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Some Fundamental Concepts in $(2, L)$-Fuzzy Topology Based on Complete Residuated Lattice-Valued Logic

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Abstract

In the present paper we introduce and study fundamental concepts in the framework of L-fuzzifying topology (so called $(2, L)$-fuzzy topology) as L-concepts where L is a complete residuated lattice. The concepts of $(2, L)$-derived, $(2, L)$-closure, $(2, L)$-interior, $(2, L)$-exterior and $(2, L)$-boundary operators are studied and some results on above concepts are obtained. Also, the concepts of an L-convergence of nets and an L-convergence of filters are introduced and some important results are obtained. Furthermore, we introduce and study bases and subbases in $(2, L)$-topology. As applications of our work the corresponding results (see [10–11]) are generalized and new consequences are obtained.

Key Words : L-fuzziying topology; convergence of nets; convergence of filters; bases; subbases and Complete residuated lattice.

1. Introduction

Recently [13] (see [2, 4, 6, 9]) the concept of (M, L)-fuzzy topology was appeared as a function $\tau : M^X \longrightarrow L$ where X is an ordinary set and M, L are some types of lattices.

The concept of $(2, L)$-fuzzy topology (L-fuzzifying topology) appeared in [2] by Höhle under the name "L-fuzzy topology" (cf. Definition 4.6, Proposition 4.11 in [2]). In the case of $L = I$ where I is the closed unit interval $[0, 1]$ the terminology "L-fuzzifying topology" traces back to Ying (cf. Definition 2.1 in [10]).

The main purpose of this paper is to introduce and study some fundamental concepts in $(2, L)$-fuzzy topology as L-concepts where L is a complete residuated lattice.

In [16], it was proved that the concept of complete residuated lattice (see [7, 12]) and the concept of strictly two-sided commutative quantale (see [3, 8]) are equivalent. Sometimes we need more conditions on L such as that the finite meet is distributive over arbitrary joins or the completely distributive law or the double negation law as we illustrate through this paper.

As applications of our work generalizations of the corresponding results in [10–11] are obtained and new consequences are obtained as we illustrate through this paper.

The contents of our paper are arranged as follows: In Section 2, we recall some basic definitions and results in complete residuated lattice and in $(2, L)$-fuzzy topology. In Section 3, we consider and study some properties of the concepts of $(2, L)$-derived, $(2, L)$-closure, $(2, L)$-interior, $(2, L)$-exterior and $(2, L)$-boundary operators in $(2, L)$-fuzzy topology. Section 4 is devoted to introduce and study an L-convergence of nets in $(2, L)$-fuzzy topology. In Section 5, we introduce and study an L-convergence of filters in $(2, L)$-topology. In Section 6, we introduce and study bases and subbases in $(2, L)$-fuzzy topology. Finally in Section 7 a conclusion is given to illustrate some applications of our work.

2. Preliminaries

First, we introduce the definition of complete residuated lattice.

Definition 2.1 [7, 12, 16]. A structure $(L, \vee, \wedge, *, \longrightarrow, \bot, \top)$ is called a complete residuated lattice if and only if

(1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,

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(2) $(L, *, \top)$ is a commutative monoid, i.e.,

(a) $*$ is a commutative and associative binary operation on L, and

(b) For every $a \in L, a * \top = a$,

(3) $\longrightarrow$ is a binary operation which couple with $*$ as: $a * b \le c$ if and only if $a \le b \longrightarrow c \ \ \forall a, b, c \in L$.

Definition 2.2 [3, 8]. A structure $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is called a strictly two-sided commutative quantale if and only if

(1) $(L, \vee, \wedge, \perp, \top)$ is a complete lattice whose greatest and least element are $\top, \perp$ respectively,

(2) $(L, *, \top)$ is a commutative monoid,

(3) (a) $*$ is distributive over arbitrary joins, i.e., $a * \bigvee_{j \in J} b_j = \bigvee_{j \in J} (a * b_j) \ \forall a \in L, \ \forall \{b_j | j \in J\} \subseteq L$,

(b) $\longrightarrow$ is a binary operation on L defined by: $a \longrightarrow b = \bigvee_{\lambda * a \le b} \lambda \ \ \forall a, b \in L$.

Theorem 2.3 [16]. A structure $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a complete residuated lattice if and only if it is a strictly two-sided commutative quantale.

Corollary 2.4 [16]. A structure $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a complete MV-algebra if and only if $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a complete residuated lattice satisfies the additional property

(MV) $(a \longrightarrow b) \longrightarrow b = a \vee b \ \ \forall a, b \in L$.

In the rest of the present paper we assume that L is a complete residuated lattice. Now, we recall the laws of completely distributive and double negation for L.

Definition 2.5 [1]. L is satisfies the completely distributive law if the following statement is satisfied: $\forall \{A_j | j \in J\} \subseteq 2^L$, where 2^L is the power set of L, we have $\bigwedge_{j \in J} \bigvee A_j = \bigvee_{f \in \prod_{j \in J} A_j} (\bigwedge_{j \in J} f(j))$.

Note that if L satisfies the completely distributive law will satisfies that finite meet is distributive over arbitrary joins but the converse not true

Definition 2.6 [4]. L is satisfies the double negation law if the follows statement is satisfied: $(a \longrightarrow \perp) \longrightarrow \perp = a \ \ \forall \ a \in L$.

Definition 2.7 [4]. Let $f, g \in L^X$. The L-equality between f and g is denoted by $[[f, g]]$ and defined as follows: $[[f, g]] = \bigwedge_{x \in X} \big((f(x) \longrightarrow g(x)) \wedge (g(x) \longrightarrow f(x))\big)$.

Definition 2.8 [15]. Let $f, g \in L^X$. The L-inclusion of f in g is denoted by $[[f, g[[$ and defined as follows: $[[f, g[[= \bigwedge_{x \in X} \big(f(x) \longrightarrow g(x)\big)$.

In the following we recall the concept of (M, L)-fuzzy topology and illustrate that the L-fuzzifying topology is in fact the $(2, L)$-fuzzy topology.

Definition 2.9 (Höhle [2], Höhle and Šostak [4], Kubiak [6], Šostak [9], [14]). An (M, L)-fuzzy topology is a mapping $\tau : M^X \longrightarrow L$ such that

(1) $\tau(1_X) = \tau(1_\emptyset) = \top$,

(2) $\tau(A \wedge B) \ge \tau(A) \wedge \tau(B) \ \forall A, B \in M^X$,

(3) $\tau(\bigvee_{j \in J} A_j) \ge \bigwedge_{j \in J} \tau(A_j) \ \forall \{A_j | j \in J\} \subseteq M^X$.

The pair (M^X, τ) is called an (M, L)-fuzzy topological space.

When $M = \{0, 1\}$, Definition 2.9 will reduce to that of $(2, L)$-fuzzy topology (L-fuzzifying topology).

Some basic concepts and results in $(2, L)$-fuzzy topology (L-fuzzifying topology) which are useful in the present paper are given as follows:

Definition 2.10 [15]. Let (X, τ) be a $(2, L)$-fuzzy topological space. The family of all $(2, L)$-closed sets will be denoted by $F_\tau \in L^{(2^X)}$, and defined as follows: $F_\tau(A) = \tau(X - A)$ where $X - A$ is the complement of A.

Definition 2.11 [3]. Let $x \in X$. The $(2, L)$-neighborhood system of x is denoted by $\varphi_x \in L^{(2^X)}$, and defined as follows: $\varphi_x(A) = \bigvee_{x \in B \subseteq A} \tau(B)$.

Remark 2.12. Höhle proved in Proposition 3.13 [3] that if L satisfies the completely distributive law, then $\tau(A) = \bigwedge_{x \in A} \varphi_x(A)$.

Proposition 2.13 [15]. Let (X, τ) be a $(2, L)$-fuzzy topological space and let $A, B \in 2^X$. Then $\forall x \in X$,

(1) $\varphi_x(X) = \top, \varphi_x(\emptyset) = \perp$,

(2) $A \subseteq B \Longrightarrow \varphi_x(A) \le \varphi_x(B)$,

(3) $\varphi_x(A \cap B) \le \varphi_x(A) \wedge \varphi_x(B)$,

(4) If the finite meet is distributive over arbitrary joins, then $\varphi_x(A \cap B) \ge \varphi_x(A) \wedge \varphi_x(B)$,

(5) $\varphi_x(A) \le \bigvee_{y \in X - B} (\varphi_y(A) \vee \varphi_x(B)) \ \forall B \in 2^X$.

Definition 2.14 [15]. The $(2, L)$-closure operator is denoted by $Cl_\tau \in (L^X)^{2^X}$, and defined as follows: $Cl_\tau(A)(x) = \varphi_x(X - A) \longrightarrow \perp$.

