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Fuzzy r-Quasi Open Set and Fuzzy r-Quasi Continuity

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Abstract

In this paper, we introduce the concept of fuzzy r-quasi open sets which are generalizations of fuzzy r-open sets, and obtain some basic properties of such fuzzy sets. Also we introduce and study the concepts of fuzzy r-quasi continuous mapping and fuzzy r-quasi open(closed) mapping.

Key Words: fuzzy quasi topological space, fuzzy r-quasi open set, fuzzy r-quasi continuous, fuzzy r-quasi open mapping, fuzzy r-quasi closed mapping

1. Introduction

Let X be a set and $I = [0, 1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively. A^c will denote the complement $1_X - A$ of a fuzzy set A of X

A Chang's fuzzy topological space [1] is an ordered pair (X, τ), where X is a non-empty set and $\tau \subseteq I^X$ satisfying the following conditions:

(O1) 0_X , $1_X \in \tau$;

(O2) $\forall A, B \in I^X$, if $A, B \in \tau$, then $(A \cap B) \in \tau$;

(O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, if $A_i \in \tau$, then $\cup_{i\in J} A_i \in \tau$.

A smooth topological space [4] is an ordered pair (X, τ), where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$;

(O2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$. Then the mapping $\tau : I^X \to I$ is called a *smooth topology* on X. The number $\tau(A)$ is called *the degree of openness* of A.

A mapping $\tau^* : I^X \to I$ is called *a smooth cotopology* [4] iff the following three conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$;

(C2) $\forall A, B \in I^X,\ \tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$;

(C3) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \tau^*(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \tau^*(A_i)$.

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Definition 1.1 ([3]). *A fuzzy quasi topological space* (simply, FQTS) is an ordered pair $(X, \mathcal{T})$, where X is a non-empty set and $\mathcal{T} : I^X \to I$ is a mapping satisfying the following conditions:

(QO1) $\mathcal{T}(0_X) = 1$;

(QO2) $\forall A, B \in I^X,\ \mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$;

(QO3) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \mathcal{T}(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \mathcal{T}(A_i)$.

Then the mapping $\mathcal{T} : I^X \to I$ is called a *fuzzy quasi topology* on X. The number $\mathcal{T}(A)$ is called *the degree of quasi openness* of A.

Chang's fuzzy topology $\Rightarrow$ smooth topology $\Rightarrow$ fuzzy quasi topology

Definition 1.2 ([3]). A mapping $\mathcal{T}^* : I^X \to I$ is called a *fuzzy quasi cotopology* if the following three conditions are satisfied:

(QC1) $\mathcal{T}^*(1_X) = 1$;

(QC2) $\forall A, B \in I^X,\ \mathcal{T}^*(A \cup B) \geq \mathcal{T}^*(A) \wedge \mathcal{T}^*(B)$;

(QC3) for every subfamily $\{A_i : i \in J\} \subseteq I^X,\ \mathcal{T}^*(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \mathcal{T}^*(A_i)$.

Then the mapping $\mathcal{T}^* : I^X \to I$ is called a *fuzzy quasi cotopology* on X. The number $\mathcal{T}^*(A)$ is called *the degree of quasi closedness* of A.

Theorem 1.3 ([3]). If $\mathcal{T}$ is a fuzzy quasi topology on X, then the mapping $\mathcal{T}^* : I^X \to I$, defined by $\mathcal{T}^*(A) = \mathcal{T}(A^c)$ where A^c denotes the complement of A, is a fuzzy quasi cotopology on X. And if $\mathcal{T}^*$ is a fuzzy quasi cotopology on X, then the mapping $\mathcal{T} : I^X \to I$, defined by $\mathcal{T}(A) = \mathcal{T}^*(A^c)$, is a fuzzy quasi topology on X.

2. Main Results

Definition 2.1. Let $(X, \mathcal{T})$ be a FQTS and $A \in I^X$. Then

(1) The r-*closure* of A, denoted by $qCl_r(A)$, is defined by

$qCl_r(A) = \cap\{K \in I^X : \mathcal{T}^*(K) \geq r, A \subseteq K\}$,

where $\mathcal{T}^*(K) = \mathcal{T}(K^c)$.

(2) The r-*interior* of A, denoted by $qInt_r(A)$, is defined by

$qInt_r(A) = \cup\{K \in I^X : \mathcal{T}(K) \geq r, K \subseteq A\}$.

A fuzzy set A is said to be *fuzzy r-quasi open* if $\mathcal{T}(A) \geq r$, A is said to be *fuzzy r-quasi closed* if $\mathcal{T}^*(A) \geq r$.

Theorem 2.2. Let $(X, \mathcal{T})$ be a FGTS and $A, B \in I^X$. Then

(1) $qInt_r(0_X) = 0_X$, $qCl_r(1_X) = 1_X$.

