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Fixed Point Theorem for Compatible Maps with Type(I) and (II) in Intuitionistic Fuzzy Metric Space

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Abstract

In this paper, we give definitions of compatible mappings of type(I) and (II) in intuitionistic fuzzy metric space and obtain common fixed point theorem and example under the conditions of compatible mappings of type(I) and (II) in complete intuitionistic fuzzy metric space. Our research generalize, extend and improve the results given by many authors.

Key Words : Common fixed point theorem, compatible mapping of type(I),(II), intuitionistic fuzzy metric space.

1. Introduction

Fang[4], Kaleva and Seikkala[5], Kramosil and Michalek[6] have introduced the concept of fuzzy metric space for each different methods, and some authors have been improved generalized and extended several properties in this space. Cho et.al.[2], Turkoglu et.al.[18] and Sharma et.al.[17] studied this concept of compatible mappings of type(α) and type(β) in fuzzy metric space. Cho et.al.[3] introduced the concept of compatible mapping type(I) and (II) in fuzzy metric spaces.

Recently, Park[7] and Park et.al.[14] defined the intuitionistic fuzzy metric space. Many authors([8], [9], [11], [12], [13] etc) obtained a fixed point theorems in this space. Also, Park[10], Park et.al.[15] introduced the concept of compatible mappings of type(α) and type(β), and obtained common fixed point theorems in intuitionistic fuzzy metric space. Furthermore, Alaca et.al.[1] obtained some results on this spaces.

In this paper, we give definitions of compatible mappings of type(I) and (II) in intuitionistic fuzzy metric space and obtain common fixed point theorem and example under the conditions of compatible mappings of type(I) and (II) in complete intuitionistic fuzzy metric space with different method of Alaca et.al.[1]. Our research generalize, extend and improve the results given by many authors.

2. Preliminaries

Throughout this paper, **N** denote the set of all positive integers. Now, we begin with some definitions, properties in intuitionistic fuzzy metric space as following:

Let us recall(see [16]) that a continuous $t-$norm is a operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following conditions: (a)$*$ is commutative and associative, (b)$*$ is continuous, (c)$a * 1 = a$ for all $a \in [0,1]$, (d)$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a,b,c,d \in [0,1])$. Also, a continuous $t-$conorm is a operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ which satisfies the following conditions: (a)$\diamond$ is commutative and associative, (b)$\diamond$ is continuous, (c)$a \diamond 0 = a$ for all $a \in [0,1]$, (d)$a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ $(a,b,c,d \in [0,1])$.

Also, let us recall (see [7]) that the following conditions are satisfied: (a)For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_4 \diamond r_2 \leq r_1$; (b)For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

Definition 2.1. ([14])The $5-$tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a)$M(x,y,t) > 0$,
(b)$M(x,y,t) = 1 \Longleftrightarrow x = y$,
(c)$M(x,y,t) = M(y,x,t)$,
(d)$M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
(e)$M(x,y,\cdot) : (0,\infty) \rightarrow (0,1]$ is continuous,
(f)$N(x,y,t) > 0$,
(g)$N(x,y,t) = 0 \Longleftrightarrow x = y$,

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(h)$N(x,y,t) = N(y,x,t)$,
(i)$N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$,
(j)$N(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Note that (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Let X be an intuitionistic fuzzy metric space. For any $t > 0$, the open ball $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1-r,\ N(x,y,t) < r\}$$

Let X be an intuitionistic fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Then τ is a topology on X(induced by the intuitionistic fuzzy metric (M,N)). A sequence $\{x_n\} \subset X$ converges to x if and only if $M(x_n,x,t) \to 1$, $N(x_n,x,t) \to 0$ as $n \to \infty$, for all $t > 0$. It is called a Cauchy sequence if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n,x_m,t) > 1-\epsilon$, $N(x_n,x_m,t) < \epsilon$ for any $m,n \geq n_0$. The intuitionistic fuzzy metric space X is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x,y,t) > 1-r$, $N(x,y,t) < r$ for all $x,y \in A$. The following lemma is necessary for Lemma 2.3.