Proposition 2.15 [15]. Let (X, τ) be a $(2, L)$-topological space, then:

(1) If L satisfies the double negation law, then $\varphi_x(A) = Cl_\tau(X - A)(x) \longrightarrow \perp \ \forall A \in 2^X, \ \forall x \in X$,

(2) $Cl_\tau(\emptyset) = 1_\emptyset$ where $1_\emptyset \in L^X$ and defined as follows: $1_\emptyset(x) = \perp \ \ \forall x \in X$,

(3) $A \le Cl_\tau(A) \ \ \forall A \in 2^X$,

(4) If $A, B \in 2^X$, $A \subseteq B$, then $Cl_\tau(A) \leq Cl_\tau(B)$,

(5) If the finite meet is distributive over arbitrary joins, then $Cl_\tau(A \cup B) \leq Cl_\tau(A) \vee Cl_\tau(B) \ \ \forall A, B \in 2^X$.

3. $(2, L)$-derived, $(2, L)$-closure, $(2, L)$-interior, $(2, L)$-exterior and $(2, L)$-boundary operators in $(2, L)$-fuzzy topology

Definition 3.1. Let (X, τ) be a $(2, L)$-fuzzy topological space. The $(2, L)$-derived operator is denoted by $d_\tau \in (L^X)^{2^X}$, and defined as follows:

$$d_\tau(A)(x) = \bigwedge_{B \cap (A - \{x\}) = \emptyset} (\varphi_x(B) \longrightarrow \bot).$$

Lemma 3.2. For every $a, b \in L$ we have $\bigwedge_{j \in J} (a_j \longrightarrow b) = (\bigvee_{j \in J} a_j) \longrightarrow b$.

Proof. For every $a, b \in L$ we have

$$\begin{aligned} (\bigvee_{j \in J} a_j) \longrightarrow b &= \bigvee_{\lambda * \bigvee_{j \in J} a_j \leq b} \lambda = \bigvee_{\bigvee_{j \in J} (\lambda * a_j) \leq b} \lambda \\ &= \bigvee_{\forall j \in J,\ \lambda * a_j \leq b} \lambda = \bigvee_{\forall j \in J, \lambda \leq a_j \longrightarrow b} \lambda \\ &= \bigvee_{\lambda \leq \bigwedge_{j \in J} (a_j \longrightarrow b)} \lambda = \bigwedge_{j \in J} (a_j \longrightarrow b). \ \Box \end{aligned}$$

Lemma 3.3. Let (X, τ) be a $(2, L)$-fuzzy topological space. Then we have, $d_\tau(A)(x) = \varphi_x((X - A) \cup \{x\}) \longrightarrow \bot$.

Proof. From Lemma 3.2 and Proposition 2.13 (2) we have

$$\begin{aligned} d_\tau(A)(x) &= \bigwedge_{B \cap (A - \{x\}) = \emptyset} (\varphi_x(B) \longrightarrow \bot) \\ &= \Big(\bigvee_{B \cap (A - \{x\}) = \emptyset} \varphi_x(B) \Big) \longrightarrow \bot \\ &= \Big(\bigvee_{B \subseteq (X - A) \cup \{x\}} \varphi_x(B) \Big) \longrightarrow \bot \\ &= \varphi_x((X - A) \cup \{x\}) \longrightarrow \bot. \ \Box \end{aligned}$$

Lemma 3.4. For every $a \in L$ we have $a \leq (a \longrightarrow \bot) \longrightarrow \bot$.

Proof. For every $a \in L$ we obtain $a \longrightarrow \bot \leq a \longrightarrow \bot$ which implies that $(a \longrightarrow \bot) * a \leq \bot$ so that $a * (a \longrightarrow \bot) \leq \bot$. Hence $a \leq (a \longrightarrow \bot) \longrightarrow \bot$. $\Box$

Theorem 3.5. Let (X, τ) be a $(2, L)$-fuzzy topological space. Then we have

(1) $\big[\big[d_\tau(\emptyset), 1_\emptyset\big]\big] = \top$,

(2) If $A, B \in 2^X$, $A \subseteq B$, then $d_\tau(A) \leq d_\tau(B)$,

(3) $\forall A, B \in 2^X$, $d_\tau(A \cup B) \geq d_\tau(A) \vee d_\tau(B)$,

(4) If the finite meet is distributive over arbitrary joins, then $d_\tau(A \cup B) = d_\tau(A) \vee d_\tau(B) \ \ \forall A, B \in 2^X$,

(5) $F_\tau(A) \leq \big[\big[d_\tau(A), A\big[\big[$,

(6) If L satisfies the double negation law and the completely distributive law, then $F_\tau(A) = \big[\big[d_\tau(A), A\big[\big[$.

Proof. (1) From Lemma 3.3 we have

$$\begin{aligned} d_\tau(\emptyset)(x) &= \varphi_x((X - \emptyset) \cup \{x\}) \longrightarrow \bot \\ &= \varphi_x(X) \longrightarrow \bot \\ &= \top \longrightarrow \bot = \bot = 1_\emptyset(x) \ \forall x \in X. \end{aligned}$$

(2) If $A \subseteq B$, then from Lemma 3.3 and Proposition 2.13 (2) we have

$$\begin{aligned} d_\tau(A)(x) &= \varphi_x((X - A) \cup \{x\}) \longrightarrow \bot \\ &\leq \varphi_x((X - B) \cup \{x\}) \longrightarrow \bot = d_\tau(B)(x). \end{aligned}$$

(3) From Proposition 2.13 (3) we have

$$\begin{aligned} & d_\tau(A \cup B)(x) \\ &= \varphi_x\big((X - (A \cup B)) \cup \{x\}\big) \longrightarrow \bot \\ &= \varphi_x\big(((X - A) \cap (X - B)) \cup \{x\}\big) \longrightarrow \bot \\ &= \varphi_x\Big(((X - A) \cup \{x\}) \cap ((X - B) \cup \{x\})\Big) \longrightarrow \bot \\ &\geq \Big(\varphi_x\big((X - A) \cup \{x\}\big) \wedge \varphi_x\big((X - B) \cup \{x\}\big)\Big) \longrightarrow \bot \\ &= \Big(\varphi_x\big((X - A) \cup \{x\}\big) \longrightarrow \bot\Big) \vee \\ &\qquad \Big(\varphi_x\big((X - B) \cup \{x\}\big) \longrightarrow \bot\Big) \\ &= d_\tau(A) \vee d_\tau(B). \end{aligned}$$

(4) From Proposition 2.13 (4) the inequality in the proof of (3) above become equality so that the result hold. (5) From Lemma 3.3 and Lemma 3.4 we have

$$\begin{aligned} & \big[\big[d_\tau(A), A\big[\big[\\ &= \bigwedge_{x \in X} (d_\tau(A)(x) \longrightarrow A(x)) \\ &= \Big(\bigwedge_{x \in X - A} (d_\tau(A)(x) \longrightarrow \bot) \Big) \wedge \Big(\bigwedge_{x \in A} (d_\tau(A)(x) \longrightarrow \top) \Big) \\ &= \Big(\bigwedge_{x \in X - A} (d_\tau(A)(x) \longrightarrow \bot) \Big) \wedge \top \\ &= \bigwedge_{x \in X - A} ((\varphi_x((X - A) \cup \{x\}) \longrightarrow \bot) \longrightarrow \bot) \\ &\geq \bigwedge_{x \in X - A} \varphi_x((X - A) \cup \{x\}) = \bigwedge_{x \in X - A} \varphi_x(X - A) \\ &= \bigwedge_{x \in X - A} \ \bigvee_{x \in B \subseteq X - A} \tau(B) \geq \tau(X - A) = F_\tau(A). \end{aligned}$$

(6) The inequalities in the proof of (5) above become equalities from the double negation law and from Remark

2.12 (since L satisfies the completely distributive law) respectively so that the result hold. □

Definition 3.6. Let $A, B \in 2^X$. The binary crisp predicat $D: 2^X \times 2^X \longrightarrow \{\bot, \top\}$, called crisp jointness, is given as follows:

$$D(A,B) = \begin{cases} \top, & \text{if } A\cap B \neq \emptyset \\ \bot, & \text{if } A\cap B = \emptyset. \end{cases}$$

Theorem 3.7. Let (X,τ) be a $(2,L)$-fuzzy topological space. Then for any x, A,

(1) $Cl_\tau(A)(x) = \bigwedge_{x\notin B\supseteq A} (F_\tau(B) \longrightarrow \bot)$,

(2) $Cl_\tau(A)(x) = \bigwedge_{B\in 2^X} (\varphi_x(B) \longrightarrow D(A,B))$,

(3) $\big[\big[Cl_\tau(A), A\cup d_\tau(A)\big]\big] = \top$,

(4) $F_\tau(A) \leq \big[\big[A, Cl_\tau(A)\big]\big]$,

(5) If L satisfies the double negation law and the completely distributive law, then $F_\tau(A) = \big[\big[A, Cl_\tau(A)\big]\big]$.

