

A NONHARMONIC FOURIER SERIES AND DYADIC SUBDIVISION SCHEMES

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ABSTRACT. In the spectral analysis, Fourier coefficients are very important to give informations for the original signal f on a finite domain, because they recover f . Also Fourier analysis has extension to wavelet analysis for the whole space R . Various kinds of reconstruction theorems are main subject to analyze signal function f in the field of wavelet analysis. In this paper, we will present a new reconstruction theorem of functions in $L^1(R)$ using a nonharmonic Fourier series. When we construct this series, we have used dyadic subdivision schemes.

1. Introduction

If a given function f has n th derivative at x_0 , then we have the Taylor polynomial of degree n at x_0 (i.e, if f is smooth enough around x_0 , the differentiability gives informations for an accurate expression of f). Maybe readers think about the local polynomial approximations like cubic splines. In analogy of the square integrable functions on R , we may guess the Fourier series expansion on $[0, 2\pi]$ or the wavelet series expansion on R (See [1], [2] and [4]).

Hence it is natural to ask whether the integrable function f on R has such a series expression like the square integrable functions on R . The answer is affirmative! See Section 2.

Let us start with the definition of the Fourier transform of f : for $f \in L^1(R)$, the Fourier transform of f is defined by

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t)e^{-2\pi iwt} dt, w \in R.$$

Here, $L^p(R)$ is the spaces, $1 \leq p < \infty$, of all measurable functions f such that

$$\|f\|_p = \left(\int_R |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

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Let f and g be measurable functions on R . The convolution of f and g is the function $f * g$ defined by $f * g(x) = \int_R f(x-y)g(y)dy$ for all x such that the integral exists.

For an approach to the main subject, we state some known convergence theorem related to convolutions. We know that even if f is in $L^1(R)$, \hat{f} is not always in $L^1(R)$. But we can recover f from \hat{f} in some reasonable style when $\hat{f} \notin L^1(R)$. Now we modify the divergent integral $\int_R \hat{f}(y)e^{2\pi ixy}dy$ with $\int_R \hat{f}(y)\Phi(ty)e^{2\pi ixy}dy$, where Φ is a continuous function which vanishes rapidly enough at infinity to make the integral converges(see [3]).

Theorem 1.1. *Suppose $\Phi \in L^1(R) \cap C_0(R)$, $\Phi(0) = 1$ and $\phi = \hat{\Phi} \in L^1(R)$. Suppose also that $|\phi(x)| \leq C(1+|x|)^{-1-\epsilon}$ for some C , $\epsilon > 0$. Then we have $\int_R \hat{f}(y)\Phi(ty)e^{2\pi ixy}dy \rightarrow f(x)$ as $t \rightarrow 0$, for every x in the Lebesgue set of f .*

See [3] to prove the theorem.

An application of Theorem 1.1 is given when we put $\Phi(t) = \max((1-|t|), 0)$. We have the Fejer Kernel $\phi(x) = (\frac{\sin \pi x}{\pi x})^2$ by a simple calculation. If we let $n\phi(nx) = \sigma_{2\pi n}(x)$, then $f * \sigma_{2\pi n}(\theta)$ converges to $f(\theta)$ for almost all θ as $n \rightarrow \infty$ by Theorem 1.1. Thus we have the following lemma related to an approximate identity, because $\sigma_{2\pi n}$ is a Dirac sequence.

Lemma 1.2. *Let $f \in L^1(R)$. Then we have $\lim_{n \rightarrow \infty} f * \sigma_{2\pi n}(\theta) = f(\theta)$, for almost all $\theta \in R$. Moreover, $f * \sigma_{2\pi n}$ converges to f uniformly on S , as $n \rightarrow \infty$, when S is a compact subset of R on which f is continuous.*

For the detail proof, see the example of Fejer kernel in [3] and [5].

2. Main results

Now we consider the kernel $K_n(x) = \sigma_{2\pi n}(x)$, that has been introduced in the above section. Also we put

$$K_{m,n}(x) = \sum_{k=-(m-1)}^{m-1} \frac{n}{m} \left(1 - \frac{|k|}{m}\right) e^{2\pi ni \frac{k}{m}x}.$$

Then we have

$$f * K_{m,n}(\theta) = \sum_{k=-(m-1)}^{m-1} \frac{n}{m} \left(1 - \frac{|k|}{m}\right) \hat{f}\left(\frac{nk}{m}\right) e^{2\pi i \frac{nk}{m}\theta},$$

where $\hat{f}\left(\frac{nk}{m}\right) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i \frac{nk}{m}x} dx$.

