limit of f_u at $\alpha \in (0,1)$, then it is well-known that

$$L_{\alpha^+}u = \overline{\{x \in R^n : u(x) > \alpha\}}.$$

3. Main Results

The L^p-metric d_p on the fuzzy number space E^n is defined as follows;

$$d_p(u,v) = \left(\int_0^1 h(L_\alpha u, L_\alpha v)^p\, d\alpha\right)^{1/p}.$$

It is well-known that (E^n, d_p) is separable but not complete. This fact seems to be natural since E^n is too small for it to be complete w.r.t. d_p. In order to achieve completeness, we need to introduce a new family of fuzzy sets that includes E^n.

For $1 \le p < \infty$, let $F_p(R^n)$ (resp. $F_{c,p}(R^n)$) be the family of all fuzzy sets $u \in F(R^n)$ (resp. $F_c(R^n)$) such that

$$\int_0^1 \|L_\alpha u\|^p\, d\alpha < \infty.$$

It is obvious that $F_\infty(R^n) \subset F_p(R^n)$ but $F_\infty(R^n) \ne F_p(R^n)$. It is easy to prove that the d_p on $F_p(R^n)$ satisfies the axioms of metric. We first prove the completeness of $(F_p(R^n), d_p)$.

Theorem 3.1. *$(F_p(R^n), d_p)$ is complete.*

Proof. Let $\{u_i\}$ be a Cauchy sequence in $(F_p(R^n), d_p)$ such that $\int_0^1 h(L_\alpha u_i, L_\alpha u_j)^p\, d\alpha \to 0$ as $i, j \to \infty$.

Step 1: First, we show that there exists a subsequence $\{u_{i_k}\}$ of $\{u_i\}$ such that $\{L_\alpha u_{i_k}\}$ is a Cauchy sequence in $(K(R^n), h)$ for almost all α.

We note that for each $\varepsilon > 0$,

$$\begin{aligned} &\mu\{\alpha : h(L_\alpha u_i, L_\alpha u_j) > \varepsilon\} \\ \le\ &\frac{1}{\varepsilon^p}\int_0^1 h(L_\alpha u_i, L_\alpha u_j)^p\, d\alpha \to 0 \end{aligned}$$

as $i, j \to \infty$, where μ denote the Lebesgue measure.

For any positive integer k, we find an integer N_k such that

$$\mu(\{\alpha : h(L_\alpha u_i, L_\alpha u_j) \ge \frac{1}{2^k}\}) < \frac{1}{2^k}$$

for $i, j \ge N_k$. Now we write

$$i_1 = N_1,\ i_k = (i_{k-1}+1) \vee N_k \text{ for k} \ge 2,$$

then $\{u_{i_k}\}$ is a subsequence of $\{u_i\}$.

Let $I_k = \{\alpha : h(L_\alpha u_{i_k}, L_\alpha u_{i_{k+1}}) \ge \frac{1}{2^k}\}$ and

$$I_0 = \limsup_{k\to\infty} I_k = \cap_{m=1}^\infty \cup_{k=m}^\infty I_k.$$

Then since

$$\mu(\cup_{k=m}^\infty I_k) \le \sum_{k=m}^\infty \mu(I_k) < \frac{1}{2^{m-1}},$$

we have that $\mu(I_0) = 0$. And if $\alpha \notin I_0$, then there exists m such that $\alpha \notin \cup_{k=m}^\infty I_k$ and so for $k, l \ge m$,

$$h(L_\alpha u_{i_k}, L_\alpha u_{i_l}) \le \sum_{k=m}^\infty h(L_\alpha u_{i_k}, L_\alpha u_{i_{k+1}}) < \frac{1}{2^{m-1}},$$

which implies $\{L_\alpha u_{i_k}\}$ is a Cauchy sequence in $(K(R^n), h)$.

Step 2: By completeness of $(K(R^n), h)$, $\{L_\alpha u_{i_k}\}$ converges to A_α for some $A_\alpha \in K(R^n)$ for each $\alpha \notin I_0$.