Proof. (1) From Lemma 3.2 we have

$$\begin{aligned} Cl_\tau(A)(x) &= \varphi_x(X-A) \longrightarrow \bot \\ &= \Big(\bigvee_{x\in X-B\subseteq X-A} \tau(X-B)\Big) \longrightarrow \bot \\ &= \Big(\bigvee_{x\notin B\supseteq A} F_\tau(B)\Big) \longrightarrow \bot \\ &= \bigwedge_{x\notin B\supseteq A} (F_\tau(B) \longrightarrow \bot). \end{aligned}$$

(2) From Lemma 3.2 we have

$$\begin{aligned} &\bigwedge_{B\in 2^X} (\varphi_x(B) \longrightarrow D(A,B)) \\ &= \Big(\bigwedge_{B\in 2^X,\, A\cap B=\emptyset} (\varphi_x(B) \longrightarrow D(A,B))\Big) \wedge \\ &\qquad \Big(\bigwedge_{B\in 2^X,\, A\cap B\neq\emptyset} (\varphi_x(B) \longrightarrow D(A,B))\Big) \\ &= \Big(\bigwedge_{A\cap B=\emptyset} (\varphi_x(B) \longrightarrow \bot)\Big) \wedge \\ &\qquad \Big(\bigwedge_{A\cap B\neq\emptyset} (\varphi_x(B) \longrightarrow \top)\Big) \\ &= \bigwedge_{A\cap B=\emptyset} (\varphi_x(B) \longrightarrow \bot) \wedge \top \\ &= \bigwedge_{B\subseteq X-A} (\varphi_x(B) \longrightarrow \bot) \\ &= \Big(\bigvee_{B\subseteq X-A} \varphi_x(B)\Big) \longrightarrow \bot \\ &= \varphi_x(X-A) \longrightarrow \bot = Cl_\tau(A)(x). \end{aligned}$$

(3) If $x \in A$, then from Proposition 2.15 (3) we have $(A\cup d_\tau(A))(x) = \top = Cl_\tau(A)(x)$. Now suppose $x \notin A$. Then from Lemma 3.3 we have

$$\begin{aligned} (A\cup d_\tau(A))(x) &= A(x) \vee d_\tau(A)(x) \\ &= d_\tau(A)(x) \\ &= \varphi_x((X-A)\cup\{x\}) \longrightarrow \bot \\ &= \varphi_x(X-A) \longrightarrow \bot = Cl_\tau(A)(x). \end{aligned}$$

(4) From Proposition 2.15 (3) and Lemma 3.4 we have

$$\begin{aligned} \big[\big[A, Cl_\tau(A)\big]\big] &= \big[\big[A, Cl_\tau(A)\big[\big[\wedge \big[\big[Cl_\tau(A), A\big[\big[\\ &= \top \wedge \big[\big[Cl_\tau(A), A\big[\big[\\ &= \bigwedge_{x\in X} (Cl_\tau(A)(x) \longrightarrow A(x)) \\ &= \Big(\bigwedge_{x\in X-A} (Cl_\tau(A)(x) \longrightarrow \bot)\Big) \wedge \\ &\qquad \Big(\bigwedge_{x\in A} (Cl_\tau(A)(x) \longrightarrow \top)\Big) \\ &= \Big(\bigwedge_{x\in X-A} (Cl_\tau(A)(x) \longrightarrow \bot)\Big) \wedge \top \\ &= \bigwedge_{x\in X-A} ((\varphi_x(X-A) \longrightarrow \bot) \longrightarrow \bot) \\ &\geq \bigwedge_{x\in X-A} \varphi_x(X-A) \\ &= \bigwedge_{x\in X-A} \bigvee_{x\in B\subseteq X-A} \tau(B) \\ &\geq \tau(X-A) = F_\tau(A). \end{aligned}$$

(5) The inequalities in the proof of (4) above become equalities from the double negation law and from Remark 2.12 (since L satisfies the completely distributive law) respectively so that the result hold. □

From Proposition 2.15 (3) and Theorem 3.7 (4), (5) we have the following result.

Corollary 3.8. Let (X,τ) be a $(2,L)$-fuzzy topological space. Then we have

(1) $F_\tau(A) \leq \big[\big[Cl_\tau(A), A\big[\big[$, and

(2) If L satisfies the double negation law and the completely distributive law, then $F_\tau(A) = \big[\big[Cl_\tau(A), A\big[\big[$.

Definition 3.9. Let (X,τ) be a $(2,L)$-fuzzy topological space and $A \subseteq X$. The $(2,L)$-interior operator is denoted by $Int_\tau \in (L^X)^{2^X}$, and defined as follows:

$Int_\tau(A)(x) = \varphi_x(A)$.

Definition 3.10. Let $A, B \in 2^X$. The binary crisp predicat $\subseteq: 2^X \times 2^X \longrightarrow \{\bot, \top\}$, called crisp inclusion, is given as follows:

$$\subseteq(A,B) = \begin{cases} \top, & \text{if } A\subseteq B \\ \bot, & \text{if } A\nsubseteq B. \end{cases}$$

Theorem 3.11. Let (X,τ) be a $(2,L)$-fuzzy topological space. Then for any x, A, B, we have

(1) $\big[\big[Int_\tau(A), Cl_\tau(X-A)\longrightarrow\bot\big[\big[=\top$,

(2) If L satisfies the double negation law, then $\big[\big[Int_\tau(A), Cl_\tau(X-A)\longrightarrow\bot\big]\big]=\top$,

(3) $\big[\big[Int_\tau(X), X\big]\big]=\top$,

(4) $\big[\big[Int_\tau(A), A\big[\big[=\top$,

(5) $(\tau(B)\wedge(\subseteq(B,A)))\leq\big[\big[B, Int_\tau(A)\big[\big[$,

(6) $\tau(A)\leq\big[\big[A, Int_\tau(A)\big]\big]$,

(7) If L satisfies the completely distributive law, then $\tau(A)=\big[\big[A, Int_\tau(A)\big]\big]$,

(8) $Int_\tau(A)(x)\leq A(x)\wedge(d_\tau(X-A)(x)\longrightarrow\bot)$,

(9) If L satisfies the double negation law, then $Int_\tau(A)(x)=A(x)\wedge(d_\tau(X-A)(x)\longrightarrow\bot)$,

(10) If $A\subseteq B$, then $\big[\big[Int_\tau(A), Int_\tau(B)\big[\big[=\top$,

(11) $Int_\tau(A\cap B)\leq Int_\tau(A)\wedge Int_\tau(B)$,

(12) If the finite meet is distributive over arbitrary joins, then $Int_\tau(A\cap B)\geq Int_\tau(A)\wedge Int_\tau(B)$.

Proof. (1) From Lemma 3.4 we have $Cl_\tau(X-A)(x)\longrightarrow\bot=((\varphi_x(A)\longrightarrow\bot)\longrightarrow\bot)\geq\varphi_x(A)=Int_\tau(A)$.

(2) Since the double negation law is satisfied, the inequality in the proof of (1) above becomes equality so that the result hold.

(3) $Int_\tau(X)=\varphi_x(X)=\top$.

(4) Using Proposition 2.15 (3) and (1) above we have

$$\begin{aligned}Int_\tau(A)(x) &\leq Cl_\tau(X-A)(x)\longrightarrow\bot\\ &\leq (X-A)(x)\longrightarrow\bot\\ &=(A\longrightarrow\bot)(x)\longrightarrow\bot\\ &\leq(A(x)\longrightarrow\bot)\longrightarrow\bot=A(x).\end{aligned}$$

(5) If $B\not\subseteq A$, then the result holds. Now suppose $B\subseteq A$. Then from Proposition 2.13 (2), we have

$$\begin{aligned}\big[\big[B, Int_\tau(A)\big[\big[&=\bigwedge_{x\in B}Int_\tau(A)(x)\\ &=\bigwedge_{x\in B}\varphi_x(A)\geq\bigwedge_{x\in B}\varphi_x(B)\\ &\geq\tau(B)=(\tau(B)\wedge(\subseteq(B,A))).\end{aligned}$$

(6) From (4) above we have

$$\begin{aligned}\big[\big[A, Int_\tau(A)\big]\big] &=\big[\big[A, Int_\tau(A)\big[\big[\ \wedge\ \big[\big[Int_\tau(A), A\big[\big[\\ &=\big[\big[A, Int_\tau(A)\big[\big[\\ &=\bigwedge_{x\in A}Int_\tau(A)(x)\\ &=\bigwedge_{x\in A}\varphi_x(A)\geq\tau(A).\end{aligned}$$

(7) From Remark 2.12 the inequality in the proof of (6) above becomes equality so that the result hold.

