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Existence of Periodic Solutions for Fuzzy Differential Equations

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Abstract

In this paper, we investigate the existence and calculation of the expression of periodic solutions for fuzzy differential equations with three types of forcing terms, by using Hukuhara derivative. In particular, Theorems 3.2, 4.2 and 5.2 are the results of existences of periodic solutions for fuzzy differential equations Ⅰ, Ⅱ and Ⅲ, respectively. These results will help us to study phenomena with periodic peculiarity such as wave or sound.

Key Words : Existence, periodic, solutions, fuzzy differential equations

1. Introduction

The concept of fuzzy set was initiated by Zadeh via membership function in 1965. Many authors have studied the fuzzy equations. Fuzzy differential equations are a field of increasing interest, due to their applicability to the analysis of phenomena where imprecision is inherent. Diamond and Kloeden [2] proved the fuzzy optimal control for fuzzy system. Nieto et al. [9] proved the existence of solution for the initial value problems associated to the fuzzy equations. Kwun et al. [4] proved nonlocal controllability for the semilinear fuzzy integrodifferential equations in n-dimensional fuzzy vector space. Our objective is also throughout the fuzzy systems, the situation is vague and uncertain to enable them to solve mathematical problems. But periodicity of solutions in the fuzzy case is difficult to study, due to the behavior of the solutions of fuzzy differential equations. So in this work, we were to be used in various fields as we show that existence of periodic solutions for fuzzy differential equations. Park et al. [6] studied for the almost periodic solutions of fuzzy systems. Bede and Gal [14] dealt with the almost periodic fuzzy-number-valued functions. Rosana Rodríguez-López [13] proved the periodic boundary value problems for impulsive fuzzy differential equations.

In this paper, we study the existence of periodic solutions for the following fuzzy differential equations with three type forcing term:

$$(1.1)\qquad \begin{cases} u'(t) = M(t)u(t) + *, \ \ t \in I = [0, T], \\ u(0) = u_0, \end{cases}$$

where $T > 0$, $u_0 \in E^1$, $M : I \to E^1, u : I \to E^1$ and $*$ are first type $f(t)$, second type $f(t, u(t))$ and last type $f(t, u(t), \int_0^t g(t, s, u(s))ds)$. We calculate the expression of the periodic solutions for fuzzy differential equations by using Hukuhara derivative.

2. Preliminaries

We consider E^1 the space of one-dimensional fuzzy numbers $u : R \to [0, 1]$, satisfying the following properties:

1. u is normal, i.e., there exists an $u_0 \in R$ such that $u(t_o) = 1$;
2. u is fuzzy convex, i.e., $u(\lambda t + (1 - \lambda)s) \geq \min\{u(t), u(s)\}$ for any $t, s \in R$, $0 \leq \lambda \leq 1$;
3. $u(t)$ is upper semi-continuous, i.e., $u(t_0) \geq \lim_{k\to\infty} u(t_k)$ for any $t_k \in R$ $(k = 0, 1, 2, \cdots)$, $t_k \to t_0$;
4. $[u]^0$ is compact.

The level sets of u, $[u]^\alpha = \{t \in R : u(t) \geq \alpha\}, \alpha \in (0, 1]$, and $[u]^0$ are nonempty compact convex sets in R ([2]).

Definition 2.1. [15] Let $u : I \to E^1$ be differentiable. Denote $u^\alpha(t) = [u_l^\alpha(t), u_r^\alpha(t)]$, $\alpha \in [0, 1]$. Then u_l^α and u_r^α are differentiable and $[u'(t)]^\alpha = [u_l'^\alpha(t), u_r'^\alpha(t)]$.

Definition 2.2. The metric d_H on E^1 is defined by

$$d_H([u]^\alpha, [v]^\alpha) = \max\{|u_l^\alpha - v_l^\alpha|, |u_r^\alpha - v_r^\alpha|\}.$$

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Definition 2.3. [2] The $supremum\ metric$ d_∞ on E^1 is defined by

$$d_\infty(u,v) = \sup_{\alpha\in[0,1]} d_H([u]^\alpha, [v]^\alpha),$$

for all $u, v \in E^1$ and is obviously a metric on E^1.

Definition 2.4. The $supremum\ metric$ H_1 on $C^1(I, E^1)$ is defined by

$$H_1(u,v) = \sup_{t\in[0,T]} d_\infty(u(t), v(t)),$$

for all $u, v \in E^1$.

Definition 2.5. [2] A mapping $u : I \to E^1$ is $Hukuhara\ differentiable$ at $t \in I$ if for some $h_0 > 0$ the Hukuhara differences

$$u(t+\Delta t) -_H u(t), u(t) -_H u(t-\Delta t)$$

exist in E^1 for all $0 < t < h$ and if there exists an $u'(t) \in E^1$ such that

$$\lim_{\Delta t\to 0_+} d_\infty((u(t+\Delta t) -_H u(t))/\Delta t, u'(t)) = 0$$

and

$$\lim_{\Delta t\to 0_+} d_\infty((u(t) -_H u(t-\Delta t))/\Delta t, u'(t)) = 0,$$

here $u'(t)$ is called the $Hukuhara\ derivative$ of u at t. In view of the definition of the metric d_∞, all the level set mappings $[u(\cdot)]^\alpha$ are $Hukuhara\ differentiable$ at t with Hukuhara derivatives $[u'(t)]^\alpha$ for each $\alpha \in [0,1]$ when $u : I \to E^1$ is Hukuhara differentiable at t with Hukuhara derivative $u'(t)$.