If $0 < \alpha \le 1$ and $\alpha \in I_0$, then we define

$$A_\alpha = \cap_{\beta<\alpha, \beta\notin I_0} A_\beta.$$

Then by Lemma 2.1, there exists a $u \in F(R^n)$ such that $L_\alpha u = A_\alpha$ for each $0 < \alpha \le 1$. Now we have to show that $u \in F_p(R^n)$ and $d_p(u_i, u) \to 0$ as $i \to \infty$.

Since $\{u_{i_k}\}$ is Cauchy sequence in $(F_p(R^n), d_p)$, there exist an M such that for $k, l \geq M$,

$$\int_0^1 h(L_\alpha u_{i_k}, L_\alpha u_{i_l})^p \, d\alpha < 1.$$

For a fixed $k \geq M$, since

$$\lim_{l\to\infty} h(L_\alpha u_{i_k}, L_\alpha u_{i_l}) = h(L_\alpha u_{i_k}, L_\alpha u)$$

for almost all α, we have that by Fatou's lemma,

$$\begin{aligned} &\int_0^1 h(L_\alpha u_{i_k}, L_\alpha u)^p \, d\alpha \\ \leq \quad & \liminf_{l\to\infty} \int_0^1 h(L_\alpha u_{i_k}, L_\alpha u_{i_l})^p \, d\alpha \leq 1. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^1 \|L_\alpha u\|^p \, d\alpha \\ \leq \quad & 2^p \int_0^1 \|L_\alpha u_{i_k}\|^p \, d\alpha + 2^p \int_0^1 h(L_\alpha u_{i_k}, L_\alpha u)^p \, d\alpha < \infty, \end{aligned}$$

which implies $u \in F_p(R^n)$.

Finally, the triangle inequality

$$d_p(u_i, u) \leq d_p(u_i, u_{i_k}) + d_p(u_{i_k}, u)$$

shows that $d_p(u_i, u) \to 0$ as $i \to \infty$. □

Corollary 3.2. *$F_{cp}(R^n)$ is a closed subspace of $(F_p(R^n), d_p)$ and so it is complete.*

Proof. Let $\{u_i\}$ be a sequence in $(F_{cp}(R^n), d_p)$ such that for some $v \in F_p(R^n)$,

$$d_p(u_i, v) \to 0 \text{ as } i \to \infty.$$

Then there exists a $I \subset (0,1]$ with Lebesgue measure 0 such that for all $\alpha \notin I$,

$$h(L_\alpha u_i, L_\alpha v) \to 0 \text{ as } i \to \infty.$$

Since $L_\alpha u_i \in K_c(R^n)$ and $K_c(R^n)$ is a closed subspace of $K(R^n)$, $L_\alpha v \in K_c(R^n)$ for all $\alpha \notin I$. If $0 < \alpha \leq 1$ and $\alpha \in I$, then we can choose a increasing sequence $\{\alpha_k\}$ with $\alpha_k \notin I$ so that $\alpha_k \to \alpha$ as $k \to \infty$. Then by left-continuity of $L_\alpha v$ as a function of α, we have $h(L_{\alpha_k} v, L_\alpha v) \to 0$ as $k \to \infty$, and so $L_\alpha v \in K_c(R^n)$. This completes the proof. □

Now we prove that $(F_p(R^n), d_p)$ is separable. To do this, we need some lemmas.

Lemma 3.3. *If $A_j, B_j \in K(R^n)$, $j = 1, 2$, then*

$$h(A_1 \cup A_2, B_1 \cup B_2) \leq \max[h(A_1, B_1), h(A_2, B_2)].$$

Proof. It follows from the fact that

$$\begin{aligned} & \sup_{a\in A_1\cup A_2} \inf_{b\in B_1\cup B_2} |a-b| \\ = \quad & \max(\sup_{a\in A_1} \inf_{b\in B_1\cup B_2} |a-b|, \sup_{a\in A_2} \inf_{b\in B_1\cup B_2} |a-b|) \\ \leq \quad & \max(\sup_{a\in A_1} \inf_{b\in B_1} |a-b|, \sup_{a\in A_2} \inf_{b\in B_2} |a-b|) \end{aligned}$$

