East Asian Mathematical Journal
Vol. 26 (2010), No. 1, pp. 81–93

CONVERGENCE THEOREMS OF MULTI-STEP ITERATIVE SCHEMES WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE NONSELF MAPPINGS

Jong Kyu Kim, G. S. Saluja, and H. K. Nashine

Abstract. In this paper, a strong convergence theorem of multi-step iterative schemes with errors for asymptotically quasi-nonexpansive type nonself mappings is established in a real uniformly convex Banach space. Our results extend the corresponding results of Wangkeeree [12], Xu and Noor [13], Kim et al.[1,6,7] and many others.

1. Introduction

Let E be a real Banach space, $F(T)$, $D(T)$ and $\mathbf{N}$ denote the set of fixed points of T, the domain of T, and the set of positive integers, respectively.

Definition 1.1. Let $T\colon D(T) \subset E \to E$ be a mapping.

(1) T is said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||$$

for all $x, y \in D(T)$.

(2) T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - p|| \leq ||x - p||$$

for all $x \in D(T)$ and $p \in F(T)$.

(3) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y||$$

for all $x, y \in D(T)$ and $n \in N$.

Received October 30, 2009; Accepted January 4, 2010.
2000 *Mathematics Subject Classification.* 47H09, 47H10.
Key words and phrases. Asymptotically quasi-nonexpansive type nonself mappings, fixed point, multi-step iteration scheme with errors, real uniformly convex Banach space.
This work was supported by National Research Foundation of Korea Grant funded by the Korean Government (2009-0076898).

(4) T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - p|| \leq k_n ||x - p||$$

for all $x \in D(T)$, $p \in F(T)$ and all $n \in N$.

(5) T is said to be asymptotically nonexpansive type if

$$\limsup_{n\to\infty}\{\sup_{x,y\in D(T)} [\|T^n x - T^n y\|^2 - \|x - y\|^2]\} \leq 0.$$

(6) T is said to be asymptotically quasi-nonexpansive type [8] if

$$\limsup_{n\to\infty}\{\sup_{x\in D(T),p\in F(T)} [\|T^n x - p\|^2 - \|x - p\|^2]\} \leq 0.$$

Remark 1.2. We know that the following implications hold from the above definitions:

$$\begin{array}{ccccc} (1) & \Longrightarrow & (3) & \Longrightarrow & (5) \\ \Downarrow F(T) \neq \emptyset & & \Downarrow F(T) \neq \emptyset & & \Downarrow F(T) \neq \emptyset \\ (2) & \Longrightarrow & (4) & \Longrightarrow & (6) \end{array}$$

Noor [8,9] introduced the three-step iterative sequences and studied the approximate solution of variational inequalities in Hilbert spaces.

The three-step iterative approximation problems were studied extensively by Glowinski and Le Tallec [3], Haubruge et al. [5], Noor [8,9] and Kim et al [6,7].

In the year 2002, Xu and Noor [13] introduced the three-step iterative for asymptotically nonexpansive mappings and proved the following strong convergence theorem in Banach spaces;

Theorem 1.3. [13] *Let X be a real uniformly convex Banach space, C be a nonempty closed bounded convex subset of X. Let T be a completely continuous asymptotically nonexpansive self mappings with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For a given $x_0 \in C$, define

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1 - \gamma_n)x_n \\ y_n &= \beta_n T^n z_n + (1 - \beta_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n. \end{aligned} \tag{1.1}$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ convergence strongly to a fixed point of T.

Recently, Wangkeeree [12] introduced the three-step iterative sequences with errors for asymptotically quasi-nonexpansive nonself mappings and proved the following strong convergence theorem in real uniformly convex Banach spaces:

Theorem 1.4. [12] *Let X be a real uniformly convex Banach space, C be a nonempty closed bounded convex subset of X and P be the nonexpansive retraction of X onto C. Let T be a uniformly L-Lipschitzian completely continuous and asymptotically quasi-nonexpansive nonself-mapping with sequence $\{k_n\}$ in $[0,\infty)$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. For an arbitrary $x_0 \in C$ the sequence $\{x_n\}$ defined by*

$$\begin{aligned} z_n &= P(\alpha_n'' T(PT)^{n-1} x_n + \beta_n'' x_n + \gamma_n'' u_n), \\ y_n &= P(\alpha_n' T(PT)^{n-1} z_n + \beta_n' x_n + \gamma_n' v_n), \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1} y_n + \beta_n x_n + \gamma_n w_n), \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$ and $\{\gamma_n''\}$ are appropriate real sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three bounded sequences in C with the following restrictions:

(i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$,
(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, $\sum_{n=1}^{\infty} \gamma_n'' < \infty$,
(iii) $0 \le \alpha < \alpha_n, \beta_n, \alpha_n', \beta_n' \le \beta < 1$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

A subset K of E is called retract of E if there exists a continuous mapping $P\colon E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P\colon E \to K$ is called retraction if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P.