(2) $qInt_r(A) \subseteq A$, $A \subseteq qCl_r(A)$.

(3) $A \subseteq B \Rightarrow qInt_r(A) \subseteq qInt_r(B)$, $qCl_r(A) \subseteq qCl_r(B)$.

Proof. Obvious. □

Theorem 2.3. Let $(X, \mathcal{T})$ be a FGTS and $A \in I^X$. Then

(1) $qCl_r(A)^c = qInt_r(A^c)$.

(2) $qInt_r(A)^c = qCl_r(A^c)$.

Proof. (1) For $A \in I^X$,

$$\begin{aligned} qCl_r(A)^c &= (\cap\{K \in I^X : \mathcal{T}^*(K) \geq r, A \subseteq K\})^c \\ &= \cup\{K^c : K \in I^X, \tau(K^c) \geq r, K^c \subseteq A^c\} \\ &= \cup\{U \in I^X : \tau(U) \geq r, U \subseteq A^c\} \\ &= qInt_r(A^c). \end{aligned}$$

(2) It is similar to the proof of (1). □

Lemma 2.4. Let $(X, \mathcal{T})$ be a FQTS. The statements are hold:

(1) If $\mathcal{T}(A_i) \geq r$ for each $i \in J$, then $\mathcal{T}(\cup_{i \in J} A_i) \geq r$.

(2) If $\mathcal{T}^*(A_i) \geq r$ for each $i \in J$, $\mathcal{T}^*(\cap_{i \in J} A_i) \geq r$.

Proof. (1) For each $i \in J$, if $\mathcal{T}(A_i) \geq r$, then $\mathcal{T}(\cup_{i \in J} A_i) \geq \wedge_{i \in J} \mathcal{T}(A_i) \geq r$.

(2) It follows from definition of fuzzy quasi cotopology. □

From Lemma 2.4, the next theorem is easily obtained.

Theorem 2.5. Let $(X, \mathcal{T})$ be a FGTS and $A \in I^X$. Then

(1) A is fuzzy r-quasi open iff $A = qInt_r(A)$.

(2) A is fuzzy r-quasi closed iff $A = qCl_r(A)$.

Theorem 2.6. Let $(X, \mathcal{T})$ be a FQTS and $A, B \in I^X$. Then

(1) $qInt_r(qInt_r(A)) = qInt_r(A)$.

(2) $gCl_r(qCl_r(A))) = qCl_r(A)$.

Proof. It follows from Theorem 2.5. □

Definition 2.7. Let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a mapping on FQTS's. Then f is said to be *fuzzy r-quasi continuous* if for every $A \in I^Y$, we have

$$\mathcal{T}_2(A) \geq r \Rightarrow \mathcal{T}_1(f^{-1}(A)) \geq r.$$

Theorem 2.8. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be FQTS's. Then the following are equivalent:

(1) f is fuzzy r-quasi continuous.

(2) For every fuzzy r-quasi open set A in Y, $f^{-1}(A)$ is fuzzy r-quasi open in X.

(3) ${\mathcal{T}_2}^*(B) \geq r \Rightarrow {\mathcal{T}_1}^*(f^{-1}(B)) \geq r$ for $B \in I^Y$.

(4) For every fuzzy r-quasi closed set A in Y, $f^{-1}(A)$ is fuzzy r-quasi closed in X.

(5) $f(qCl_r(A)) \subseteq qCl_r(f(A)$ for $A \in I^X$,

(6) $qCl_r(f^{-1}(B)) \subseteq f^{-1}(qCl_r(B))$ for $B \in I^Y$.

(7) $f^{-1}(qInt_r(B)) \subseteq qInt_r(f^{-1}(B))$ for $B \in I^Y$.

Proof. (1) ⇒ (2) Let A be a fuzzy r-quasi open set. Then $\mathcal{T}_2(A) \geq r$ and so by fuzzy r-quasi continuity, $\mathcal{T}_1(f^{-1}(A)) \geq r$. Hence $f^{-1}(A)$ is fuzzy r-quasi open.

(2) ⇒ (3) For $B \in I^Y$, if ${\mathcal{T}_2}^*(B) \geq r$, then $\mathcal{T}_2(B^c) \geq r$, so B^c is fuzzy r-quasi open. By (2), $f^{-1}(B^c)$ is fuzzy r-quasi open, and this implies

$$\mathcal{T}_1(f^{-1}(B^c)) = \mathcal{T}_1((f^{-1}(B))^c) = {\mathcal{T}_1}^*(f^{-1}(B)) \geq r.$$

So ${\mathcal{T}_1}^*(f^{-1}(B)) \geq r$.

(3) ⇒ (4) Obvious.