□

Lemma 3.4. *If $u \in F(R^n)$ and $0 < \beta < 1$, then there exists a partition $\beta = \beta_0 < \cdots < \beta_m = 1$ of $[\beta, 1]$ such that*

$$h(L_{\beta_k} u, L_{\beta_{k-1}^+} u) < \varepsilon \text{ for all } k = 1, \cdots, m.$$

Proof. Let $\varepsilon > 0$ be given. By applying Lemma 2.1, for each $\beta < \alpha < 1$, we can take $\delta_\alpha > 0$ so that

$$h(L_\alpha u, L_{\alpha-\delta_\alpha} u) < \varepsilon$$

and

$$h(L_{\alpha^+} u, L_{\alpha+\delta_\alpha} u) < \varepsilon.$$

Also, we can choose $\delta_\beta, \delta_1 > 0$ so that

$$h(L_{\beta^+} u, L_{\beta+\delta_\beta} u) < \varepsilon$$

and

$$h(L_1 u, L_{1-\delta_1} u) < \varepsilon.$$

Let $I_\beta = [\beta, \beta+\delta_\beta), I_1 = (1-\delta_1, 1]$ and for each $\beta < \alpha < 1$,

$$I_\alpha = (\alpha-\delta_\alpha, \alpha-\delta_\alpha).$$

Then by the compactness of $[\beta, 1]$, there exists $\alpha_1, \cdots, \alpha_N \in (\beta, 1)$ such that

$$[0,1] = I_\beta \cup I_1 \cup (\cup_{i=1}^N I_{\alpha_i}).$$

The collection of points $\{\beta, \beta+\delta_\beta, 1-\delta_1, 1\} \cup \{\alpha_i - \delta_{\alpha_i}, \alpha_i, \alpha_i + \delta_{\alpha_i} : i = 1, \cdots, N\}$ forms a partition of $[\beta, 1]$. We denote these points in ascending order by

$$\beta = \beta_0 < \beta_1 < \cdots < \beta_m = 1.$$

Then it is obvious that for all $k = 1, 2, \cdots, m$,

$$h(L_{\beta_k} u, L_{\beta_{k-1}^+} u) < \varepsilon.$$

□

Theorem 3.5. *$(F_p(R^n), d_p)$ is separable.*

Proof. Since $(K(R^n),h)$ is separable, there exists a countable dense subclass $\mathcal{K}$ of $K(R^n)$.

Now let $\mathcal{F}$ be the family of fuzzy sets v which for some positive m, there exist a finite unions $A_1 \supset \cdots \supset A_m$ of sets in $\mathcal{K}$ and rational points $0 < \alpha_1 \le \cdots \le \alpha_{m-1} < 1$ such that

$$v(x) = \sum_{k=1}^{m-1} \alpha_k I_{A_k \setminus A_{k+1}}(x) + I_{A_m}(x),$$

where I_A denotes the indicator function of A.

Then it is obvious that $\mathcal{F}$ is countable subset of $F_p(R^n)$.

Now it suffices to prove that $\mathcal{F}$ is dense in $(F_p(R^n), d_p)$. Let $u \in F_p(R^n)$ and $\varepsilon > 0$ be given. First we choose $0 < \beta < 1$ so that

$$\int_0^\beta \|L_\alpha u\|^p \, d\alpha < (\varepsilon/16)^p. \tag{1}$$

And then, by applying Lemma 3.4, we choose a partition $\beta = \beta_0 < \cdots < \beta_m = 1$ of $[\beta, 1]$ such that

$$h(L_{\beta_k} u, L_{\beta^+_{k-1}} u) < \varepsilon/8 \text{ for all } k = 1, \cdots, m.$$

If we take $B_k \in \mathcal{K}, k = 1, 2, \cdots, m$ so that

$$h(B_k, L_{\beta_k} u) < \varepsilon/8 \text{ for each } k.$$

and let $A_k = \cup_{i=k}^m B_i$, then by lemma 3.3,

$$h(L_{\beta_k} u, A_k) < \varepsilon/8,$$

and

$$h(L_{\beta^+_{k-1}} u, A_k) \le h(L_{\beta^+_{k-1}} u, L_{\beta_k} u) + h(L_{\beta_k} u, A_k) < \varepsilon/4. \tag{2}$$