(8) If $x\notin A$, then $\varphi_x(A)=\bot$. Hence, $Int_\tau(A)(x)=\bot=A(x)\wedge(d_\tau(X-A)(x)\longrightarrow\bot)$. If $x\in A$, then from Lemma 3.3 and Lemma 3.4 we have

$$\begin{aligned}&A(x)\wedge(d_\tau(X-A)(x)\longrightarrow\bot)\\ &=d_\tau(X-A)(x)\longrightarrow\bot\\ &=(\varphi_x(A\cup\{x\})\longrightarrow\bot)\longrightarrow\bot\geq\varphi_x(A)=Int_\tau(A)(x).\end{aligned}$$

(9) Since the double negation law is satisfied, the inequality in the proof of (8) above becomes equality so that the result hold.

(10) Follows from Proposition 2.13 (2).

(11) Follows from Proposition 2.13 (3).

(12) Follows from Proposition 2.13 (4). □

From Theorem 3.11 (4), (6) and (7) we have the following result.

Corollary 3.12. Let (X,τ) be a $(2,L)$-fuzzy topological space. Then we have

(1) $\tau(A)\leq\big[\big[A, Int_\tau(A)\big[\big[$, and

(7) If L satisfies the completely distributive law, then $\tau(A)=\big[\big[A, Int_\tau(A)\big[\big[$.

Definition 3.13. Let (X,τ) be a $(2,L)$-fuzzy topological space. The $(2,L)$-exterior operator is denoted by $Ext_\tau(A)\in(L^X)^{2^X}$, and defined as follows: $Ext_\tau(A)(x)=\varphi_x(X-A)$.

Theorem 3.14. Let (X,τ) be a $(2,L)$-fuzzy topological space. Then for any x, A, B, we have

(1) $\big[\big[Ext_\tau(A), Cl_\tau(A)\longrightarrow\bot\big[\big[=\top$,

(2) If L satisfies the double negation law, then $\big[\big[Ext_\tau(A), Cl_\tau(A)\longrightarrow\bot\big]\big]=\top$,

(3) $\big[\big[Ext_\tau(\emptyset), X\big]\big]=\top$,

(4) $\big[\big[Ext_\tau(A), X-A\big[\big[=\top$,

(5) $(F_\tau(B)\wedge(\subseteq(A,B)))\leq\big[\big[X-B, Ext_\tau(A)\big[\big[$,

(6) $F_\tau(A)\leq\big[\big[X-A, Ext_\tau(A)\big]\big]$,

(7) If L satisfies the completely distributive law, then $F_\tau(A)=\big[\big[X-A, Ext_\tau(A)\big]\big]$,

(8) $Ext_\tau(A)(x)\leq(X-A)(x)\wedge(d_\tau(A)(x)\longrightarrow\bot)$,

(9) If L satisfies the double negation law, then $Ext_\tau(A)(x)=(X-A)(x)\wedge(d_\tau(A)(x)\longrightarrow\bot)$,

(10) If $A\subseteq B$, then $\big[\big[Ext_\tau(B), Ext_\tau(A)\big[\big[=\top$,

(11) $Ext_\tau(A\cup B)\leq Ext_\tau(A)\wedge Ext_\tau(B)$,

(12) If the finite meet is distributive over arbitrary joins, then $Ext_\tau(A\cup B)\geq Ext_\tau(A)\wedge Ext_\tau(B)$,

(13) $\big[\big[Int_\tau(A), Ext_\tau(X-A)\big]\big]=\top$.

Proof. The proof is obtained from Theorem 3.11 in a straightforward manner. □

Definition 3.15. For any $A\subseteq X$, the $(2,L)$-boundary operator is denoted by $b_\tau\in(L^X)^{2^X}$, and defined as follows: $b_\tau(A)(x)=\big(\varphi_x(A)\longrightarrow\bot\big)\wedge\big(\varphi_x(X-A)\longrightarrow\bot\big)$.

Lemma 3.16. Let (X,τ) be a $(2,L)$-fuzzy topological space. Then for any x, A,

$b_\tau(A)(x) = \bigwedge_{B\in 2^X} \big(\varphi_x(B) \longrightarrow D(B,A)\wedge D(B,(X-A))\big)$.

Proof. Applying Proposition 2.13 (2), we have

$$\begin{aligned}
&\bigwedge_{B\in 2^X} \big(\varphi_x(B) \longrightarrow D(B,A)\wedge D(B,(X-A))\big)\\
&=\Big(\bigwedge_{B\in 2^X,\ A\cap B\neq\emptyset,\ B\cap X-A=\emptyset} \big(\varphi_x(B)\longrightarrow D(B,A)\wedge D(B,(X-A))\big)\Big)\\
&\wedge\Big(\bigwedge_{B\in 2^X,\ A\cap B\neq\emptyset,\ B\cap X-A\neq\emptyset} \big(\varphi_x(B)\longrightarrow D(B,A)\wedge D(B,(X-A))\big)\Big)\\
&\wedge\Big(\bigwedge_{B\in 2^X,\ A\cap B=\emptyset,\ B\cap X-A=\emptyset} \big(\varphi_x(B)\longrightarrow D(B,A)\wedge D(B,(X-A))\big)\Big)\\
&\wedge\Big(\bigwedge_{B\in 2^X,\ A\cap B=\emptyset,\ B\cap X-A\neq\emptyset} \big(\varphi_x(B)\longrightarrow D(B,A)\wedge D(B,(X-A))\big)\Big)\\
&=\Big(\bigwedge_{B\in 2^X,\ A\cap B\neq\emptyset,\ B\cap X-A=\emptyset} \big(\varphi_x(B)\longrightarrow \top\wedge\bot\big)\Big)\\
&\wedge\Big(\bigwedge_{B\in 2^X,\ A\cap B\neq\emptyset,\ B\cap X-A\neq\emptyset} \big(\varphi_x(B)\longrightarrow \top\wedge\top\big)\Big)\\
&\wedge\Big(\bigwedge_{B\in 2^X,\ A\cap B=\emptyset,\ B\cap X-A=\emptyset} \big(\varphi_x(B)\longrightarrow \bot\wedge\bot\big)\Big)\\
&\wedge\Big(\bigwedge_{B\in 2^X,\ A\cap B=\emptyset,\ B\cap X-A\neq\emptyset} \big(\varphi_x(B)\longrightarrow \bot\wedge\top\big)\Big)\\
&=\Big(\bigwedge_{B\subseteq A}(\varphi_x(B)\longrightarrow\bot)\Big)\wedge\top\wedge\top\wedge \Big(\bigwedge_{B\subseteq X-A}(\varphi_x(B)\longrightarrow\bot)\Big)\\
&=\Big(\bigvee_{B\subseteq A}\varphi_x(B)\longrightarrow\bot\Big)\wedge\Big(\bigvee_{B\subseteq X-A}\varphi_x(B)\longrightarrow\bot\Big)\\
&=\Big(\varphi_x(A)\longrightarrow\bot\Big)\wedge\Big(\varphi_x(X-A)\longrightarrow\bot\Big)\\
&=b_\tau(A)(x).\ \square
\end{aligned}$$

Theorem 3.17. Let (X,τ) be a $(2,L)$-fuzzy topological space. Then for any A,

(1) (a) $\big[\big[b_\tau(X),1_\emptyset\big]\big]=\top$, (b) $\big[\big[b_\tau(\emptyset),1_\emptyset\big]\big]=\top$,

(2) (a) $\big[\big[b_\tau(A),Cl_\tau(A)\cap Cl_\tau(X-A)\big]\big]=\top$, (b) $b_\tau(A)=b_\tau(X-A)$,

(3) $\big[\big[Int_\tau(A)\cup Ext_\tau(A),b_\tau(A)\longrightarrow\bot\big[\big[=\top$,

(4) If L satisfies the double negation law, then $\big[\big[Int_\tau(A)\cup Ext_\tau(A),b_\tau(A)\longrightarrow\bot\big]\big]=\top$,

(5) (a) $\big[\big[Cl_\tau(A),A\cup b_\tau(A)\big]\big]=\top$, (b) $F_\tau(A)\le\big[\big[b_\tau(A),A\big[\big[$,

(6) (a) $\big[\big[Int_\tau(A),A\cap(b_\tau(A)\longrightarrow\bot)\big[\big[=\top$, (b) $\tau(A)\le\big[\big[b_\tau(A)\cap A,1_\emptyset\big]\big]$,

(7) If L satisfies the double negation law, then $\big[\big[Int_\tau(A),A\cap(b_\tau(A)\longrightarrow\bot)\big]\big]=\top$.