Lemma 2.2. Let X be an intuitionistic fuzzy metric space. If we define $E_r : X^2 \to R^+ \cup \{0\}$ by $E_r(x,y) = \inf\{t > 0 : M(x,y,t) > 1-r, N(x,y,t) < r\}$ for all $r \in (0,1)$ and $x,y \in X$. Then we have

(a)For all $\lambda \in (0,1)$, there exists $r \in (0,1)$ such that $E_\lambda(x_1,x_n) \leq E_r(x_1,x_2) + E_r(x_2,x_3) + \cdots + E_r(x_{n-1},x_n)$ for all $x_1,x_2,\cdots,x_n \in X$.

(b)$\{x_n\}_{n\in\mathbf{N}}$ is convergent in intuitionistic fuzzy metric space X. if and only if $E_r(x_n,x) \to 0$. Also, $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence iff it is a Cauchy sequence with E_r.

Proof. (a)For any $\lambda \in (0,1)$, we can find a $r \in (0,1)$ such that $(1-r)*(1-r)*\cdots*(1-r) \geq 1-\lambda$, $r\diamond r\diamond\cdots\diamond r \leq \lambda$ and so we have by triangular inequality,

$$\begin{aligned}
&M(x_1,x_n,E_r(x_1,x_2)+\cdots+E_r(x_{n-1},x_n)+n\delta)\\
&\geq M(x_1,x_2,E_r(x_1,x_2)+\delta)*\\
&\qquad\cdots*M(x_1n-1,x_n,E_r(x_{n-1},x_n)+\delta)\\
&\geq (1-r)*(1-r)*\cdots*(1-r) \geq 1-\lambda,\\
&N(x_1,x_n,E_r(x_1,x_2)+\cdots+E_r(x_{n-1},x_n)+n\delta)\\
&\leq N(x_1,x_2,E_r(x_1,x_2)+\delta)\diamond\\
&\qquad\cdots\diamond N(x_{n-1},x_n,E_r(x_{n-1},x_n)+\delta)\\
&\leq r\diamond r\diamond\cdots\diamond r \leq \lambda
\end{aligned}$$

for all $\delta > 0$ which implies that $E_\lambda(x_1,x_n) \leq E_r(x_1,x_2)+E_r(x_2,x_3)+\cdots+E_r(x_{n-1},x_n)+n\delta$. Since $\delta > 0$ is arbitrary, we have $E_\lambda(x_1,x_n) \leq E_r(x_1,x_2) + E_r(x_2,x_3) + \cdots + E_r(x_{n-1},x_n)$.

(b)Since M,N are continuous and $E_r(x,y) = \inf\{t > 0 : M(x,y,t) > 1-r,\ N(x,y,t) < r\}$, we have $M(x_n,x,\mu) > 1-r,\ N(x_n,x,\mu) < r$ iff $E_r(x_n,x) < \mu$ for all $\mu > 0$. □

Lemma 2.3. Let X be an intuitionistic fuzzy metric space. If a sequence $\{x_n\} \subset X$ is such that for any $n \in \mathbf{N}$,

$$M(x_n,x_{n+1},t) \geq M(x_0,x_1,k^n t),$$
$$N(x_n,x_{n+1},t) \leq N(x_0,x_1,k^n t)$$

for all $k > 1$, then the sequence $\{x_n\} \subset X$ is a Cauchy sequence.

Proof. For all $r \in (0,1)$ and $x_n,x_{n+1} \in X$, we have for all $t > 0$,

$$\begin{aligned}
&E_r(x_{n+1},x_n)\\
&= \inf\{t : M(x_{n+1},x_n,t) > 1-r,\ N(x_{n+1},x_n,t) < r\}\\
&\leq \inf\{t : M(x_0,x_1,k^n t) > 1-r,\ N(x_0,x_1,k^n t) < r\}\\
&= \inf\{\frac{t}{k^n} : M(x_0,x_1,t) > 1-r,\ N(x_0,x_1,t) < r\}\\
&= \frac{1}{k^n}\inf\{t : M(x_0,x_1,t) > 1-r,\ N(x_0,x_1,t) < r\}\\
&= \frac{1}{k^n}E_r(x_0,x_1).
\end{aligned}$$