(4) ⇒ (5) For $A \in I^X$,

$$\begin{aligned} f^{-1}(qCl_r f(A)) &= f^{-1}[\cap\{F \in I^Y : f(A) \subseteq F, \\ &\quad F \text{ is fuzzy } r\text{-quasi closed}\}] \\ &= \cap\{f^{-1}(F) \in I^X : A \subseteq f^{-1}(F), \\ &\quad f^{-1}(F) \text{ is fuzzy } r\text{-quasi closed}\}. \end{aligned}$$

Thus from definition of operator of closure on a FQTS, $qCl_r(A) \subseteq f^{-1}(qCl_r f(A))$. So $f(qCl_r(A)) \subseteq qCl_r f(A)$.

(5) ⇒ (6) Obvious.

(6) ⇒ (7) Obvious.

(7) ⇒ (1) For $B \in I^Y$, if $\mathcal{T}_2(B) \geq r$, then B is fuzzy r-quasi open, and

$$f^{-1}(B) = f^{-1}(qInt_r(B)) \subseteq qInt_r(f^{-1}(B)).$$

This implies $f^{-1}(B)$ is fuzzy r-quasi open, that is, $\mathcal{T}_1(f^{-1}(B)) \geq r$. Hence f is fuzzy r-quasi continuous. □

Definition 2.9. Let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a mapping on FQTS's. Then f is said to be *fuzzy r-quasi open* if for every fuzzy r-quasi open set A in X, $f(A)$ is fuzzy r-quasi open in Y.

Theorem 2.10. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be FQTS's. Then the following are equivalent:

(1) f is fuzzy r-quasi open.
(2) For $A \in I^X$, $\mathcal{T}_1(A) \geq r \Rightarrow \mathcal{T}_2(f(A)) \geq r$.
(3) $f(qInt_r(A)) \subseteq qInt_r(f(A))$ for $A \in I^X$.
(4) $qInt_r(f^{-1}(B)) \subseteq f^{-1}(qInt_r(B))$ for $B \in I^Y$.

Proof. (1) $\Leftrightarrow$ (2) It is obvious from definition of fuzzy r-quasi open set.

(1) $\Rightarrow$ (3) For $A \in I^X$, $qInt_r(A)$ is fuzzy r-quasi open. Since f is fuzzy r-quasi open, $f(qInt_r(A))$ is fuzzy r-quasi open. So

$$f(qInt_r(A)) = qInt_r(f(qInt_r(A))) \subseteq qInt_r(f(A)).$$

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (1) Let A be a fuzzy r-quasi open set. Then from (4), it follows

$$qInt_r(A) \subseteq qInt_r(f^{-1}(f(A))) \subseteq f^{-1}(qInt_r(f(A))).$$

Since $A = qInt_r(A)$, we have $f(A) \subseteq qInt_r(f(A))$, and hence from Theorem 2.5, $f(A)$ is fuzzy r-quasi open. □

Definition 2.11. Let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a mapping on FQTS's. Then f is said to be *fuzzy r-quasi closed* if for every fuzzy r-quasi closed set A in X, $f(A)$ is fuzzy r-quasi closed in Y.

Theorem 2.12. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be FQTS's. Then the following are equivalent:

(1) f is fuzzy r-quasi closed.
(2) For $A \in I^X$, ${\mathcal{T}_1}^*(A) \geq r \Rightarrow {\mathcal{T}_2}^*(f(A)) \geq r$.
(3) $qCl_r(f(A)) \subseteq f(qCl_r(A))$ for $A \in I^X$.

Proof. (1) $\Leftrightarrow$ (2) Obvious.

(1) $\Rightarrow$ (3) For $A \in I^X$, $gCl_r(A)$ is fuzzy r-quasi closed. Since f is fuzzy r-quasi closed, $f(qCl_r(A))$ is fuzzy r-quasi closed. So

$$qCl_r(f(A)) \subseteq qCl_r(f(qCl_r(A))) = f(qCl_r(A)).$$

(3) $\Rightarrow$ (1) Let A be a fuzzy r-quasi closed set. Then from (3) and $qCl_r(A) = A$,

$$qCl_r(f(A)) \subseteq f(qCl_r(A)) = f(A).$$

Thus $f(A)$ is fuzzy r-quasi closed. □

References

[1] C. L. Chang, "Fuzzy topological spaces", *J. Math. Anal. Appl.*, vol. 24, pp. 182-190, 1968.

[2] S. J. Lee and E. P. Lee, "Fuzzy r-continuous and fuzzy r-semicontinuous maps" , *Int. J. Math. Math. Sci.*, vol. 27, pp. 53-63, 2001.

[3] W. K. Min, "Fuzzy quasi topological spaces", submitted.

[4] A. A. Ramadan, "Smooth topological spaces", *Fuzzy Sets and Systems*, vol. 48, pp. 371-375, 1992.

[5] L. A. Zadeh, "Fuzzy sets", *Information and Control*, vol. 8, pp. 338-353, 1965.

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