Proof. (1) (a) From Proposition 2.13 (1) we have

$$\begin{aligned}
b_\tau(X)(x)&=\big(\varphi_x(X)\longrightarrow\bot\big)\wedge\big(\varphi_x(\emptyset)\longrightarrow\bot\big)\\
&=(\top\longrightarrow\bot)\wedge(\bot\longrightarrow\bot)\\
&=\bot\wedge\top=\bot=1_\emptyset(x).
\end{aligned}$$

(1) (b) The proof is similar to (1) (a).

(2) (a)
$$\begin{aligned}
&(Cl_\tau(A)\cap Cl_\tau(X-A))(x)\\
&=Cl_\tau(A)(x)\wedge Cl_\tau(X-A)(x)\\
&=\big(\varphi_x(A)\longrightarrow\bot\big)\wedge\big(\varphi_x(X-A)\longrightarrow\bot\big)\\
&=b_\tau(A)(x).
\end{aligned}$$

(2) (b) Obvious.

(3) From (2) above and Lemma 3.4 we have

$$\begin{aligned}
&b_\tau(A)(x)\longrightarrow\bot\\
&=(Cl_\tau(A)\cap Cl_\tau(X-A))(x)\longrightarrow\bot\\
&=\big(Cl_\tau(A)(x)\longrightarrow\bot\big)\vee\big(Cl_\tau(X-A)(x)\longrightarrow\bot\big)\\
&=\big((\varphi_x(X-A)\longrightarrow\bot)\longrightarrow\bot\big)\vee\big((\varphi_x(A)\longrightarrow\bot)\longrightarrow\bot\big)\\
&\ge\varphi_x(X-A)\vee\varphi_x(A)\\
&=Ext_\tau(A)(x)\vee Int_\tau(A)(x)=(Int_\tau(A)\cup Ext_\tau(A))(x).
\end{aligned}$$

(4) Under the condition that L satisfies the double negation law, the inequality in the proof of (3) above becomes equality so that the result holed.

(5) (a) If $x\in A$, then from Proposition 2.15 (3) we have $Cl_\tau(A)(x)=(A\cup b_\tau(A))(x)=\top$. If $x\notin A$, then

$$\begin{aligned}
(A\cup b_\tau(A))(x)&=b_\tau(A)(x)\\
&=\big(\varphi_x(A)\longrightarrow\bot\big)\wedge\big(\varphi_x(X-A)\longrightarrow\bot\big)\\
&=\varphi_x(X-A)\longrightarrow\bot=Cl_\tau(A)(x).
\end{aligned}$$

(5) (b) From Corollary 3.8 (1) and Theorems 3.7 (3) we have

$$F_\tau(A)\le\big[\big[Cl_\tau(A),A\big[\big[=\big[\big[A\cup b_\tau(A),A\big[\big[\le\big[\big[b_\tau(A),A\big[\big[.$$

(6) (a) If $x\notin A$, then $Int_\tau(A)=A(x)\cap(b_\tau(A)(x)\longrightarrow\bot)=\bot$. If $x\in A$, then

$$\begin{aligned}
&A(x)\cap(b_\tau(A)(x)\longrightarrow\bot)\\
&=b_\tau(A)(x)\longrightarrow\bot\\
&=\big((\varphi_x(A)\longrightarrow\bot)\wedge\big(\varphi_x(X-A)\longrightarrow\bot\big)\big)\longrightarrow\bot\\
&=\big(\varphi_x(A)\longrightarrow\bot\big)\longrightarrow\bot\\
&\ge\varphi_x(A)=Int_\tau(A)(x).
\end{aligned}$$

(6) (b) From (6) (a) and Corollary 3.12 (1) we obtain

$$
\begin{aligned}
&\big[\big[b_\tau(A)\cap A,1_\emptyset\big]\big]\\
&=\big[\big[b_\tau(A)\cap A,1_\emptyset\big[\big[\wedge\big[\big[1_\emptyset,b_\tau(A)\cap A\big[\big[\\
&=\big[\big[b_\tau(A)\cap A,1_\emptyset\big[\big[\wedge\top\\
&=\bigwedge_{x\in X}\big((b_\tau(A)\cap A)(x)\longrightarrow\bot\big)\\
&=\bigwedge_{x\in A}\big(b_\tau(A)(x)\longrightarrow\bot\big)\\
&\geq\bigwedge_{x\in A}Int_\tau(A)(x)\\
&=\bigwedge_{x\in X}\big(A(x)\longrightarrow Int_\tau(A)(x)\big)\\
&=\big[\big[A,Int_\tau(A)\big[\big[\geq\tau(A).
\end{aligned}
$$

(7) Since L satisfies the double negation law, the inequality in the proof of (6) (a) above becomes equality so that the result holed. □

4. L-convergence of nets in $(2,L)$-fuzzy topology

Definition 4.1. Let (X,τ) be a $(2,L)$-fuzzy topological space. The class of all nets in X is denoted by $N(X)=\{S|S:D\longrightarrow X$, where $(D,\geq)$ is a directed set$\}$.

Definition 4.2. The binary L-predicates $\rhd,\propto\in L^{(N(X)\times X)}$, are defined as follows:

$$S\rhd x=\bigwedge_{S\not\subsetneqq A}(\varphi_x(A)\longrightarrow\bot),$$

$$S\propto x=\bigwedge_{S\not\lessapprox A}(\varphi_x(A)\longrightarrow\bot),\quad S\in N(X)$$

where $S\rhd x$ stands for the degree in L to which S is an L-convergent to x and $S\propto x$ stands for the degree in L to which x is an L-accumulation point of S. Also, $\subsetneqq$ and $\lessapprox$ are the binary crisp predicates " almost in " and " often in" respectively.

Definition 4.3. The L-sets lim S, adh $S\in L^X$ defined as follows:

lim $S(x)=S\rhd x$ and adh $S(x)=S\propto x$

are called L-limit and L-adherence of S, respectively.

Definition 4.4. Let $S,T\in N(X)$ The binary crisp predicat $<:2^X\times 2^X\longrightarrow\{\bot,\top\}$, is given as follows:

$$<(T,S)=\begin{cases}\top, & \text{if } T<S,\\ \bot, & \text{if } T\not< S,\end{cases}$$

where $T<S$ stand for T is a subnet of S.

Theorem 4.5. Let (X,τ) be a $(2,L)$-fuzzy topological space. Then we have

(1) $d_\tau(A)(x)\geq\bigvee_{S\in N(X)}((\subseteq(S,A-\{x\}))\wedge(S\rhd x))$,

(2) If L is totally ordered and that the finite meet is distributive over arbitrary joins, then $d_\tau(A)(x)\leq\bigvee_{S\in N(X)}((\subseteq(S,A-\{x\}))\wedge(S\rhd x))$,

(3) $Cl_\tau(A)(x)\geq\bigvee_{S\in N(X)}((\subseteq(S,A))\wedge(S\rhd x))$,

(4) If L is totally ordered and that the finite meet is distributive over arbitrary joins, then $Cl_\tau(A)(x)=\bigvee_{S\in N(X)}((\subseteq(S,A))\wedge(S\rhd x))$,

(5) If L satisfies the double negation law, then $F_\tau(A)\leq\bigwedge_{S\in N(X)}((\subseteq(S,A))\longrightarrow\big[\big[\text{lim}S,A\big[\big[)$,

(6) If L is totally ordered and satisfies the double negation law, the completely distributive law, then $F_\tau(A)=\bigwedge_{S\in N(X)}((\subseteq(S,A))\longrightarrow\big[\big[\text{lim}S,A\big[\big[)$,

(7) $\bigvee_{T\in N(X)}((<(T,S))\wedge(T\rhd x))\leq S\propto x$.