3. Existence of Periodic Solutions for Fuzzy Differential Equations I

In this section we study the existence of fuzzy strong solutions and periodic solutions for the following problem:

$$\text{(3.1)}\qquad \begin{cases} u'(t) = M(t)u(t) + f(t), & t \in I = [0,T], \\ u(0) = u_0, \end{cases}$$

where $T > 0$, $u_0 \in E^1$, $M : I \to E^1, u : I \to E^1$ and $f : I \to E^1$ are continuous.

Definition 3.1. [3] The fuzzy process $u : I \to E^1$ is a fuzzy solution of equation (3.1) if and only if

$$\begin{aligned}
&(u_l'^\alpha)(t) = \min\{M_i^\alpha(t)u_j^\alpha(t) + f_l^\alpha(t), i,j = l,r\}, \\
&(u_r'^\alpha)(t) = \max\{M_i^\alpha(t)u_j^\alpha(t) + f_r^\alpha(t), i,j = l,r\}, \\
&(u_l^\alpha)(0) = u_{0l}^\alpha, (u_r^\alpha)(0) = u_{0r}^\alpha.
\end{aligned}$$

Theorem 3.2. For every $u_0 \in E^1$, problem (3.1) has a unique fuzzy solution $u \in C(I, E^1)$.

Proof. Assume that the value u_0 and $M(t), f(t)$ are positive fuzzy numbers. From the definition of fuzzy solution,

$$(u_l'^\alpha)(t) = M_l^\alpha(t)u_l^\alpha(t) + f_l^\alpha(t),$$

$$(u_r'^\alpha)(t) = M_r^\alpha(t)u_r^\alpha(t) + f_r^\alpha(t)$$

and

$$\begin{aligned}
(u_l^\alpha)(t) &= u_{0l}^\alpha e^{\int_0^t M_l^\alpha(s)ds} + \int_0^t e^{\int_s^t M_l^\alpha(\tau)d\tau} f_l^\alpha(s)ds, \\
(u_r^\alpha)(t) &= u_{0r}^\alpha e^{\int_0^t M_r^\alpha(s)ds} + \int_0^t e^{\int_s^t M_r^\alpha(\tau)d\tau} f_r^\alpha(s)ds.
\end{aligned}$$

Let $S(t)$ is fuzzy number

$$\begin{aligned}
[S(t)]^\alpha &= [S_l^\alpha(t), S_r^\alpha(t)] \\
&= [e^{\int_0^t M_l^\alpha(s)ds}, e^{\int_0^t M_r^\alpha(s)ds}]
\end{aligned}$$

and $S_i^\alpha(t)$ $(i = l, r)$ is continuous. That is, there exists a constant $c > 0$ such that $|S_i^\alpha(t)| \le c$ for all $t \in I$. The equation (3.1) is related to the following fuzzy integral equations;

$$\text{(3.2)}\qquad u(t) = u_0 S(t) + \int_0^t S(t-s)f(s)ds.$$

For each $\rho(t) \in C^1(I, E^1), t \in I$. Define

$$(\Pi\rho)(t) = u_0 S(t) + \int_0^t S(t-s)f(s)ds.$$

Thus, $(\Pi\rho)(t) : I \to C^1(I, E^1)$ is continuous, and $\Pi : C^1(I, E^1) \to C^1(I, E^1)$. It is obvious that fixed point of Π is solution for the problem (3.1). That is, the (3.1) has a unique fuzzy solution $u \in C(I, E^1)$.

Now we show that $u(t)$ is Hukuhara differentiable. Let's $t \in I, h > 0$, for every $\alpha \in [0,1]$,

$$\begin{aligned}
&\frac{[u(t+h) -_H u(t)]^\alpha}{h} \\
&= \frac{1}{h}\Big[u_{0l}^\alpha S_l^\alpha(t+h) + \int_0^{t+h} S_l^\alpha(t+h-s)f_l^\alpha(s)ds \\
&\qquad - u_{0l}^\alpha S_l^\alpha(t) - \int_0^t S_l^\alpha(t-s)f_l^\alpha(s)ds, \\
&\qquad u_{0r}^\alpha S_r^\alpha(t+h) + \int_0^{t+h} S_r^\alpha(t+h-s)f_r^\alpha(s)ds \\
&\qquad - u_{0r}^\alpha S_r^\alpha(t) - \int_0^t S_r^\alpha(t-s)f_r^\alpha(s)ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}\Big[u_{0l}^{\alpha} S_l^{\alpha}(t) S_l^{\alpha}(h) \\
&\quad + \int_t^{t+h} S_l^{\alpha}(t+h-t) S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds \\
&\quad + \int_0^t S_l^{\alpha}(t+h-t) S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds \\
&\qquad - u_{0l}^{\alpha} S_l^{\alpha}(t) - \int_0^t S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds, \\
&\quad u_{0r}^{\alpha} S_r^{\alpha}(t) S_r^{\alpha}(h) \\
&\quad + \int_0^{t+h} S_r^{\alpha}(t+h-t) S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds \\
&\quad + \int_0^t S_r^{\alpha}(t+h-t) S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds \\
&\qquad - u_{0r}^{\alpha} S_r^{\alpha}(t) - \int_0^t S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds\Big] \\
&= \frac{1}{h}\Big[u_{0l}^{\alpha} S_l^{\alpha}(t)(S_l^{\alpha}(h)-1) \\
&\quad + S_l^{\alpha}(h) \int_t^{t+h} S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds \\
&\qquad + \int_0^t S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds (S_l^{\alpha}(h)-1), \\
&\quad u_{0r}^{\alpha} S_r^{\alpha}(t)(S_r^{\alpha}(h)-1) \\
&\quad + S_r^{\alpha}(h) \int_0^{t+h} S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds \\
&\qquad + \int_0^t S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds (S_r^{\alpha}(h)-1)\Big] \\
&= \frac{1}{h}\Big\{[u_{0l}^{\alpha} S_l^{\alpha}(t)(S_l^{\alpha}(h)-1), u_{0r}^{\alpha} S_r^{\alpha}(t)(S_r^{\alpha}(h)-1)] \\
&\quad + \Big[S_l^{\alpha}(h) \int_t^{t+h} S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds, \\
&\qquad S_r^{\alpha}(h) \int_0^{t+h} S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds\Big] \\
&\quad + \Big[\int_0^t S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds (S_l^{\alpha}(h)-1), \\
&\qquad \int_0^t S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds (S_r^{\alpha}(h)-1)\Big]\Big\}.
\end{aligned}
$$