By Lemma 2.2, for all $\lambda \in (0,1)$, there exists $r \in (0,1)$ such that

$$\begin{aligned}
&E_r(x_n,x_m)\\
&\leq E_r(x_n,x_{n+1}) + \cdots + E_r(x_{m-1},x_m)\\
&\leq \frac{1}{k^n}E_r(x_0,x_1) + \cdots + \frac{1}{k^{m-1}}E_r(x_0,x_1)\\
&= E_r(x_0,x_1)\sum_{j=n}^{m-1}\frac{1}{k^j} \to 0.
\end{aligned}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence in X. □

3. Some Properties of Compatible Mappings

Definition 3.1. [12]Let A,B be mappings from an intuitionistic fuzzy metric space X into itself. Then the mappings A and B are said to be compatible if for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n,BAx_n,t) = 1,$$
$$\lim_{n\to\infty} N(ABx_n,BAx_n,t) = 0$$

whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$.

Definition 3.2. [12]Let A,B be mappings from an intuitionistic fuzzy metric space X into itself. Then the mappings are said to be weak-compatible if $Ax = Bx$ implies $ABx = BAx$.

Remark 3.3. Let (A, B) be pair of self mappings of intuitionistic fuzzy metric space X. Then (A, B) is commuting implies (A, B) is compatible. Also, (A, B) is compatible implies (A, B) is weak-compatible but the converse is not true.

Example 3.4. Let (X, d) be the metric space with $X = [0, 2]$. Denote $a * b = ab$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and let M_d, N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows :

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X and $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space. Define self mappings A, B on X by

$$A(X) = \begin{cases} 0 & \text{if } 0 \le x \le 1 \\ \frac{x}{2} & \text{if } 1 < x \le 2 \end{cases}$$

$$B(X) = \begin{cases} 2x & \text{if } x = 1 \\ \frac{x+3}{5} & \text{otherwise} \end{cases}.$$

Then $A1 = B1 = 2$. Also, $AB1 = BA1 = 1$ and $AB2 = BA2 = 2$. Thus (A, B) is weak compatible. Again, define $Ax_n = 1 - \frac{1}{4n}$ and $Bx_n = 1 - \frac{1}{10n}$. Then $\lim_{n\to\infty} Ax_n = 1$, $\lim_{n\to\infty} Bx_n = 1$, but (A, B) is not compatible.

Definition 3.5. [10]Let A, B be mappings from intuitionistic fuzzy metric space X into itself. Then the mappings A, B are said to be compatible of type(α) if for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1,$$
$$\lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0$$

and

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1,$$
$$\lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$.

Definition 3.6. [10]Let A, B be mappings from intuitionistic fuzzy metric space X into itself. Then the mappings A, B are said to be compatible of type(β) if for all $t > 0$,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1,$$
$$\lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$.

Proposition 3.7. Let X be an intuitionistic fuzzy metric space with $t * t \ge t$, $t \diamond t \le t$ for all $t \in [0, 1]$ and A, B be continuous mappings from X into itself. Then

(a)A and B are compatible if and only if they are compatible of type(α)

(b)A and B are compatible if and only if they are compatible of type(β)

(c)A and B are compatible of type(α) if and only if they are compatible of type(β)

Definition 3.8. Let A, B be mappings from intuitionistic fuzzy metric space X into itself. Then the mappings A, B are said to be compatible of type(I) if for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, x, t) \le M(Bx, x, t),$$
$$\lim_{n\to\infty} N(ABx_n, x, t) \ge N(Bx, x, t)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$.

Definition 3.9. Let A, B be mappings from intuitionistic fuzzy metric space X into itself. Then the mappings A, B are said to be compatible of type(II) if and only if B, A are said to be compatible of type(I).

Proposition 3.10. Let X be an intuitionistic fuzzy metric space. Suppose that A, B are compatible of type(I)(respectively, (II)) and $Ax = Bx$ for some $x \in X$. Then for all $t > 0$,

$$M(Ax, BBx, t) \ge M(Ax, ABx, t),$$
$$N(Ax, BBx, t) \le N(Ax, ABx, t)$$

(respectively, $M(Bx, AAx, t) \ge M(Bx, BAx, t)$, $N(Bx, AAx, t) \le N(Bx, BAx, t)$).