Motivated by Wangkeeree [12], Xu and Noor [13] and some others, the purpose of this paper is to construct a multi step iteration scheme with errors for approximating fixed point of asymptotically quasi-nonexpansive type nonself mappings (when such a fixed point exists) and to prove strong convergence theorem for such maps.

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E and $T\colon K \to E$ be an asymptotically quasi-nonexpansive type nonself mapping.

For a given $x_1 \in K$, and a fixed $N \in \mathbf{N}$, define the sequence $\{x_n\}$ by

$$\begin{aligned} x_{n+1} = x_n^{(N)} &= P(\alpha_n^{(N)} T(PT)^{n-1} x_n^{(N-1)} + \beta_n^{(N)} x_n + \gamma_n^{(N)} u_n^{(N)}) \\ x_n^{(N-1)} &= P(\alpha_n^{(N-1)} T(PT)^{n-1} x_n^{(N-2)} + \beta_n^{(N-1)} x_n + \gamma_n^{(N-1)} u_n^{(N-1)}) \\ &\cdots \\ x_n^{(3)} &= P(\alpha_n^{(3)} T(PT)^{n-1} x_n^{(2)} + \beta_n^{(3)} x_n + \gamma_n^{(3)} u_n^{(3)}) \\ x_n^{(2)} &= P(\alpha_n^{(2)} T(PT)^{n-1} x_n^{(1)} + \beta_n^{(2)} x_n + \gamma_n^{(2)} u_n^{(2)}) \\ x_n^{(1)} &= P(\alpha_n^{(1)} T(PT)^{n-1} x_n + \beta_n^{(1)} x_n + \gamma_n^{(1)} u_n^{(1)}) \end{aligned} \tag{1.3}$$

where $\{u_n^{(1)}\}, \{u_n^{(2)}\}, \dots, \{u_n^{(N)}\}$ are bounded sequences in K and $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$ are appropriate real sequences in $[0,1]$ such that $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$ for each $i \in \{1, 2, \dots, N\}$.

In this paper, we will prove the convergence theorems of the iteration scheme (1.3) for asymptotically quasi-nonexpansive type nonself mappings. The results presented in this paper generalize and extend the corresponding results of Wangkeeree [12], Xu and Noor [13], Kim et al. [6,7] and many others.

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1. [2] Let E be a real normed linear space, K a nonempty subset of E. Let $P\colon E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T\colon K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that the following inequality holds:

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le (1+k_n)\left\|x-y\right\|; \forall x, y \in K, \forall n \ge 1. \quad (2.1)$$

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that:

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le L\left\|x-y\right\|; \ \forall x, y \in K, \ \forall n \ge 1. \quad (2.2)$$

Definition 2.2. [12] A nonself mapping $T\colon K \to E$ is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} k_n = 0$ such that:

$$\left\|T(PT)^{n-1}x - x^*\right\| \le (1+k_n)\left\|x-x^*\right\|; \forall x \in K, x^* \in F(T), \forall n \ge 1. \quad (2.3)$$

Definition 2.3. [4] A Banach space E is said to be uniformly convex if the modulus of convexity of E

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\} > 0$$

for all $0 < \varepsilon \le 2$ (i.e., $\delta_E(\varepsilon)$ is a function $(0,2] \to (0,1)$).

Lemma 2.4. [11] *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.5. [10] *Let E be a real uniformly convex Banach space and $0 < a \le t_n \le b < 1$ for all $n \ge 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying*

$$\limsup_{n\longrightarrow\infty} \|x_n\| \le r, \qquad \limsup_{n\longrightarrow\infty} \|y_n\| \le r,$$

and

$$\lim_{n\longrightarrow\infty}\|t_nx_n+(1-t_n)y_n\|=r,$$

for some $r\geq 0$*. Then*

$$\lim_{n\longrightarrow\infty}\|x_n-y_n\|=0.$$

3. Main results

In this section, we define an asymptotically quasi-nonexpansive type nonself-mapping and prove the convergence theorems of the iterative sequences which is generated by the mapping.