Proof. (1) We know that $S\rhd x=\bigwedge_{S\not\subsetneqq A}(\varphi_x(A)\longrightarrow\bot)$. Also,

$$
\begin{aligned}
&\bigvee_{S\in N(X)}((\subseteq(S,A-\{x\}))\wedge(S\rhd x))\\
&=\bigvee_{S\in N(X),\,S\subseteq A-\{x\}}\ \bigwedge_{B\in 2^X,\,S\not\subsetneqq B}(\varphi_x(B)\longrightarrow\bot).
\end{aligned}
$$

Since for any $S\in N(X)$ such that $S\subseteq A-\{x\}$, one can prove that $S\not\subsetneqq(X-A)\cup\{x\}$, as follows:
Suppose $S\subsetneqq(X-A)\cup\{x\}$. Then there exist $m\in D$ and $n\in D$ such that $n\geq m$ and $S(n)\in(X-A)\cup\{x\}$. So, $S(n)\notin X-((X-A)\cup\{x\})=A-\{x\}$. Thus $S\not\subseteq A-\{x\}$, we have a contradiction. Therefore from Lemma 3.3 we have

$$
\begin{aligned}
&\bigvee_{S\in N(X),\,S\subseteq A-\{x\}}\ \bigwedge_{B\in 2^X,\,S\not\subsetneqq B}(\varphi_x(B)\longrightarrow\bot)\\
&\leq\bigvee_{S\in N(X),\,S\subseteq A-\{x\}}\Big(\varphi_x((X-A)\cup\{x\})\longrightarrow\bot\Big)\\
&=\varphi_x((X-A)\cup\{x\})\longrightarrow\bot=d_\tau(A)(x).
\end{aligned}
$$

(2) We want to prove that $d_\tau(A)(x)\leq\bigvee_{S\in N(X)}((\subseteq(S,A-\{x\}))\wedge(S\rhd x))$. Now, if $d_\tau(A)(x)=\bot$ then the result holds. Suppose $d_\tau(A)(x)\geq\bot$ and suppose $(\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}=\{B\in 2^X|\varphi_x(B)>d_\tau(A)(x)\longrightarrow\bot\}$ is the strong $(d_\tau(A)(x)\longrightarrow\bot)$-cut of φ_x. Now, for any $B\in(\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$, we have $B\cap(A-\{x\})\neq\emptyset$ (Indeed, from Lemma 3.2 we have

$d_\tau(A)(x) = \big(\bigvee_{B\cap(A-\{x\})=\emptyset} \varphi_x(B)\big) \longrightarrow \bot$ then

$$\begin{aligned} &d_\tau(A)(x) \longrightarrow \bot \\ &= \Big(\big(\bigvee_{B\cap(A-\{x\})=\emptyset} \varphi_x(B)\big) \longrightarrow \bot\Big) \longrightarrow \bot \\ &\geq \bigvee_{B\cap(A-\{x\})=\emptyset} \varphi_x(B), \end{aligned}$$

i.e., for every $B \in 2^X$ such that $B \cap (A - \{x\}) = \emptyset$, $\varphi_x(B) \leq d_\tau(A)(x) \longrightarrow \bot$, then $\varphi_x(B) \not> d_\tau(A)(x) \longrightarrow \bot$ so that $B \notin (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$.). Then for any $B \in (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$, there exists $x_B \in B \cap (A - \{x\})$. In addition, since L is totally ordered one can prove that $\big((\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}, \supseteq\big)$ is a directed set from Proposition 2.13 (4). Now, we consider the net $S^* : (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)} \longrightarrow A - \{x\}$ defined as follows: $S^*(B) = x_B$ for every $B \in (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$. Then we have

$$\bigvee_{S\in N(X)} ((\subseteq (S, A-\{x\}))\wedge(S\rhd x)) \geq \bigwedge_{S^* \not\lesssim B} (\varphi_x(B) \longrightarrow \bot).$$

Now, one can show that if $S^* \not\lesssim B$ then $B \notin (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$ as follows: Suppose $B \in (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$. Then for any $C \in (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$ such that $C \subseteq B$, we have $S^*(C) = x_C \in C \subseteq B$. So, $S^* \lesssim B$. Therefore,

$$\begin{aligned} &\bigwedge_{S^* \not\lesssim B} (\varphi_x(B) \longrightarrow \bot) \\ &\geq \bigwedge_{B\notin(\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}} (\varphi_x(B) \longrightarrow \bot) \geq d_\tau(A)(x), \end{aligned}$$

because $(\varphi_x(B) \longrightarrow \bot) \geq d_\tau(A)(x)$ for any $B \notin (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$ (Indeed, if $B \notin (\varphi_x)_{(d_\tau(A)(x)\longrightarrow\bot)}$ i.e., $\varphi_x(B) \not> d_\tau(A)(x) \longrightarrow \bot$ then $\varphi_x(B) \leq d_\tau(A)(x) \longrightarrow \bot$, because L is totally ordered so that $\varphi_x(B) \longrightarrow \bot \geq (d_\tau(A)(x) \longrightarrow \bot) \longrightarrow \bot \geq d_\tau(A)(x)$.). i.e., $d_\tau(A)(x) \leq \bigvee_{S\in N(X)} ((\subseteq (S, A-\{x\}))\wedge (S \rhd x))$.

(3) If $x \in A$, then the result holds. If $x \notin A$, then from (1) above and Theorem 3.7 (3) we have

$$\begin{aligned} Cl_\tau(A)(x) &= d_\tau(A)(x) \\ &\geq \bigvee_{S\in N(X)} \Big((\subseteq (S, A-\{x\}))\wedge (S \rhd x)\Big) \\ &= \bigvee_{S\in N(X)} \Big((\subseteq (S, A))\wedge (S \rhd x)\Big), \end{aligned}$$

because $A = A - \{x\}$.

(4) Since L is totally ordered and that the finite meet is distributive over arbitrary joins, the inequality in the proof of (3) above becomes equality so that the result holed.

(5) From Lemmas 3.2, 3.4 and the double negation law we have

$$\begin{aligned} &\bigwedge_{S\in N(X)} ((\subseteq (S, A)) \longrightarrow [[\lim S, A[[) \\ &= \bigwedge_{S\subseteq A} \bigwedge_{x\in X-A} \Big(\big(\bigwedge_{S \not\lesssim B} (\varphi_x(B) \longrightarrow \bot)\big) \longrightarrow \bot\Big) \\ &= \bigwedge_{S\subseteq A} \bigwedge_{x\in X-A} \Big(\big((\bigvee_{S \not\lesssim B} \varphi_x(B)) \longrightarrow \bot\big) \longrightarrow \bot\Big) \\ &= \bigwedge_{S\subseteq A} \bigwedge_{x\in X-A} \bigvee_{S \not\lesssim B} \varphi_x(B). \end{aligned}$$

So, from (3) above, Corollary 3.8 (1) and the double negation law we have

$$\begin{aligned} F_\tau(A) &\leq [[Cl_\tau(A), A[[= [[X - A, Cl_\tau(A) \longrightarrow \bot[[\\ &= \bigwedge_{x\in X-A} \Big(Cl_\tau(A)(x) \longrightarrow \bot\Big) \\ &\leq \bigwedge_{x\in X-A} \Big(\big(\bigvee_{S\subseteq A} \bigwedge_{S \not\lesssim B} (\varphi_x(B) \longrightarrow \bot)\big) \longrightarrow \bot\Big) \\ &= \bigwedge_{x\in X-A} \bigwedge_{S\subseteq A} \Big(\big(\bigwedge_{S \not\lesssim B} (\varphi_x(B) \longrightarrow \bot)\big) \longrightarrow \bot\Big) \\ &= \bigwedge_{x\in X-A} \bigwedge_{S\subseteq A} \Big(\big((\bigvee_{S \not\lesssim B} \varphi_x(B)) \longrightarrow \bot\big) \longrightarrow \bot\Big) \\ &= \bigwedge_{x\in X-A} \bigwedge_{S\subseteq A} \bigvee_{S \not\lesssim B} \varphi_x(B) \\ &= \bigwedge_{S\in N(X)} \Big((\subseteq (S, A)) \longrightarrow [[\lim S, A[[\Big). \end{aligned}$$

(6) Under the conditions that L is totally ordered and satisfies the double negation law, the completely distributive law, the inequalities in the proof of (5) above becomes equalities so that the result holed.