As we have seen in the above equation, one interesting part is that $f * K_{m,n}$ has nonharmonic Fourier coefficients, $\hat{f}\left(\frac{nk}{m}\right)$.

Also, the kernel $K_{m,n}$ can be expressed by

$$K_{m,n}(x) = \frac{n}{m} \left[1 + 2 \sum_{k=1}^{m-1} \left(1 - \frac{k}{m}\right) \cos 2\pi n \frac{k}{m}x \right].$$

Thus we obtain

$$f * K_{m,n}(\theta) = \frac{n}{m} \left[a(0) + 2 \sum_{k=1}^{m-1} \left(1 - \frac{k}{m} \right) \left(a \left(\frac{nk}{m} \right) \cos 2\pi \frac{nk}{m} \theta + b \left(\frac{nk}{m} \right) \sin 2\pi \frac{nk}{m} \theta \right) \right],$$

where $a(\frac{nk}{m}) = \int_{-\infty}^{\infty} f(x) \cos 2\pi \frac{nk}{m} x dx$ and $b(\frac{nk}{m}) = \int_{-\infty}^{\infty} f(x) \sin 2\pi \frac{nk}{m} x dx$.

Before getting into the further calculation, we need to specify $K_{m,n}$ as the closed form. The following lemma shows what it is:

Lemma 2.1. *We get $K_{m,n}(x) = \frac{n(\sin \pi nx)^2}{m^2(\sin \pi \frac{n}{m} x)^2}$, and we obtain*

$$\begin{aligned} K_{m,n}(x) - K_n(x) &= K_{m,n}(x) \left(1 - \left(\frac{\sin \pi \frac{n}{m} x}{\pi \frac{n}{m} x} \right)^2 \right) \\ &\leq n \left(1 - \left(\frac{\sin \pi \frac{n}{m} x}{\pi \frac{n}{m} x} \right)^2 \right). \end{aligned}$$

Proof. We put $F_n(x) = \frac{1}{2\pi} \sum_{k=-(n-1)}^{n-1} \left(1 - \frac{|k|}{n} \right) e^{ikx}$. Because F_n is the Fejer kernel, $F_n(x) = \frac{(\sin \frac{n}{2} x)^2}{2\pi n (\sin \frac{x}{2})^2}$. For the detail calculation of F_n , see [5]. Then we have $K_{m,n}(x) = 2\pi \frac{n}{m} F_m(2\pi \frac{n}{m} x)$. Hence the above inequality can be derived from $K_n(x) = \frac{(\sin \pi nx)^2}{\pi^2 nx^2}$ by a direct calculation. \square

From Lemma 2.1, we have some close relationship between $K_{m,n}$ and K_n . Since $f * K_n(x)$ converges to $f(x)$ by Lemma 1.2, we wonder if $f * K_{m,n}$ converges to f . If we replace $K_{m,n}$ by $K_{n^\alpha, n}$, then we obtain some convergence theorem for the recovery of functions in $L^1(R)$.

Theorem 2.2. *Let $f \in L^1(R)$ and let f_n be the n th truncation of f i.e., $f_n = \chi_{[-n,n]} f$. Then we get*

$$\lim_{n \rightarrow \infty} f_n * K_{n^\alpha, n}(\theta) = f(\theta),$$

for almost all $\theta \in R$, where α is integer greater than 2.

Proof. Since $f * K_n(\theta)$ converges to $f(\theta)$ for almost all $\theta \in R$, it is enough to show that for given θ and $\epsilon > 0$ there exists a positive large integer n depending on θ such that $|f_n * K_{n^\alpha, n}(\theta) - f * K_n(\theta)| \leq \epsilon$.