4. Main Results

Theorem 4.1. Let X be an intuitionistic fuzzy metric space with $t * t \ge t$ and $t \diamond t \le t$ for all $t \in [0, 1]$. Let A, B, S and T be self mappings of a complete intuitionistic fuzzy metric space satisfying

(a)$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,

(b)There exists a constant $k \in (0, \frac{1}{2})$ such that for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$,

$$\begin{aligned} &M(Ax, By, kt) \\ &\ge \min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ &\qquad M(Ax, Ty, \alpha t), M(By, Sx, (2-\alpha)t)\}, \\ &N(Ax, By, kt) \\ &\le \max\{N(Sx, Ty, t), N(Ax, Sx, t), N(By, Ty, t), \\ &\qquad N(Ax, Ty, \alpha t), N(By, Sx, (2-\alpha)t)\}. \end{aligned}$$

If the mappings A, B, S and T satisfy any one of the following conditions:

(c)The pair (A, S) and (B, T) are compatible of type(I) and S or T is continuous.

(d)The pair (A, S) and (B, T) are compatible of type(II) and A or B is continuous.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary point. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. We can construct the sequences $\{x_n\}, \{y_n\} \subset X$ such that $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, $y_{2n} = Ax_{2n} = Tx_{2n+1}$ for $n = 0, 1, 2, \cdots$. Then by $\alpha = 1 - q$ and $q \in (\frac{1}{2}, 1]$, using (b) with $x = x_{2n}, y = x_{2n+1}$,

$$\begin{aligned}
&M(y_{2n}, y_{2n+1}, kt)\\
&= M(Ax_{2n}, Bx_{2n+1}, kt)\\
&\geq \min\{M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t),\\
&\quad M(Bx_{2n+1}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, (1-q)t),\\
&\quad M(Bx_{2n+1}, Sx_{2n}, (1+q)t)\}\\
&= \cdots\cdots\\
&\geq M(y_{2n}, y_{2n+1}, qt)\\
&\qquad \cdots\cdots\\
&\geq M(y_0, y_1, \frac{q}{k^n}t),\\
&N(y_{2n}, y_{2n+1}, kt)\\
&= N(Ax_{2n}, Bx_{2n+1}, kt)\\
&\leq \max\{N(Sx_{2n}, Tx_{2n+1}, t), N(Ax_{2n}, Sx_{2n}, t),\\
&\quad N(Bx_{2n+1}, Tx_{2n+1}, t), N(Ax_{2n}, Tx_{2n+1}, (1-q)t),\\
&\quad N(Bx_{2n+1}, Sx_{2n}, (1+q)t)\}\\
&= \cdots\cdots\\
&\leq N(y_{2n}, y_{2n+1}, qt)\\
&\qquad \cdots\cdots\\
&\leq N(y_0, y_1, \frac{q}{k^n}t).
\end{aligned}$$

Now, setting $2n = m$ and for any $p \in N$,

$$\begin{aligned}
&M(y_m, y_{m+p}, kt)\\
&\geq M(y_m, y_{m+1}, \frac{t}{p}) * \cdots * M(y_{m+p-1}, y_{m+p}, \frac{t}{p}),\\
&N(y_m, y_{m+p}, kt)\\
&\leq N(y_m, y_{m+1}, \frac{t}{p}) \diamond \cdots \diamond N(y_{m+p-1}, y_{m+p}, \frac{t}{p}).
\end{aligned}$$

Hence by Lemma 2.3, $\{y_n\}$ is a Cauchy sequence in X which is complete. Let $\lim_{n\to\infty} y_n = z$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1}\\
&= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1}\\
&= \lim_{n\to\infty} Sx_{2n+2} = z.
\end{aligned}$$

Suppose that the condition (c) is satisfied and T is continuous. Then we have $\lim_{n\to\infty} TTx_{2n+1} = Tz$ and

$$\begin{aligned}
&\lim_{n\to\infty} M(BTx_{2n+1}, z, t) \leq M(Tz, z, t),\\
&\lim_{n\to\infty} N(BTx_{2n+1}, z, t) \geq N(Tz, z, t).
\end{aligned}$$