Let $\alpha_m = \beta_m = 1$ and for each $k = 1, \cdots, m-1$, we choose rational points α_k so that

$$\beta_{k-1} < \alpha_k \le \beta_k, \; h(L_{\alpha_k} u, L_{\beta_k} u) < \varepsilon/8$$

and

$$\sum_{k=1}^m \int_{\alpha_k}^{\beta_k} (\|L_\alpha u\| + \|A_1\|)^p \, d\alpha < \varepsilon^p/4. \tag{3}$$

Then

$$h(L_{\alpha_k} u, A_k) \le h(L_{\alpha_k} u, L_{\beta_k} u) + h(L_{\beta_k} u, A_k) < \varepsilon/4. \tag{4}$$

Now if we define

$$v(x) = \sum_{k=1}^{m-1} \alpha_k I_{A_k \setminus A_{k+1}}(x) + I_{A_m}(x),$$

then

$$L_\alpha v = \begin{cases} A_1 & if \quad 0 < \alpha \le \alpha_1, \\ A_k & if \quad \alpha_{k-1} < \alpha \le \alpha_k, k = 2, \cdots, n. \end{cases}$$

Since for $0 < \alpha \le \beta$,

$$\begin{aligned} h(L_\alpha u, L_\alpha v) &\le h(L_\alpha u, L_{\beta^+} u) + h(L_{\beta^+} u, A_1) \\ &\le 2\|L_\alpha u\| + \varepsilon/4 \quad \text{by (2)}, \end{aligned}$$

we have

$$\begin{aligned} &\int_0^\beta h(L_\alpha u, L_\alpha v)^p \, d\alpha \\ \le\; & 2^p [4^p \int_0^\beta \|L_\alpha u\|^p \, d\alpha + (\varepsilon/4)^p \beta] \\ \le\; & (\varepsilon/2)^p (1+\beta) \quad \text{by (1)}. \end{aligned}$$

And for $1 \le k \le m$,

$$\begin{aligned} &\int_{\beta_{k-1}}^{\beta_k} h(L_\alpha u, L_\alpha v)^p \, d\alpha \\ =\; & \int_{\beta_{k-1}}^{\alpha_k} h(L_\alpha u, L_\alpha v)^p \, d\alpha + \int_{\alpha_k}^{\beta_k} h(L_\alpha u, L_\alpha v)^p \, d\alpha \\ \le\; & [h(L_{\beta^+_{k-1}} u, A_k) \vee h(L_{\alpha_k} u, A_k)]^p (\alpha_k - \beta_{k-1}) \\ & \quad + \int_{\alpha_k}^{\beta_k} (\|L_\alpha u\| + \|A_k\|)^p \, d\alpha \\ \le\; & (\varepsilon/4)^p (\beta_k - \beta_{k-1}) + \int_{\alpha_k}^{\beta_k} (\|L_\alpha u\| + \|A_k\|)^p \, d\alpha \\ & \quad \text{by (2) and (4)}. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} & d_p^p(u,v) \\ =\; & \int_0^\beta h(L_\alpha u, A_1)^p \, d\alpha + \sum_{k=1}^m \int_{\beta_{k-1}}^{\beta_k} h(L_\alpha u, A_k)^p \, d\alpha \\ <\; & (\varepsilon/2)^p (1+\beta) + (\varepsilon/4)^p (1-\beta) \\ & \quad + \sum_{k=1}^m \int_{\alpha_k}^{\beta_k} (\|L_\alpha u\| + \|A_1\|)^p \, d\alpha \\ \le\; & 2\varepsilon^p \quad \text{by (3)}. \end{aligned}$$

This completes the proof. □

We note that $\mathcal{F} \subset F_\infty(R^n)$ in the proof of Theorem 3.5. This means that $F_p(R^n)$ is the completion of $(F_\infty(R^n), d_p)$.

Remark. The results established in the above are valid even though R^n is replaced by any real separable Banach space.

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