In the previous lemma, we know that

$$\begin{aligned} f_n * K_{n^\alpha, n}(\theta) - f * K_n(\theta) &= \int_{\theta-n}^{\theta+n} f(\theta-y) K_{n^\alpha, n}(y) \left(1 - \left(\frac{\sin \frac{\pi}{n^{\alpha-1}} y}{\frac{\pi}{n^{\alpha-1}} y} \right)^2 \right) dy \\ &\quad + \left(\int_{-\infty}^{\theta-n} + \int_{\theta+n}^{\infty} \right) f(\theta-y) K_n(y) dy \\ &= (1) + (2). \end{aligned}$$

The main part of the calculation is :

$$\begin{aligned}
(1) &\leq \int_{\theta-n}^{\theta+n} |f(\theta-y)| 2n \left(1 - \frac{\sin \frac{\pi}{n^{\alpha-1}} y}{\frac{\pi}{n^{\alpha-1}} y}\right) dy \\
&\leq \frac{n}{3} \int_{\theta-n}^{\theta+n} |f(\theta-y)| \left(\frac{\pi y}{n^{\alpha-1}}\right)^2 dy \\
&\leq \frac{\pi^2}{3} \frac{(|\theta|+n)^2}{n^{2\alpha-3}} \|f\|_1 \\
&\leq \frac{4\pi^2}{3n^{2\alpha-5}} \|f\|_1 \quad \text{if } n > |\theta| + 1.
\end{aligned}$$

For a calculation of the other part, choose n large enough such that $n > |\theta| + 1$. Then we have $|(\int_{-\infty}^{\theta-n} + \int_{\theta+n}^{\infty}) f(\theta-y) K_n(y) dy| \leq \frac{1}{n} \|f\|_1$ because $|K_n(y)| \leq \frac{1}{\pi^2 n}$ if $|y| \geq 1$.

Then we prove our theorem. \square

If we also discretize the $K_{m,n}$ properly using typical subdivision schemes, to be performed in Section 3, we have a nonharmonic Fourier series for functions in $L^1(\mathbb{R})$:

Theorem 2.3. *Let $f \in L^1(\mathbb{R})$ and let f_n be the n th truncation of f i.e., $f_n = \chi_{[-n,n]} f$. Then we obtain*

$$f(x) = \lim_{n \rightarrow \infty} \sum_{l=0}^n \sum_{k=0}^{2^{2l+1}-1} \frac{d_{l,k}}{2^l} \left(\tilde{a}_{l,k}^n \cos \frac{k\pi}{2^l} x + \tilde{b}_{l,k}^n \sin \frac{k\pi}{2^l} x \right),$$

for almost all $x \in \mathbb{R}$, where $\tilde{a}_{l,k}^n = \int_{-\infty}^{\infty} f_n(x) \cos \frac{k\pi}{2^l} x dx$, $\tilde{b}_{l,k}^n = \int_{-\infty}^{\infty} f_n(x) \sin \frac{k\pi}{2^l} x dx$, and where we put $d_{0,0} = 1, d_{0,1} = \frac{1}{2}$, and $d_{l,k}$ is exactly the same index as in Lemma 3.1.

The corollary below, to be proved in Section 3, gives some further convergence theorem for the functions of bounded support:

Corollary 2.4. *Suppose that f is integrable function which is zero outside of a bounded interval on \mathbb{R} . Then we have*

$$f(\theta) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^{2n+1}-1} \frac{d_{n,k}}{2^n} \left(a_{n,k} \cos \frac{k\pi}{2^n} \theta + b_{n,k} \sin \frac{k\pi}{2^n} \theta \right), \quad \text{for almost all } \theta \in \mathbb{R},$$

where $a_{n,k} = \int_{-\infty}^{\infty} f(x) \cos \frac{k\pi}{2^n} x dx$, $b_{n,k} = \int_{-\infty}^{\infty} f(x) \sin \frac{k\pi}{2^n} x dx$, and where we put $d_{0,0} = 1, d_{0,1} = \frac{1}{2}$, and $d_{n,k}$ is exactly the same index as in Lemma 3.1.

Moreover, the above series converges to f uniformly on K , when f belongs to $C_c(\mathbb{R})$ with the compact support K .

Remark 2.5. Observe that, in the above theorem and corollary, we have a convergent nonharmonic Fourier series simply localizing harmonic frequencies

of Fourier series. Without any localization of time domain, we get series representation like wavelets. We can compare these results to wavelet series and other filters in the digital signal processing (see [6] and [7]). For the useful application of $K_{m,n}$, see [8].

3. Some technical proofs and subdivision schemes

Now we are ready to think about some discretization of kernels to get an infinite series by means of natural subdivision schemes. Actually, we will discretize some singular kernel into pieces to get a nonharmonic Fourier series.