The limits of these functions as $h \to 0^+$, respectively,

$$
\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h}[u_{0l}^{\alpha} S_l^{\alpha}(t) S_l^{\alpha}(h)-1), u_{0r}^{\alpha} S_r^{\alpha}(t)(S_r^{\alpha}(h)-1)] \\
&= [u_{0l}^{\alpha} M_l^{\alpha}(t) S_l^{\alpha}(t), u_{0r}^{\alpha} M_r^{\alpha}(t) S_r^{\alpha}(t)],
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h}\Big[S_l^{\alpha}(h) \int_t^{t+h} S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds, \\
&\qquad S_r^{\alpha}(h) \int_0^{t+h} S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds\Big] \\
&= [f_l^{\alpha}(t), f_r^{\alpha}(t)],
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h}\Big[\int_0^t S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds (S_l^{\alpha}(h)-1), \\
&\qquad \int_0^t S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds (S_r^{\alpha}(h)-1)\Big] \\
&= \Big[M_l^{\alpha}(t) \int_0^t S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds, \\
&\qquad M_r^{\alpha}(t) \int_0^t S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds\Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\lim_{h\to 0^+} \frac{[u(t+h) -_H u(t)]^{\alpha}}{h} \\
&= \Big[u_{0l}^{\alpha} M_l^{\alpha}(t) S_l^{\alpha}(t) + f_l^{\alpha}(t) \\
&\qquad + M_l^{\alpha}(t) \int_0^t S_l^{\alpha}(t-s) f_l^{\alpha}(s) ds, \\
&\quad u_{0r}^{\alpha} M_r^{\alpha}(t) S_r^{\alpha}(t) + f_r^{\alpha}(t) \\
&\qquad + M_r^{\alpha}(t) \int_0^t S_r^{\alpha}(t-s) f_r^{\alpha}(s) ds\Big] \\
&= [M_l^{\alpha}(t) u_l^{\alpha}(t) + f_l^{\alpha}(t), \;\; M_r^{\alpha}(t) u_r^{\alpha}(t) + f_r^{\alpha}(t)].
\end{aligned}
$$

The same behavior can be checked for the left-sided Hukuhara quotients

$$
\frac{[u(t) -_H u(t-h)]^{\alpha}}{h}, \;\; h > 0.
$$

This proves that

$$
\begin{aligned}
&d_H\Big(\Big[\frac{[u(t+h) -_H u(t)]^{\alpha}}{h}\Big], \\
&\quad [M_l^{\alpha}(t) u_l^{\alpha}(t) + f_l^{\alpha}(t), M_r^{\alpha}(t) u_r^{\alpha}(t) + f_r^{\alpha}(t)]\Big) \\
&\quad \to 0,
\end{aligned}
$$

as $h \to 0^+$, uniformly in α, so that

$$
d_{\infty}\Big(\frac{u(t+h) -_H u(t)}{h}, u'(t)\Big) \to 0,
$$

where, for $t \in I$, $u'(t)$ given levelwise by

$$
[u'(t)]^{\alpha} = [M_l^{\alpha}(t) u_l^{\alpha}(t) + f_l^{\alpha}(t), M_r^{\alpha}(t) u_r^{\alpha}(t) + f_r^{\alpha}(t)]
$$

is a fuzzy number. Thus $u(t)$ is fuzzy strong solution of equation (3.1).

Now we study the existence of periodic solutions for fuzzy differential equation (3.1). For this purpose, assume that the following conditions hold.

(H1) For a constant $T > 0, t \in I = [0, T]$,

$$
f(t+T) = f(t), \; Pu_0 = u(T).
$$

And we define a map P along the solution in such a way that, for $u(\cdot,\phi)$ a solution of equation (3.1) with the initial function ϕ,

$$P\phi = u_T(\cdot,\phi), \phi \in C([0,T],E^1), \tag{3.3}$$

and then examine whether the map P has a fixed point. We note that a fixed point of P gives rise to periodic solutions. Because if $P\phi = \phi$, then for the solution $u(\cdot) = u(\cdot,\phi)$ with $u_0(\cdot,\phi) = \phi$, we can define

$$y(t) = u(t+T). \tag{3.4}$$

Now, for $t \geq 0$, we can use the known formulas[10]

$$S(0) = I, S(t+s) = S(t)S(s), \tag{3.5}$$

to obtain

$$\begin{aligned}
y(t) &= u(t+T) \\
&= S(t+T)u_0 + \int_0^{t+T} S(t+T-s)f(s)ds \\
&= S(t)S(T)u_0 + \int_0^{T} S(t)S(T-s)f(s)ds \\
&\quad + \int_T^{t+T} S(t+T-s)f(s)ds \\
&= S(t)\Big[S(T)u_0 + \int_0^{T} S(T-s)f(s)ds\Big] \\
&\quad + \int_0^{t} S(t+T-s-T)f(s+T)ds \\
&= S(t)u(T) + \int_0^{t} S(t-s)f(s)ds \\
&= S(t)u_0 + \int_0^{t} S(t-s)f(s)ds.
\end{aligned}$$

This implies that y is also a solution and $y_0 = u_T(\phi) = P\phi = \phi$. Then, the uniqueness implies that $(u(t+T))y(t) = u(t)$, so that $u(\phi)$ is a periodic solution.