(7) Set $\Re_S = \{A : S \not\lesssim A\}$ and $\beta_T = \{A : T \not\lesssim A\}$. Then for any $T < S$ (for the definition of the subnet see [5]), one can deduce that $\Re_S \subseteq \beta_T$ as follows:
Suppose $T = S \circ K$. If $S \not\lesssim A$, then there exists $m \in D$ such that $S(n) \notin A$ when $n \geq m$, where $\geq$ directs the domain D of S. Now, we will show that $T \not\lesssim A$. If not, then there exists $p \in E$ such that $T(q) \in A$ when $q \geq p$, where $\geq$ directs the domain E of T. Now, for $p \in E$ and $q \geq p$ we have $K(q) \geq m$, because $T < S$. Moreover, since $S \not\lesssim A$ and $K(q) \geq m$, we have $S(K(q)) \notin A$. But $S(K(q)) = T(q) \in A$. They are contrary. Hence,

$\Re_S \subseteq \beta_T$. Therefore

$$\begin{aligned}
&\bigvee_{T\in N(X)} \Big((<(T,S)) \wedge (T \rhd x)\Big)\\
&= \bigvee_{T<S} \bigwedge_{T \not\lesssim A} (\varphi_x(A) \longrightarrow \bot)\\
&= \bigvee_{T<S} \bigwedge_{A\in\beta_T} (\varphi_x(A) \longrightarrow \bot)\\
&\leq \bigwedge_{A\in\Re_S} (\varphi_x(A) \longrightarrow \bot)\\
&= \bigwedge_{S \not\lesssim A} (\varphi_x(A) \longrightarrow \bot) = S \propto x. \ \square
\end{aligned}$$

Definition 4.6. Let $S \in N(X)$ and $A, \in 2^X$. The binary crisp predicat $\lesssim : 2^X \times 2^X \longrightarrow \{\bot, \top\}$, is given as follows:

$$\lesssim(S,A) = \begin{cases} \top, & \text{if } \ S\lesssim A\\ \bot, & \text{if } \ S\not\lesssim A. \end{cases}$$

Lemma 4.7. Let (X, τ) be a $(2, L)$-fuzzy topological space. Then we have

$$S\rhd x = \bigwedge_{A\in 2^X} \Big(A(x)\wedge\tau(A) \longrightarrow \lesssim(S,A)\Big).$$

Proof. If $B \subseteq A$ and $S \not\lesssim A$, then $S \not\lesssim B$. Therefore

$$\begin{aligned}
S \rhd x &= \bigwedge_{S \not\lesssim A} (\varphi_x(A) \longrightarrow \bot)\\
&= \bigvee_{S \not\lesssim A} \varphi_x(A) \longrightarrow \bot\\
&= \bigvee_{S \not\lesssim A} \bigvee_{x\in B\subseteq A} \tau(B) \longrightarrow \bot\\
&\geq \bigvee_{S \not\lesssim B,\, x\in B} \tau(B) \longrightarrow \bot\\
&= \bigwedge_{S \not\lesssim B,\, x\in B} (\tau(B) \longrightarrow \bot)\\
&= \bigwedge_{B\in 2^X} \Big(B(x) \wedge \tau(B) \longrightarrow \lesssim(S,B)\Big)\\
&= \bigwedge_{A\in 2^X} \Big(A(x) \wedge \tau(A) \longrightarrow \lesssim(S,A)\Big).
\end{aligned}$$

Conversely, since $\varphi_x(A) \geq \tau(A)$, then we have

$$\begin{aligned}
&\bigwedge_{A\in 2^X} \Big(A(x) \wedge \tau(A) \longrightarrow \lesssim(S,A)\Big)\\
&= \bigwedge_{S \not\lesssim A,\, x\in A} (\tau(A) \longrightarrow \bot)\\
&\geq \bigwedge_{S \not\lesssim A} (\varphi_x(A) \longrightarrow \bot) = S \rhd x. \ \square
\end{aligned}$$

In the following theorem we prove that for a universal nets in $(2, L)$-fuzzy topological space $\lim S(x) = \text{adh}S(x)\ \forall x \in X$.

Theorem 4.8. If S is a universal net, then $\big[\big[\lim S, \text{adh}S\big]\big] = \top$.

Proof. For any net $S \in N(X)$ and any $A \in 2^X$ one can obtain that if $S \not\lesssim A$, then $S \not\lesssim A$. Suppose S is a universal net in X and $S \not\lesssim A$. Then, $S \lesssim X - A$. So $S \not\lesssim A$ (Indeed, $S \lesssim X - A$ if and only if there exists $m_1 \in D$ such that for every $n \in D, n \geq m_1, S(n) \in X - A$ if and only if there exists $m_1 \in D$ such that for every $n \in D, n \geq m_1, S(n) \notin A$ if and only if $S \not\lesssim A$.). Hence for any universal net S in X, we have

$$\begin{aligned}
\lim S(x) &= \bigwedge_{S \not\lesssim A} (\varphi_x(A) \longrightarrow \bot)\\
&= \bigwedge_{S \not\lesssim A} (\varphi_x(A) \longrightarrow \bot) = \text{adh}S(x). \ \square
\end{aligned}$$

5. L-convergence of filters in $(2, L)$-fuzzy topology

Definition 5.1. Let $F(X)$ be the set of all filters on X. The binary L-predicates $\rhd, \propto \in L^{(F(X)\times X)}$, are respectively defined as follows:

$$K \rhd x = \bigwedge_{A\in 2^X} (\varphi_x(A) \longrightarrow K(A)),$$

$$K \propto x = \bigwedge_{A\in 2^X} (K(A) \longrightarrow Cl_\tau(A)(x)),\ K \in F(X).$$

Definition 5.2. The L-sets $\lim K,\ \text{adh } K \in L^X$ defined as follows:

$\lim K(x) = K \rhd x$ and $\text{adh}K(x) = K \propto x$.

are called L-limit and L-adherence of K, respectively.

Theorem 5.3. (1) If $S \in N(X)$ and K^S is the filter corresponding to S,

i.e., $K^S = \{A : S \lesssim A\}$, then

(a) $\big[\big[\lim K^S, \lim S\big]\big] = \top$, and

(b) $\big[\big[\text{adh}K^S, \text{adh}S\big]\big] = \top$.

(2) If $K \in F(X)$ and S^K is the net corresponding to K,

i.e., $S^K : D \longrightarrow X, (x, A) \longmapsto x, (x, A) \in D$,

where $D = \{(x, A) : x \in A \in K\}$, $(x, A) \geq (y, B)$ if and only if $A \subseteq B$, then

(a) $\big[\big[\lim S^K, \lim K\big]\big] = \top$, and

(b) $\big[\big[\text{adh}S^K, \text{adh}K\big]\big] = \top$.

Proof. (1) (a) For any $x \in X$, we have

$$\begin{aligned}\lim K^S(x) &= \bigwedge_{A \notin K^S} (\varphi_x(A) \longrightarrow \bot) \\ &= \bigwedge_{S \not\underset{\sim}{\subset} A} (\varphi_x(A) \longrightarrow \bot) = \lim S(x).\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \mathrm{adh}K^S(x) &= \bigwedge_{A \in K^S} Cl_\tau(A)(x) \\ &= \bigwedge_{S \underset{\sim}{\subset} A} (\varphi_x(X-A) \longrightarrow \bot) \\ &= \bigwedge_{S \not\underset{\sim}{\subset} (X-A)} (\varphi_x(X-A) \longrightarrow \bot) \\ &= \mathrm{adh}S(x).\end{aligned}$$

(2) (a) First we prove that $S^K \underset{\sim}{\subset} A$ if and only if $A \in K$. If $A \in K$, then $A \neq \emptyset$ and so there exists at least an element $x \in A$. So for $(x,A) \in D$ and $(y,B) \in D$ such that $(y,B) \geq (x,A), B \subseteq A$ and so $S^K(y,B) = y \in B \subseteq A$. Thus $S^K \underset{\sim}{\subset} A$.

Conversely, suppose $S^K \underset{\sim}{\subset} A$. Then there exists $(y,B) \in D$ such that $(z,C) \geq (y,B)$ and we have $S^K(z,C) \in A$. So for every $z \in B, (z,B) \geq (y,B)$ and $S^K(z,B) = z \in A$ implies $B \subseteq A$. Then $A \in K$. Thus $A \notin K$ if and only if $S^K \not\underset{\sim}{\subset} A$. Now,

$$\begin{aligned}\lim S^K(x) &= S^K \rhd x \\ &= \bigwedge_{S^K \not\underset{\sim}{\subset} A} (\varphi_x(A) \longrightarrow \bot) \\ &= \bigwedge_{A \notin K} (\varphi_x(A) \longrightarrow \bot) = \lim K(x).\end{aligned}$$

(b) First we prove that $X - A \in K$ if and only if $S^K \not\underset{\sim}{\subset} A$. Suppose $S^K \not\underset{\sim}{\subset} A$. Then there exists $(z,B) \in D$ such that for every $(y,C) \in D$ with $(y,C) \geq (z,B), S^K(y,C) \notin A$. Now, for every $x \in B, (x,B) \geq (z,B)$ and $S^K(x,B) = x \notin A$, i.e., $B \cap A = \emptyset$ so $B \subseteq X - A$ and then $X - A \in K$.