First of all, we discretize the kernel K_n as $K_{m,n}$ using a typical dyadic subdivision scheme. Hence we consider dyadically dilated kernel :

$$\begin{aligned} K_{2^n}(x) &= 2^{n+1} \int_0^1 (1-t) \cos 2\pi 2^n t x dt \\ &= \int_0^{2^{n+1}} \left(1 - \frac{s}{2^{n+1}}\right) \cos \pi s x ds. \end{aligned}$$

Also, we let $g_n(s, x) = \left(1 - \frac{s}{2^{n+1}}\right) \cos \pi s x$. Then we consider a projection $P_n(x)$ as sum of finite subdivision for $K_{2^n}(x)$ defined by $P_n(x) = \sum_{k=0}^{2^{n+1}-1} g_n(k, x)$. Thus we obtain

$$\begin{aligned} P_n(x) - P_{n-1}(x) &= \sum_{k=0}^{2^{n+1}-1} \left(1 - \frac{k}{2^{n+1}}\right) \cos \pi k x - \sum_{k=0}^{2^n-1} \left(1 - \frac{k}{2^n}\right) \cos \pi k x \\ &= \sum_{k=1}^{2^{n+1}-1} \frac{c_k}{2^{n+1}} \cos \pi k x, \end{aligned}$$

where

$$c_k = \begin{cases} k, & \text{if } k \leq 2^n \\ 2^{n+1} - k & \text{if } k > 2^n. \end{cases}$$

Roughly speaking, we are informed that $\lim_{n \rightarrow \infty} K_{2^n}(x)$ can be approximated to the limit of $P_n(x)$ by the following sum :

$$\begin{aligned} \lim_{n \rightarrow \infty} P_n(x) &= P_0(x) + \sum_{n=1}^{\infty} (P_n(x) - P_{n-1}(x)) \\ &= \lim_{n \rightarrow \infty} \left(1 + \sum_{k=0}^n \sum_{i=1}^{2^{k+1}-1} \frac{c_i}{2^{k+1}} \cos \pi i x \right) \\ &= \sum_{n=0}^{\infty} \cos n \pi x. \end{aligned}$$

Formally, we obtain the Fourier series from a dyadic subdivision scheme for the kernel K_{2^n} :

$$\begin{aligned} \lim_{n \rightarrow \infty} f * P_n(x) &= \sum_{n=0}^{\infty} \int f(y) \cos \pi n(x-y) dy \\ &= a_0 + \sum_{n=1}^{\infty} (a_n \cos \pi n x + b_n \sin \pi n x), \end{aligned}$$

where $a_n = \int f(y) \cos n\pi y dy$, for $n = 0, 1, 2, \dots$ and $b_n = \int f(y) \sin n\pi y dy$, for $n = 1, 2, \dots$

But the above projection P_n of the kernel K_{2^n} doesn't guarantee the convergence to functions in $L^1(R)$. Hence we need some slightly different projections $P_{2^n}^n$ to the same kernel K_{2^n} .

Since we have $K_{2^n}(x) = 2^n \int_0^2 (1 - \frac{s}{2}) \cos 2^n \pi s x ds$, we put $\tilde{g}_n(s, x) = (1 - s/2) \cos 2^n \pi s x$. Thus we can consider projections $P_{2^n}^n$ as the discretization of the kernel $K_{2^n}(x)$:

$$\begin{aligned} P_{2^n}^l(x) &= 2^n \left[\frac{1}{2^{n+l}} \sum_{k=0}^{2^{n+l+1}-1} \tilde{g}_n \left(\frac{k}{2^{n+l}}, x \right) \right] \\ &= \frac{1}{2^l} \sum_{k=0}^{2^{n+l+1}-1} \left(1 - \frac{k}{2^{n+l+1}} \right) \cos \frac{k\pi}{2^l} x. \end{aligned}$$

Therefore, we can find the closed form for projections $P_{2^n}^n(x) - P_{2^{n-1}}^{n-1}(x)$ by a simple calculation:

Lemma 3.1. *We obtain*

$$\begin{aligned} P_{2^l}^l(x) - P_{2^{l-1}}^{l-1}(x) &= \sum_{k=0}^{2^{2l+1}-1} \frac{d_{l,k}}{2^l} \cos \frac{k\pi}{2^l} x, \\ P_{2^n}^n(x) &= \sum_{l=0}^n \sum_{k=0}^{2^{2l+1}-1} \frac{d_{l,k}}{2^l} \cos \frac{k\pi}{2^l} x, \end{aligned}$$

where

$$c_{l,k} = \begin{cases} k, & \text{if } k \leq 2^{2l-1} \\ 2^{2l} - k & \text{if } k > 2^{2l-1}, \end{cases}$$

$d_{0,0} = 1, d_{0,1} = \frac{1}{2}, d_{l,2k} = (-1) \left(1 - \frac{2k}{2^{2l+1}} \right) + \frac{2c_{l,k}}{2^{2l}}$, and $d_{l,2k+1} = \left(1 - \frac{2k+1}{2^{2l+1}} \right)$, $l = 1, 2, 3, \dots, k = 0, 1, 2, 3, \dots$

Proof. Now we calculate

$$\begin{aligned}
P_{2^l}^l(x) - P_{2^{l-1}}^{l-1} &= (P_{2^l}^l(x) - P_{2^l}^{l-1}(x)) + (P_{2^l}^{l-1}(x) - P_{2^{l-1}}^{l-1}(x)) \\
&= \frac{1}{2^l} \sum_{k=0}^{2^{2l+1}-1} (-1)^{k+1} \left(1 - \frac{k}{2^{2l+1}}\right) \cos \frac{k\pi}{2^l} x \\
&\quad + \frac{1}{2^{l-1}} \sum_{k=1}^{2^{2l}-1} \frac{c_{l,k}}{2^{2l}} \cos \frac{k\pi}{2^{l-1}} x \\
&= \frac{1}{2^l} \sum_{k=0}^{2^{2l+1}-1} d_{l,k} \cos \frac{k\pi}{2^l} x
\end{aligned}$$

and thus, we have

$$\begin{aligned}
P_{2^n}^n(x) &= P_{2^0}^0(x) + \sum_{l=1}^n (P_{2^l}^l(x) - P_{2^{l-1}}^{l-1}(x)) \\
&= \sum_{l=0}^n \sum_{k=0}^{2^{2l+1}-1} \frac{d_{l,k}}{2^l} \cos \frac{k\pi}{2^l} x,
\end{aligned}$$

where

$$c_{l,k} = \begin{cases} k, & \text{if } k \leq 2^{2l-1} \\ 2^{2l} - k & \text{if } k > 2^{2l-1}, \end{cases}$$

$d_{0,0} = 1, d_{0,1} = \frac{1}{2}, d_{l,2k} = (-1)(1 - \frac{2k}{2^{2l+1}}) + \frac{2c_{l,k}}{2^{2l}}$, and $d_{l,2k+1} = (1 - \frac{2k+1}{2^{2l+1}})$, $l = 1, 2, 3, \dots, k = 0, 1, 2, 3, \dots$

Hence we complete our lemma. \square

In a formal manner without proofs, we obtain

$$\begin{aligned}
\lim_{n \rightarrow \infty} f * P_{2^n}^n(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{2^{2n+1}-1} \frac{d_{n,k}}{2^n} \int f(y) \cos \frac{k\pi}{2^n} (x-y) dy \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{2^{2n+1}-1} \frac{d_{n,k}}{2^n} \left(a_{n,k} \cos \frac{k\pi}{2^n} x + b_{n,k} \sin \frac{k\pi}{2^n} x \right)
\end{aligned}$$

where $a_{n,k} = \int_{-\infty}^{\infty} f(x) \cos \frac{k\pi}{2^n} x dx$ and $b_{n,k} = \int_{-\infty}^{\infty} f(x) \sin \frac{k\pi}{2^n} x dx$.

Therefore, we consider the convergence of the above nonharmonic Fourier series for functions in $L^1(\mathbb{R})$. See Theorem 2.3:

Proof of Theorem 2.3. Since the n th partial sum of the above series is $P_{2^n}^n$ by the construction of the series itself or by Lemma 3.1, it is enough to show that

$$\lim_{n \rightarrow \infty} f_n * P_{2^n}^n(x) = f(x)$$

for almost all $x \in R$. By the definition of projections, we get

$$\begin{aligned} P_{2^n}^n(x) &= \frac{1}{2^n} \sum_{k=0}^{2^{2n+1}-1} \left(1 - \frac{k}{2^{2n+1}}\right) \cos \frac{k\pi}{2^n} x \\ &= \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} \left[1 + 2 \sum_{k=1}^{2^{2n+1}-1} \left(1 - \frac{k}{2^{2n+1}}\right) \cos \frac{k\pi}{2^n} x \right] \\ &= \frac{1}{2^{n+1}} + K_{2^{2n+1}, 2^n}(x) \end{aligned}$$