4. Existence of Periodic Solutions for Fuzzy Differential Equations II

In this section we study the existence of fuzzy strong solutions and periodic fuzzy solutions for the following fuzzy differential equation:

$$\begin{cases} u'(t) = M(t)u(t) + f(t,u(t)), \\ \qquad\qquad\qquad\qquad t \in I = [0,T], \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where $T > 0$, initial value $u_0 \in E^1$, fuzzy coefficient $M : I \to E^1$, and $f : I \times E^1 \to E^1$ satisfies a global Lipschitz condition, i.e., there exists a finite constant $k_1 > 0$ such that

$$d_H([f(s,x(s))]^\alpha, [f(s,y(s))]^\alpha) \leq k_1 d_H([x(s)]^\alpha, [y(s)]^\alpha)$$

for all $x(s), y(s) \in E^1$.

Definition 4.1. [3] The fuzzy process $u : I \to E^1$ is a fuzzy solution of equation (4.1) if and only if

$$\begin{aligned}
(u_l'^\alpha)(t) &= \min\{M_i^\alpha(t)u_j^\alpha(t) + f_l^\alpha(t,u_j^\alpha(t)),\}, \\
(u_r'^\alpha)(t) &= \max\{M_i^\alpha(t)u_j^\alpha(t) + f_r^\alpha(t,u_j^\alpha(t)),\}, \\
(u_l^\alpha)(0) &= u_{0l}^\alpha, (u_r^\alpha)(0) = u_{0r}^\alpha,
\end{aligned}$$

where $i,j = l,r$.

Theorem 4.2. For every $u_0 \in E^1$, problem (4.1) has a unique fuzzy strong solution $u \in C^1(I,E^1)$.

Proof. Assume that the value u_0 and $M(t), f(t)$ are positive fuzzy numbers. From the definition of fuzzy solution,

$$(u_l'^\alpha)(t) = M_l^\alpha(t)u_l^\alpha(t) + f_l^\alpha(t,u_l^\alpha(t)),$$

$$(u_r'^\alpha)(t) = M_r^\alpha(t)u_r^\alpha(t) + f_r^\alpha(t,u_r^\alpha(t))$$

and

$$\begin{aligned}
(u_l^\alpha)(t) &= u_{0l}^\alpha e^{\int_0^t M_l^\alpha(s)ds} \\
&\quad + \int_0^t e^{\int_s^t M_l^\alpha(\tau)d\tau} f_l^\alpha(s,u_l^\alpha(s))ds, \\
(u_r^\alpha)(t) &= u_{0r}^\alpha e^{\int_0^t M_r^\alpha(s)ds} \\
&\quad + \int_0^t e^{\int_s^t M_r^\alpha(\tau)d\tau} f_r^\alpha(s,u_r^\alpha(s))ds.
\end{aligned}$$

The equation (4.1) is related to the following fuzzy integral equations;

$$u(t) = u_0 S(t) + \int_0^t S(t-s)f(s,u(s))ds \tag{4.2}$$

For each $\xi(t) \in C^1(I,E^1), t \in I$ define

$$(\Phi\xi)(t) = S(t)u_0 + \int_0^t S(t-s)f(s,\xi(s))ds.$$

Thus, $(\Phi\xi)(t) : I \to C^1(I,E^1)$ is continuous, and $\Phi : C^1(I,E^1) \to C^1(I,E^1)$. It is obvious that fixed point of Φ is solution for the problem (4.1) [3]. That is, the (4.1) has a unique fuzzy solution $u \in C(I,E^1)$.

Then, to be show $u(t)$ is fuzzy strong solution, we show that $u(t)$ is Hukuhara differentiable. Let's $t \in I, h > 0$, for every $\alpha \in [0,1]$,

$$\frac{[u(t+h) -_H u(t)]^\alpha}{h}$$

$$
\begin{aligned}
&= \frac{1}{h}\Big[u_{0l}^{\alpha}S_l^{\alpha}(t+h) \\
&\qquad + \int_0^{t+h} S_l^{\alpha}(t+h-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds \\
&\qquad - u_{0l}^{\alpha}S_l^{\alpha}(t) - \int_0^t S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds, \\
&\qquad u_{0r}^{\alpha}S_r^{\alpha}(t+h) \\
&\qquad + \int_0^{t+h} S_r^{\alpha}(t+h-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds \\
&\qquad - u_{0r}^{\alpha}S_r^{\alpha}(t) - \int_0^t S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds\Big] \\
&= \frac{1}{h}\Big[u_{0l}^{\alpha}S_l^{\alpha}(t)S_l^{\alpha}(h) \\
&\qquad + \int_t^{t+h} S_l^{\alpha}(t+h-t)S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds \\
&\qquad + \int_0^t S_l^{\alpha}(t+h-t)S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds \\
&\qquad - u_{0l}^{\alpha}S_l^{\alpha}(t) - \int_0^t S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds, \\
&\qquad u_{0r}^{\alpha}S_r^{\alpha}(t)S_r^{\alpha}(h) \\
&\qquad + \int_0^{t+h} S_r^{\alpha}(t+h-t)S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds \\
&\qquad + \int_0^t S_r^{\alpha}(t+h-t)S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds \\
&\qquad - u_{0r}^{\alpha}S_r^{\alpha}(t) - \int_0^t S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds\Big] \\
&= \frac{1}{h}\Big[u_{0l}^{\alpha}S_l^{\alpha}(t)(S_l^{\alpha}(h)-1) \\
&\qquad + S_l^{\alpha}(h)\int_t^{t+h} S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds \\
&\qquad + \int_0^t S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds(S_l^{\alpha}(h)-1), \\
&\qquad u_{0r}^{\alpha}S_r^{\alpha}(t)(S_r^{\alpha}(h)-1) \\
&\qquad + S_r^{\alpha}(h)\int_0^{t+h} S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds \\
&\qquad + \int_0^t S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds(S_r^{\alpha}(h)-1)\Big] \\
&= \frac{1}{h}\Big\{[u_{0l}^{\alpha}S_l^{\alpha}(t)(S_l^{\alpha}(h)-1),\ u_{0r}^{\alpha}S_r^{\alpha}(t)(S_r^{\alpha}(h)-1)] \\
&\qquad + \Big[S_l^{\alpha}(h)\int_t^{t+h} S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds, \\
&\qquad\quad S_r^{\alpha}(h)\int_0^{t+h} S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds\Big] \\
&\qquad + \Big[\int_0^t S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds(S_l^{\alpha}(h)-1), \\
&\qquad\quad \int_0^t S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds(S_r^{\alpha}(h)-1)\Big]\Big\}.
\end{aligned}
$$