Now, for $\alpha = 1$, letting $x = x_{2n}$, $y = Tx_{2n+1}$ in condition (b), we obtain

$$\begin{aligned}
&M(Ax_{2n}, BTx_{2n+1}, kt)\\
&\geq \min\{M(Sx_{2n}, TTx_{2n+1}, t), M(Ax_{2n}, S_{2n}, t),\\
&\quad M(BTx_{2n+1}, TTx_{2n+1}, t), M(Ax_{2n}, TTx_{2n+1}, t),\\
&\quad M(BTx_{2n+1}, Sx_{2n}, t)\},\\
&N(Ax_{2n}, BTx_{2n+1}, kt)\\
&\leq \max\{N(Sx_{2n}, TTx_{2n+1}, t), N(Ax_{2n}, S_{2n}, t),\\
&\quad N(BTx_{2n+1}, TTx_{2n+1}, t), N(Ax_{2n}, TTx_{2n+1}, t),\\
&\quad N(BTx_{2n+1}, Sx_{2n}, t)\}.
\end{aligned}$$

Letting $n \to \infty$, since

$$\begin{aligned}
&\lim_{n\to\infty} M(BTx_{2n+1}, Tz, t) \geq \lim_{n\to\infty} M(BTx_{2n+1}, z, \frac{t}{2}),\\
&\lim_{n\to\infty} N(BTx_{2n+1}, Tz, t) \leq \lim_{n\to\infty} N(BTx_{2n+1}, z, \frac{t}{2}).
\end{aligned}$$

Hence

$$\begin{aligned}
&M(z, \lim_{n\to\infty} BTx_{2n+1}, kt) > M(z, \lim_{n\to\infty} BTx_{2n+1}, \frac{t}{2}),\\
&N(z, \lim_{n\to\infty} BTx_{2n+1}, kt) < N(z, \lim_{n\to\infty} BTx_{2n+1}, \frac{t}{2}).
\end{aligned}$$

which is a contradiction for compatible of type(I). It follows that $\lim_{n\to\infty} BTx_{2n+1} = z$. Therefore, we have

$$\begin{aligned}
1 &= \lim_{n\to\infty} M(z, BTx_{2n+1}, kt) \leq M(Tz, z, t),\\
0 &= \lim_{n\to\infty} N(z, BTx_{2n+1}, kt) \geq N(Tz, z, t).
\end{aligned}$$

Hence $Tz = z$.

Again, letting $x = x_{2n}$ and $y = z$ in condition (b), we have for all $\alpha = 1$,

$$\begin{aligned}
&M(Ax_{2n}, Bz, kt)\\
&\geq \min\{M(Sx_{2n}, Tz, t), M(Ax_{2n}, Sx_{2n}, t),\\
&\quad M(Bz, Tz, t), M(Ax_{2n}, Tz, t), M(Bz, Sx_{2n}, t)\},\\
&N(Ax_{2n}, Bz, kt)\\
&\leq \max\{N(Sx_{2n}, Tz, t), N(Ax_{2n}, Sx_{2n}, t),\\
&\quad N(Bz, Tz, t), N(Ax_{2n}, Tz, t), N(Bz, Sx_{2n}, t)\}
\end{aligned}$$

and so we have for $n \to \infty$,

$$\begin{aligned}
&M(Bz, z, kt) > M(Bz, z, t),\\
&N(Bz, z, kt) < N(Bz, z, t)
\end{aligned}$$

which implies that $Bz = z$.

Also, since $B(X) \subseteq S(X)$, there exist $w \in X$ such that $Sw = z = Bz$. So, we have for $\alpha = 1$,

$$\begin{aligned} &M(Aw, Bz, kt) \\ &\geq \min\{M(Sw, Tz, t), M(Aw, Sw, t), M(Bz, Tz, t), \\ &\quad M(Aw, Tz, t), M(Bz, Sw, t)\}, \\ &N(Aw, Bz, kt) \\ &\leq \max\{N(Sw, Tz, t), N(Aw, Sw, t), N(Bz, Tz, t), \\ &\quad N(Aw, Tz, t), N(Bz, Sw, t)\}. \end{aligned}$$

Therefore

$$\begin{aligned} M(Aw, z, kt) > M(z, Aw, t), \\ N(Aw, z, kt) < N(z, Aw, t) \end{aligned}$$

which implies that $Aw = z$. Since (A, S) is compatible of type(I) and $Aw = Sw = z$, we have by Proposition 3.10,

$$\begin{aligned} M(Aw, SSw, t) \geq M(Aw, ASw, t), \\ N(Aw, SSw, t) \leq N(Aw, ASw, t) \end{aligned}$$

and so

$$\begin{aligned} M(z, Sz, t) \geq M(z, Az, t), \\ N(z, Sz, t) \leq N(z, Az, t). \end{aligned}$$