Conversely, suppose $X - A \in K$ then $X - A \neq \emptyset$ and thus it contains at least an element x. Now, for any $(z,C) \in D$ such that $(z,C) \geq (x, X-A)$, one can have that $S^K(z,C) = z \notin A$. Hence, $S^K \not\underset{\sim}{\subset} A$. Now,

$$\begin{aligned}\mathrm{adh}S^K(x) &= S^K \propto x \\ &= \bigwedge_{S^K \not\underset{\sim}{\subset} A} (\varphi_x(A) \longrightarrow \bot) \\ &= \bigwedge_{X-A \in K} Cl_\tau(X-A) \\ &= \bigwedge_{B \in K} Cl_\tau(B)(x) = \mathrm{adh}K(x). \ \square\end{aligned}$$

6. Bases and subbases in $(2,L)$-fuzzy topology

Definition 6.1. Let (X,τ) a $(2,L)$-topological space. A map $\beta : 2^X \longrightarrow L$ is called a base of τ if and only if the following statements hold:

(1) $[[\beta, \tau[[= \top$, and

(2) $\varphi_x(A) \leq \bigvee_{x \in B \subseteq A} \beta(B) \ \forall A \in 2^X$.

Theorem 6.2. Let (X,τ) a $(2,L)$-topological space. Then we have

(1) If $\tau = \beta^{(\cup)}$, where $\beta^{(\cup)}(A) = \bigvee_{\bigcup_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \beta(B_\lambda) \quad \forall A \in 2^X$ then β is a base of τ, and

(2) If L satisfies the completely distributive law and β is a base of τ, then $\tau = \beta^{(\cup)}$.

Proof. (1) We assume $\tau = \beta^{(\cup)}$ and will prove that β is a base of τ. Since $\tau = \beta^{(\cup)}$, hence $[[\beta, \tau[[= \top$. Now, we will prove that for any $A \in 2^X$, $\varphi_x(A) \leq \bigvee_{x \in B \subseteq A} \beta(B)$. Assume that $x \in B \subseteq A$ and $\bigcup_{\lambda \in \Lambda} B_\lambda = B$, then there exists $\lambda_0 \in \Lambda$ such that $x \in B_{\lambda_0}$, and furthermore $\bigwedge_{\lambda \in \Lambda} \beta(B_\lambda) \leq \beta(B_{\lambda_0}) \leq \bigvee_{x \in B \subseteq A} \beta(B)$ so that $\bigvee_{\bigcup_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \beta(B_\lambda) \leq \bigvee_{x \in B \subseteq A} \beta(B)$. Hence,

$$\begin{aligned}\varphi_x(A) &= \bigvee_{x \in B \subseteq A} \tau(B) \\ &= \bigvee_{x \in B \subseteq A} \ \bigvee_{\bigcup_{\lambda \in \Lambda} B_\lambda = B} \ \bigwedge_{\lambda \in \Lambda} \beta(B_\lambda) \leq \bigvee_{x \in B \subseteq A} \beta(B).\end{aligned}$$

(2) Assume that β is a base of τ, and we want to prove that for every $A \in 2^X$,

$\tau(A) = \bigvee_{\bigcup_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \beta(B_\lambda)$. Since $\bigcup_{\lambda \in \Lambda} B_\lambda = A$, then $\tau(A) = \tau(\bigcup_{\lambda \in \Lambda} B_\lambda) \geq \bigwedge_{\lambda \in \Lambda} \tau(B_\lambda) \geq \bigwedge_{\lambda \in \Lambda} \beta(B_\lambda)$. Hence, $\tau(A) \geq \bigvee_{\bigcup_{\lambda \in \Lambda} B_\lambda = A} \bigwedge_{\lambda \in \Lambda} \beta(B_\lambda)$.

Conversely, we note that $\varphi_x(A) \leq \bigvee_{x \in B \subseteq A} \beta(B)$ and from Remark 2.12 we have

$$\begin{aligned}\tau(A) &= \bigwedge_{\lambda \in \Lambda} \varphi_x(A) \\ &\leq \bigwedge_{\lambda \in \Lambda} \ \bigvee_{x \in B \subseteq A} \beta(B) = \bigvee_{f \in \prod_{x \in A} M_x} \ \bigwedge_{\lambda \in \Lambda} \beta(f(x)),\end{aligned}$$

where $M_x = \{B|x \in B \subseteq A\}$. Since for any $f \in \prod_{x\in A} \beta_x$, $\bigcup_{x\in A} f(x) = A$, we have that $\tau(A) \leq \bigvee_{\bigcup_{\lambda\in\Lambda} B_\lambda = A} \bigwedge_{\lambda\in\Lambda} \beta(B_\lambda)$. □

Definition 6.3. Let (X, τ) a $(2, L)$-topological space. Then $\psi \in L^{(2^X)}$ is called a subbase of τ if $\psi^{(\cap_f)}$ is a base of τ where $\psi^{(\cap_f)}(A) = \bigvee_{\bigcap_{\lambda_1\in\Lambda_1 (\text{finite})} B_{\lambda_1} = A} \bigwedge_{\lambda_1\in\Lambda_1} \beta(B_{\lambda_1})\ \forall A \in 2^X$, where $\psi^{(\cap_f)}$ stand for the finite intersection of ψ.

7. Conclusion

(1) Let $L = [0, 1]$ and let $* \in [0, 1]^{([0,1]\times[0,1])}$ is defined as follows: $\alpha * \beta = max(0, \alpha + \beta - 1)$, then the structure $(L, \vee, \wedge, *, \longrightarrow, 0, 1)$ is a completely distributive complete MV-algebra so that Lemma 5.1 [10] (resp. Theorem 5.1 [10], Theorem 5.2 [10], Theorem 2.2 [11], Lemma 2.1 [11], Theorem 2.3 [11] Theorem 6.1 [10], Theorem 6.2 [10], Lemma 6.1 [10], Theorem 7.1 [10], Theorem 4.1 [10]) is obtained as a special case of Lemma 3.3 (resp. Theorem 3.5, Theorem 3.7, Theorem 3.11, Lemma 3.16, Theorem 3.17, Theorem 4.5, Theorem 4.8, Lemma 4.7, Theorem 5.3, Theorem 6.2) above.

(2) Let $L = [0, 1]$ and let $* \in [0, 1]^{([0,1]\times[0,1])}$ is defined as follows: $\alpha * \beta = \alpha\beta$. Then $(L, \vee, \wedge, *, \longrightarrow, 0, 1)$ is a completely distributive complete residuated lattice. Note that the double negation law is not satisfied since $(\alpha \longrightarrow 0) \longrightarrow 0 = 0 \longrightarrow 0 = 1 \neq \alpha$ if $\alpha \in (0, 1)$. Hence, Lemmas 3.2, 3.3, 3.4, 3.16, 4.7, Theorems 3.5 (1)–(5), 3.7 (1)–(4), 3.11 (1), (3)–(8) and (10)–(12), 3.14 (1), (3)–(8), (10)–(13), 3.17 (1)–(3), (5) and (6), 4.5 (1)–(4) and (7), 4.8, 5.3, 6.2 and Corollaries 3.8 (1), 3.12 are satisfied as corollaries from our results.

(3) If $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a complete MV-algebra, then Lemmas 3.2, 3.3, 3.4, 3.16, 4.7, Theorems 3.5 (1)–(3) and (5), 3.7 (1)–(4), 3.11 (1)–(6) and (8)–(11), 3.14 (1)–(6), (8)–(11) and (13), 3.17, 4.5 (1), (3), (5) and (7), 4.8, 5.3, 6.2 (1) and Corollaries 3.8 (1), 3.12 (1) are satisfied as corollaries from our results because from Corollary (1) [16], any complete MV-algebra, is a complete residuated lattice. Furthermore any complete $MV-$algebra satisfies the double negation law.

(4) If $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a complete MV-algebra, such that the finite meet is distributive over arbitrary joins, then Lemmas 3.2, 3.3, 3.4, 3.16, 4.7, Theorems 3.5 (1)–(5), 3.7 (1)–(4), 3.11 (1)–(6) and (8)–(12), 3.14 (1)–(6) and (8)–(13), 3.17, 4.5 (1), (3), (5) and (7), 4.8, 5.3, 6.2 (1) and Corollaries 3.8 (1), 3.12 (1) are satisfied as corollaries from our results.

(5) If $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a completely distributive complete $MV-$algebra, then Lemmas 3.2, 3.3, 3.4, 3.16, 4.7, Theorems 3.5, 3.7, 3.11, 3.14, 3.17, 4.5 (1), (3), (5) and (7), 4.8, 5.3, 6.2 and Corollaries 3.8, 3.12 are satisfied as corollaries from our results.

(6) If $(L, \vee, \wedge, *, \longrightarrow, \perp, \top)$ is a completely distributive complete $MV-$algebra and L is totally ordered then all results in Section 3, 4, 5 and 6 are satisfied as corollaries from our results.

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