The limits of these functions as $h \to 0^+$, respectively,

$$
\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}[u_{0l}^{\alpha}S_l^{\alpha}(t)(S_l^{\alpha}(h)-1), \\
&\qquad\qquad u_{0r}^{\alpha}S_r^{\alpha}(t)(S_r^{\alpha}(h)-1)] \\
&= [u_{0l}^{\alpha}M_l^{\alpha}(t)S_l^{\alpha}(t), u_{0r}^{\alpha}M_r^{\alpha}(t)S_r^{\alpha}(t)],
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\Big[S_l^{\alpha}(h)\int_t^{t+h} S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds, \\
&\qquad S_r^{\alpha}(h)\int_0^{t+h} S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds\Big] \\
&= [f_l^{\alpha}(t,u_l^{\alpha}(t)),\ \ f_r^{\alpha}(t,u_r^{\alpha}(t)],
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\Big[\int_0^t S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds(S_l^{\alpha}(h)-1), \\
&\qquad \int_0^t S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds(S_r^{\alpha}(h)-1)\Big] \\
&= \Big[M_l^{\alpha}(t)\int_0^t S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds, \\
&\qquad M_r^{\alpha}(t)\int_0^t S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds\Big]
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\lim_{h\to 0^+}\frac{[u(t+h) -_H u(t)]^{\alpha}}{h} \\
&= \Big[u_{0l}^{\alpha}M_l^{\alpha}(t)S_l^{\alpha}(t) + f_l^{\alpha}(t,u_l^{\alpha}(t)) \\
&\qquad + M_l^{\alpha}(t)\int_0^t S_l^{\alpha}(t-s)f_l^{\alpha}(s,u_l^{\alpha}(s))ds, \\
&\quad u_{0r}^{\alpha}M_r^{\alpha}(t)S_r^{\alpha}(t) + f_r^{\alpha}(t,u_r^{\alpha}(t)) \\
&\qquad + M_r^{\alpha}(t)\int_0^t S_r^{\alpha}(t-s)f_r^{\alpha}(s,u_r^{\alpha}(s))ds\Big] \\
&= [M_l^{\alpha}(t)u_l^{\alpha}(t) + f_l^{\alpha}(t,u_l^{\alpha}(t)), \\
&\qquad M_r^{\alpha}(t)u_r^{\alpha}(t) + f_r^{\alpha}(t,u_r^{\alpha}(t))].
\end{aligned}
$$

The same behavior can be checked for the left-sided Hukuhara quotients

$$\frac{[u(t) -_H u(t-h)]^{\alpha}}{h},\ \ h > 0.$$

This proves that

$$
\begin{aligned}
&d_H\Big(\Big[\frac{[u(t+h) -_H u(t)]^{\alpha}}{h}\Big], [M_l^{\alpha}(t)u_l^{\alpha}(t) \\
&\qquad + f_l^{\alpha}(t,u_l^{\alpha}(t)), M_r^{\alpha}(t)u_r^{\alpha}(t) + f_r^{\alpha}(t,u_r^{\alpha}(t))]\Big) \\
&\qquad \to 0,
\end{aligned}
$$

as $h \to 0^+$, uniformly in α, so that

$$d_{\infty}\Big(\frac{u(t+h) -_H u(t)}{h}, u'(t)\Big) \to 0,$$

where, for $t \in I,\ u'(t)$ given levelwise by

$$[u'(t)]^\alpha = [M_l^\alpha(t)u_l^\alpha(t) + f_l^\alpha(t, u_l^\alpha(t)), M_r^\alpha(t)u_r^\alpha(t) + f_r^\alpha(t, u_r^\alpha(t))]$$

is a fuzzy number. Thus $u(t)$ is fuzzy strong solution of equation (4.1).

Now we study the periodic solutions for the fuzzy differential equation (4.1). For this purpose, assume that the following conditions hold.