Hence we copy Theorem 2.2 with the same method. Let $\theta \in R$ be given and $\epsilon > 0$ be given. We are going to choose a positive large integer n depending on θ such that $|f_n * P_{2^n}^n(\theta) - f * K_{2^n}(\theta)| \leq \epsilon$. By a simple calculation we have

$$\begin{aligned} |f_n * P_{2^n}^n(\theta) - f * K_{2^n}(\theta)| &\leq |f_n * K_{2^{2n+1}, 2^n}(\theta) - f * K_{2^n}(\theta)| + \frac{\|f\|_1}{2^{n+1}} \\ &\leq \int_{\theta-n}^{\theta+n} |f(\theta-y)| 2^{n+1} \left| 1 - \frac{\sin \frac{\pi}{2^{n+1}} y}{\frac{\pi}{2^{n+1}} y} \right| dy \\ &\quad + \left(\int_{-\infty}^{\theta-n} + \int_{\theta+n}^{\infty} \right) |f(\theta-y)| K_{2^n}(y) dy + \frac{\|f\|_1}{2^{n+1}} \\ &\leq \frac{\pi^2}{3!} \frac{(|\theta|+n)^2}{2^{n+1}} \|f\|_1 + \frac{1}{\pi^2 2^n} \|f\|_1 + \frac{\|f\|_1}{2^{n+1}} \\ &\leq c \frac{n^2}{2^n} \|f\|_1 \end{aligned}$$

for some constant c when we choose n large enough such that $n > |\theta| + 1$.

Since $f * K_{2^n}$ converges to f for almost all $\theta \in R$ by Lemma 1.2, this gives our proof complete.

Proof of Corollary 2.4. Choose an interval $I = [-M, M]$ such that f is zero outside of this interval I . Also we get $f = f_n$ for all $n > M$. The first part of theorem is proved by the copy of Theorem 2.3. The remaining part of our theorem for the uniform convergence is related to error estimation. See Remark 3.2 in the below for the detail proof.

Remark 3.2. An error estimation For applications of recovering functions $f \in L^1(R)$, we state some error estimation of the above convergent series. As in Theorem 2.3, we use similar calculations to estimate error.

Let K be a compact support to the given function $f \in C_c(R)$. Then we take large $M > 0$ such that $K \subset [-M, M]$. Then $f(\theta-y) = 0$ for all $y \in [-3M, 3M]^c$

and for all $\theta \in K$. Hence we get

$$\begin{aligned} |f * P_{2^n}^n(\theta) - f * K_{2^n}(\theta)| &\leq \frac{\|f\|_1}{2^{n+1}} + |f * K_{2^{2n+1}, 2^n}(\theta) - f * K_{2^n}(\theta)| \\ &\leq \int_{-3M}^{3M} |f(\theta - y)| 2^{n+1} \left| 1 - \frac{\sin \frac{\pi}{2^{n+1}} y}{\frac{\pi}{2^{n+1}} y} \right| dy + \frac{\|f\|_1}{2^{n+1}} \\ &\leq \frac{cM^2}{2^n} \|f\|_1 \end{aligned}$$

for some constant $c > 0$.

Also, for a given $\epsilon > 0$, we choose $\delta > 0$ such that $|f(\theta - y) - f(\theta)| < \epsilon$ whenever $\theta \in K$ and $|y| < \delta$ since f is uniformly continuous on $[-3M, 3M]$. Thus we have, for all $\theta \in K$,

$$\begin{aligned} |f * K_{2^n}(\theta) - f(\theta)| &\leq \int_{-\delta}^{\delta} |f(\theta - y) - f(\theta)| K_{2^n}(y) dy \\ &\quad + \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) \|f\|_{\infty} K_{2^n}(y) dy \\ &\leq \epsilon + 2 \int_{\delta}^{\infty} \frac{1}{\pi^2 2^n y^2} dy \\ &\leq \epsilon + \frac{2\|f\|_{\infty}}{\pi^2 2^n \delta}. \end{aligned}$$

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