Also, we have for $\alpha = 1$,

$$\begin{aligned} &M(Az, Bz, kt) \\ &\geq \min\{M(Sz, Tz, t), M(Az, Sz, t), M(Bz, Tz, t), \\ &\quad M(Az, Tz, t), M(Bz, Sz, t)\}, \\ &N(Az, Bz, kt) \\ &\leq \max\{N(Sz, Tz, t), N(Az, Sz, t), N(Bz, Tz, t), \\ &\quad N(Az, Tz, t), N(Bz, Sz, t)\}\}. \end{aligned}$$

Since

$$\begin{aligned} M(Az, Sz, t) \geq M(z, Az, \frac{t}{2}), \\ N(Az, Sz, t) \leq N(z, Az, \frac{t}{2}), \end{aligned}$$

therefore

$$\begin{aligned} &M(Az, z, kt) \\ &\geq \min\{M(Az, z, \frac{t}{2}), M(Az, z, \frac{t}{2}), M(Az, z, \frac{t}{2}), \\ &\quad M(z, Az, \frac{t}{2}), M(z, Az, \frac{t}{2})\} \\ &\geq M(z, Az, \frac{t}{2}), \\ &N(Az, z, kt) \\ &\leq \max\{N(Az, z, \frac{t}{2}), N(Az, z, \frac{t}{2}), N(Az, z, \frac{t}{2}), \\ &\quad N(z, Az, \frac{t}{2}), N(z, Az, \frac{t}{2})\} \\ &\leq N(z, Az, \frac{t}{2}). \end{aligned}$$

So $Az = z$. Thus $Az = Bz = Sz = Tz = z$ and z is a common fixed point of the self-mappings A, B, S and T.

Furthermore, if u be another fixed point of A, B, S and T, then we have for $\alpha = 1$,

$$\begin{aligned} &M(z, u, kt) = M(Az, Bu, kt) \\ &\geq \min\{M(Sz, Tu, t), M(Az, Sz, t), M(Bu, Tu, t), \\ &\quad M(Az, Tu, t), M(Bu, Sz, t)\}, \\ &N(z, u, kt) = N(Az, Bu, kt) \\ &\leq \max\{N(Sz, Tu, t), N(Az, Sz, t), N(Bu, Tu, t), \\ &\quad N(Az, Tu, t), N(Bu, Sz, t)\}. \end{aligned}$$

Therefore

$$\begin{aligned} M(z, u, kt) \geq M(z, u, t), \\ N(z, u, kt) \leq N(z, u, t). \end{aligned}$$

Hence $z = u$. □

Example 4.2. Let (X, d) be the metric space with $X = [0, 1]$. Denote $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and let M_d, N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows :

$$\begin{aligned} M_d(x, y, t) = \frac{t}{t + d(x, y)}, \\ N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}. \end{aligned}$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X and $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space. Define self mappings A, B, S and T by

$$\begin{aligned} A(X) &= 1 \\ B(X) &= 1 \\ S(X) &= \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{x is irrational} \end{cases}, \\ T(X) &= \tfrac{x+1}{2}. \end{aligned}$$

If we define $\{x_n\} \subset X$ by $x_n = 1 - \frac{1}{n}$, then we have for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$,

$$\begin{aligned} \lim_{n\to\infty} M(SAx_n, 1, t) \leq M(A1, 1, t) = 1, \\ \lim_{n\to\infty} N(SAx_n, 1, t) \geq N(A1, 1, t) = 0. \end{aligned}$$

Also, for $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 1$,

$$\begin{aligned} \lim_{n\to\infty} M(TBx_n, 1, t) \leq M(B1, 1, t) = 1, \\ \lim_{n\to\infty} N(TBx_n, 1, t) \geq N(B1, 1, t) = 0. \end{aligned}$$

Therefore, (A, S) and (B, T) are compatible of type(II) and A, B are continuous mappings. Then all the conditions of Theorem 4.1 are satisfied and 1 is a unique common fixed point of A, B, S and T on X.

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