(H2) For a constant $T > 0, t \in I = [0, T]$,

$$f(t+T, x) = f(t, x),\ Pu_0 = u(T).$$

Then by (3.4)-(3.5) and (H2), the following to obtain

$$\begin{aligned} y(t) &= u(t+T) \\ &= S(t+T)u_0 + \int_0^{t+T} S(t+T-s)f(s, u(s))ds \\ &= S(t)S(T)u_0 + \int_0^T S(t)S(T-s)f(s, u(s))ds \\ &\quad + \int_T^{t+T} S(t+T-s)f(s, u(s))ds \\ &= S(t)\Big[S(T)u_0 + \int_0^T S(T-s)f(s, u(s))ds\Big] \\ &\quad + \int_0^t S(t+T-s-T)f(s+T, u(s+T))ds \\ &= S(t)u(T) + \int_0^t S(t-s)f(s, u(s))ds \\ &= S(t)u_0 + \int_0^t S(t-s)f(s, u(s))ds. \end{aligned}$$

This implies that y is also a solution and $y_0 = u_T(\phi) = P\phi = \phi$. Then, the uniqueness implies that $(u(t+T))y(t) = u(t)$, so that $u(\phi)$ is a periodic solutions.

5. Existence of Periodic Solutions for Fuzzy Differential Equations III

In this section we study the existence of fuzzy strong solutions and periodic solutions for the following fuzzy differential equation with forcing term with memory.

(5.1)

$$\begin{cases} u'(t) = M(t)u(t) \\ \qquad +f(t, u(t), \int_0^t g(t, s, u(s))ds), \quad t \in I = [0, T], \\ u(0) = u_0, \end{cases}$$

where $T > 0,\ u_0 \in E^1$, and $M : I \to E^1,\ f : I \times E^1 \times E^1 \to E^1,\ g : I \times I \times E^1 \to E^1$.

Assume that the following hypotheses :

(H3) The function $f : I \times E^1 \times E^1 \to E^1$ satisfies a global Lipschitz condition

$$\begin{aligned} & d_H([f(s, x_1(s), y_1(s))]^\alpha, [f(s, x_2(s), y_2(s))]^\alpha) \\ \le\ & k_2(d_H([x_1(s)]^\alpha, [x_2(s)]^\alpha) \\ & \qquad + d_H([y_1(s)]^\alpha, [y_2(s)]^\alpha)), \end{aligned}$$

for all $x_i(\cdot), y_i(\cdot) \in E^1, (i = 1, 2)$ and a finite positive constant $k_2 > 0$.

(H4) The function $g : I \times I \times E^1 \to E^1$ satisfies a global Lipschitz condition

$$\begin{aligned} & d_H\Big(\Big[\int_0^t g(t, s, x(s))ds\Big]^\alpha, \Big[\int_0^t g(t, s, y(s))ds\Big]^\alpha\Big) \\ \le\ & k_3 \int_0^t d_H([x(s)]^\alpha, [y(s)]^\alpha)ds, \end{aligned}$$

for all $x(\cdot), y(\cdot) \in E^1$, and a finite positive constant $k_3 > 0$.

Definition 5.1. [3] The fuzzy process $u : I \to E^1$ is a fuzzy solution of equation (5.1) if and only if, for $i, j = l, r$,

$$\begin{aligned} (u_l'^\alpha)(t) &= \min\{M_i^\alpha(t)u_j^\alpha(t) \\ &\qquad + f_l^\alpha(t, u_j^\alpha(t), \int_0^t g_l^\alpha(t, s, u_j^\alpha(s))ds)\}, \\ (u_r'^\alpha)(t) &= \max\{M_i^\alpha(t)u_j^\alpha(t) \\ &\qquad + f_r^\alpha(t, u_j^\alpha(t), \int_0^t g_r^\alpha(t, s, u_j^\alpha(s))ds)\}, \\ (u_l^\alpha)(0) &= u_{0l}^\alpha, (u_r^\alpha)(0) = u_{0r}^\alpha. \end{aligned}$$

Theorem 5.2. Let $T > 0$, and hypotheses (H3)-(H5) hold. Then, for every $u_0 \in E^1$, problem (5.1) has a unique fuzzy strong solution $u \in C^1(I, E^1)$.

Proof. Assume that the value u_0 and $M(t), f(t)$ are positive fuzzy numbers. From the definition of fuzzy solution,

$$\begin{aligned} (u_l'^\alpha)(t) &= M_l^\alpha(t)u_l^\alpha(t) \\ &\qquad + f_l^\alpha\Big(t, u_l^\alpha(t), \int_0^t g_l^\alpha(t, s, u_l^\alpha(s))ds\Big), \\ (u_r'^\alpha)(t) &= M_r^\alpha(t)u_r^\alpha(t) \\ &\qquad + f_r^\alpha\Big(t, u_r^\alpha(t), \int_0^t g_r^\alpha(t, s, u_r^\alpha(s))ds\Big) \end{aligned}$$

and

$$\begin{aligned} (u_l^\alpha)(t) &= u_{0l}^\alpha e^{\int_0^t M_l^\alpha(s)ds} + \int_0^t e^{\int_s^t M_l^\alpha(\tau)d\tau} \\ &\qquad \times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s, \sigma, u_l^\alpha(\sigma))d\sigma\Big)ds, \\ (u_r^\alpha)(t) &= u_{0r}^\alpha e^{\int_0^t M_r^\alpha(s)ds} + \int_0^t e^{\int_s^t M_r^\alpha(\tau)d\tau} \\ &\qquad \times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s, \sigma, u_r^\alpha(\sigma))d\sigma\Big)ds. \end{aligned}$$

The equation (5.1) is related to the following fuzzy integral equations;

$$\begin{aligned} u(t) =& u_0 S(t) + \int_0^t S(t-s) \\ & \times f\Big(s, u(s), \int_0^s g(s,\tau,u(\tau))d\tau\Big)ds \end{aligned} \tag{5.2}$$

For each $\zeta(t) \in C^1(I, E^1), t \in I$ define

$$\begin{aligned} (\Psi\zeta)(t) \;=\; & S(t)u_0 + \int_0^t S(t-s) \\ & \times f\Big(s, \zeta(s), \int_0^s g(s,\tau,\zeta(\tau))d\tau\Big)ds. \end{aligned}$$

Thus, $(\Psi\zeta)(t) : I \to C^1(I, E^1)$ is continuous, and $\Psi : C^1(I, E^1) \to C^1(I, E^1)$. It is obvious that fixed point of Ψ is solution for the problem (5.1) [3].

Then, to be show $u(t)$ is fuzzy strong solution, we show $u(t)$ is Hukuhara differentiable. Let's $t \in I, h > 0$, for every $\alpha \in [0, 1]$,

$$\begin{aligned} & \frac{[u(t+h) -_H u(t)]^\alpha}{h} \\ &= \frac{1}{h}\Big[u_{0l}^\alpha S_l^\alpha(t+h) + \int_0^{t+h} S_l^\alpha(t+h-s) \\ &\qquad \times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds \\ &\qquad - u_{0l}^\alpha S_l^\alpha(t) - \int_0^t S_l^\alpha(t-s) \\ &\qquad \times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds, \\ &\quad u_{0r}^\alpha S_r^\alpha(t+h) + \int_0^{t+h} S_r^\alpha(t+h-s) \\ &\qquad \times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds \\ &\qquad - u_{0r}^\alpha S_r^\alpha(t) - \int_0^t S_r^\alpha(t-s) \\ &\qquad \times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{h}\Big[u_{0l}^\alpha S_l^\alpha(t)(S_l^\alpha(h)-1) \\ &\qquad + S_l^\alpha(h) \int_t^{t+h} S_l^\alpha(t-s) \\ &\qquad \times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds \\ &\qquad + (S_l^\alpha(h)-1) \int_0^t S_l^\alpha(t-s) \\ &\qquad \times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds, \\ &\quad u_{0r}^\alpha S_r^\alpha(t)(S_r^\alpha(h)-1) \\ &\qquad + S_r^\alpha(h) \int_t^{t+h} S_r^\alpha(t-s) \\ &\qquad \times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds \\ &\qquad + (S_r^\alpha(h)-1) \int_0^t S_r^\alpha(t-s) \\ &\qquad \times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big] \\ &= \frac{1}{h}\Big\{[u_{0l}^\alpha S_l^\alpha(t)(S_l^\alpha(h)-1), \;\; u_{0r}^\alpha S_r^\alpha(t)(S_r^\alpha(h)-1)] \\ &\quad + \Big[S_l^\alpha(h) \int_t^{t+h} S_l^\alpha(t-s) \\ &\qquad \times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds, \\ &\quad S_r^\alpha(h) \int_t^{t+h} S_r^\alpha(t-s) \\ &\qquad \times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big] \\ &\quad + \Big[(S_l^\alpha(h)-1) \int_0^t S_l^\alpha(t-s) \\ &\qquad \times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds, \\ &\quad (S_r^\alpha(h)-1) \int_0^t S_r^\alpha(t-s) \\ &\qquad \times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big]\Big\} \end{aligned}$$

The limits of these functions as $h \to 0^+$, respectively,

$$\begin{aligned} & \lim_{h\to 0^+} \frac{1}{h}[u_{0l}^\alpha S_l^\alpha(t)(S_l^\alpha(h)-1), \\ &\qquad\qquad u_{0r}^\alpha S_r^\alpha(t)(S_r^\alpha(h)-1)] \\ &= [u_{0l}^\alpha M_l^\alpha(t) S_l^\alpha(t), \;\; u_{0r}^\alpha M_r^\alpha(t) S_r^\alpha(t)], \end{aligned}$$

$$\lim_{h\to 0^+} \frac{1}{h}\Big[S_l^\alpha(h) \int_t^{t+h} S_l^\alpha(t-s)$$

$$
\begin{aligned}
&\times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds,\\
&S_r^\alpha(h)\int_t^{t+h} S_r^\alpha(t-s)\\
&\times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big]\\
= &\Big[M_l^\alpha(t)\int_0^t S_l^\alpha(t-s)\\
&\times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds,\\
&M_r^\alpha(t)\int_0^t S_r^\alpha(t-s)\\
&\times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{h\to 0^+}\frac{1}{h}\Big[(S_l^\alpha(h)-1)\int_0^t S_l^\alpha(t-s)\\
&\times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds,\\
&(S_r^\alpha(h)-1)\int_0^t S_r^\alpha(t-s)\\
&\times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big]\\
= &\Big[f_l^\alpha\Big(t, u_l^\alpha(t), \int_0^t g_l^\alpha(t,s,u_l^\alpha(s))ds\Big),\\
&f_r^\alpha\Big(t, u_r^\alpha(t), \int_0^t g_r^\alpha(t,s,u_r^\alpha(s))ds\Big)\Big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\lim_{h\to 0^+}\frac{[u(t+h)-_H u(t)]^\alpha}{h}\\
&= [u_{0l}^\alpha M_l^\alpha(t)S_l^\alpha(t),\ \ u_{0r}^\alpha M_r^\alpha(t)S_r^\alpha(t)]\\
&\quad+\Big[M_l^\alpha(t)\int_0^t S_l^\alpha(t-s)\\
&\quad\times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds,\\
&\quad M_r^\alpha(t)\int_0^t S_r^\alpha(t-s)\\
&\quad\times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big]\\
&\quad+\Big[f_l^\alpha\Big(t, u_l^\alpha(t), \int_0^t g_l^\alpha(t,s,u_l^\alpha(s))ds\Big),\\
&\quad f_r^\alpha\Big(t, u_r^\alpha(t), \int_0^t g_r^\alpha(t,s,u_r^\alpha(s))ds\Big)\Big]\\
&=\Big[u_{0l}^\alpha M_l^\alpha(t)S_l^\alpha(t)\\
&\quad+f_l^\alpha\Big(t, u_l^\alpha(t), \int_0^t g_l^\alpha(t,s,u_l^\alpha(s))ds\Big)\\
&\quad+M_l^\alpha(t)\int_0^t S_l^\alpha(t-s)\\
&\quad\times f_l^\alpha\Big(s, u_l^\alpha(s), \int_0^s g_l^\alpha(s,\tau,u_l^\alpha(\tau))d\tau\Big)ds,\\
&\quad u_{0r}^\alpha M_r^\alpha(t)S_r^\alpha(t)\\
&\quad+f_r^\alpha\Big(t, u_r^\alpha(t), \int_0^t g_r^\alpha(t,s,u_r^\alpha(s))ds\Big)\\
&\quad+M_r^\alpha(t)\int_0^t S_r^\alpha(t-s)\\
&\quad\times f_r^\alpha\Big(s, u_r^\alpha(s), \int_0^s g_r^\alpha(s,\tau,u_r^\alpha(\tau))d\tau\Big)ds\Big]\\
&=\Big[M_l^\alpha(t)u_l^\alpha(t)\\
&\quad+f_l^\alpha\Big(t, u_l^\alpha(t), \int_0^t g_l^\alpha(t,s,u_l^\alpha(s))ds\Big),\\
&\quad M_r^\alpha(t)u_r^\alpha(t)\\
&\quad+f_r^\alpha\Big(t, u_r^\alpha(t), \int_0^t g_r^\alpha(t,s,u_r^\alpha(s))ds\Big)\Big].
\end{aligned}
$$

The same behavior can be checked for the left-sided Hukuhara quotients. This proves that

$$
\begin{aligned}
&d_H\Big(\Big[\frac{[u(t+h)-_H u(t)]^\alpha}{h}\Big],\\
&\Big[M_l^\alpha(t)u_l^\alpha(t)\\
&\quad+f_l^\alpha\Big(t, u_l^\alpha(t), \int_0^t g_l^\alpha(t,s,u_l^\alpha(s))ds\Big),\\
&M_r^\alpha(t)u_r^\alpha(t)\\
&\quad+f_r^\alpha\Big(t, u_r^\alpha(t), \int_0^t g_r^\alpha(t,s,u_r^\alpha(s))ds\Big)\Big]\Big)\to 0,
\end{aligned}
$$

as $h\to 0^+$, uniformly in α, so that

$$
d_\infty\Big(\frac{u(t+h)-_H u(t)}{h}, u'(t)\Big)\to 0,
$$

where, for $t\in I,\ u'(t)$ given levelwise by

$$
\begin{aligned}
&[u'(t)]^\alpha\\
&=\Big[M_l^\alpha(t)u_l^\alpha(t)+f_l^\alpha\Big(t, u_l^\alpha(t), \int_0^t g_l^\alpha(t,s,u_l^\alpha(s))ds\Big),\\
&\quad M_r^\alpha(t)u_r^\alpha(t)+f_r^\alpha\Big(t, u_r^\alpha(t), \int_0^t g_r^\alpha(t,s,u_r^\alpha(s))ds\Big)\Big]
\end{aligned}
$$

is a fuzzy number. Thus $u(t)$ is fuzzy strong solution of equation (5.1).

Now we study the periodic solutions for the fuzzy differential equation (5.1). For this purpose, assume that the following conditions hold.

(H5) For a constant $T>0, t\in I=[0,T], x,y\in E^1$, $f(t+T,x,y)=f(t,x,y)$, $g(t+t,s+T,x)=g(t,s,x)$, $Pu_0=u(T)$.

Then by (3.4)-(3.5) and (H5), the following to obtain

$$
\begin{aligned}
y(t) &= u(t+T) \\
&= S(t+T)u_0 + \int_0^{t+T} S(t+T-s) \\
&\quad \times f\Big(s, u(s), \int_0^s g(s,\tau,u(\tau))d\tau\Big)ds \\
&= S(t)S(T)u_0 + \int_0^T S(t)S(T-s) \\
&\quad \times f\Big(s, u(s), \int_0^s g(s,\tau,u(\tau))d\tau\Big)ds \\
&\quad + \int_T^{t+T} S(t+T-s) \\
&\quad \times f\Big(s, u(s), \int_0^s g(s,\tau,u(\tau))d\tau\Big)ds \\
&= S(t)\Big[S(T)u_0 + \int_0^T S(T-s) \\
&\quad \times f\Big(s, u(s), \int_0^s g(s,\tau,u(\tau))d\tau\Big)ds\Big] \\
&\quad + \int_0^t S(t+T-s-T) \\
&\quad \times f\Big(s+T, u(s+T), \\
&\qquad \int_0^{s+T} g(s+T,\tau,u(\tau))d\tau\Big)ds \\
&= S(t)u(T) + \int_0^t S(t-s) \\
&\quad \times f\Big(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\Big)ds \\
&= S(t)u_0 + \int_0^t S(t-s) \\
&\quad \times f\Big(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau\Big)ds.
\end{aligned}
$$

This implies that y is also a solution and $y_0 = u_T(\phi) = P\phi = \phi$. Then, the uniqueness implies that $(u(t+T))y(t) = u(t)$, so that $u(\phi)$ is a periodic solutions.

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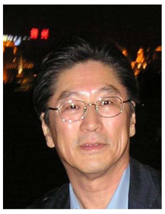

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