Definition 3.1. Let K be a nonempty subset of a Banach space E. Let $P\colon E\to K$ be a nonexpansive retraction of E onto K. A nonself mapping $T\colon K\to E$ is said to be asymptotically quasi-nonexpansive type if the following inequality hold:

$$\limsup_{n\to\infty}\{\sup_{x\in K,p\in F(T)}[\left\|T(PT)^{n-1}x-p\right\|^2-\|x-p\|^2]\}\quad\leq\quad 0. \tag{3.1}$$

Remark 3.2. If T is a self-map, then $PT=T$, so that (3.1) coincides with (6) in Definition 1.1.

Lemma 3.3. *Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E and P a nonexpansive retraction of E onto K. Let $T\colon K\to E$ be an asymptotically quasi-nonexpansive type nonself mapping with $F(T)\neq\emptyset$. Let $\{x_n\}$ be a sequence in (1.3) with the following conditions:*

(i) $\alpha_n^{(i)}+\beta_n^{(i)}+\gamma_n^{(i)}=1$ *for all* $i=1,2,\cdots,N$,

(ii) $\sum_{n=1}^{\infty}\gamma_n^{(i)}<\infty$ *for all* $i=1,2,\cdots,N$.

Then for each $x^*\in F(T)$, $\lim_{n\to\infty}\|x_n-x^*\|$ *exists.*

Proof. Let $x^*\in F(T)$. Since $\{u_n^{(i)}\}$ for $i=1,2,\ldots,N$ is bounded sequence in E, so we put

$$M=\max\{\sup_{n\geq 1}\left\|u_n^{(i)}-x^*\right\|:i=1,2,\ldots,N\},$$

it follows from (3.1) that

$$\begin{aligned}&\limsup_{n\to\infty}\left\{\sup_{x\in K,x^*\in F(T)}\left[(\left\|T(PT)^{n-1}x-x^*\right\|-\|x-p\|)\right.\right.\\&\qquad\qquad\left.\left.\times(\left\|T(PT)^{n-1}x-x^*\right\|+\|x-x^*\|)\right]\right\}\\&=\limsup_{n\to\infty}\{\sup_{x\in K,x^*\in F(T)}[\left\|T(PT)^{n-1}x-x^*\right\|^2-\|x-x^*\|^2]\}\\&\leq 0.\end{aligned}$$

Therefore we have

$$\limsup_{n\to\infty}\{\sup_{x\in K, x^*\in F(T)}[\left\|T(PT)^{n-1}x-x^*\right\|-\|x-x^*\|]\}\leq 0.$$

This implies that for any given $\varepsilon_n>0$, there exists a positive integer n_0 such that for $n\geq n_0$, we have

$$\sup_{x\in K, x^*\in F(T)}\{\left\|T(PT)^{n-1}x-x^*\right\|-\|x-x^*\|\}<\varepsilon_n.$$

Since $\{x_n\},\{x_n^{(1)}\},\cdots,\{x_n^{(N-1)}\}\subset E$, we have

$$\begin{aligned}
\left\|T(PT)^{n-1}x_n-x^*\right\|-\|x_n-x^*\| &< \varepsilon_n,\ \forall p\in F(T),\forall n\geq n_0,\\
\left\|T(PT)^{n-1}x_n^{(1)}-x^*\right\|-\left\|x_n^{(1)}-x^*\right\| &< \varepsilon_n,\ \forall p\in F(T),\forall n\geq n_0,\\
\left\|T(PT)^{n-1}x_n^{(2)}-x^*\right\|-\left\|x_n^{(2)}-x^*\right\| &< \varepsilon_n,\ \forall p\in F(T),\forall n\geq n_0,\\
&\cdots \qquad (3.2)\\
\left\|T(PT)^{n-1}x_n^{(N-1)}-x^*\right\|-\left\|x_n^{(N-1)}-x^*\right\| &< \varepsilon_n,\ \forall p\in F(T),\forall n\geq n_0.
\end{aligned}$$

Thus for any $x^*\in F(T)$, using (1.3) and (3.2), we know that

$$\begin{aligned}
\left\|x_n^{(1)}-x^*\right\| &= \left\|P(\alpha_n^{(1)}T(PT)^{n-1}x_n+\beta_n^{(1)}x_n+\gamma_n^{(1)}u_n^{(1)})-Px^*\right\|\\
&= \left\|\alpha_n^{(1)}T(PT)^{n-1}x_n+\beta_n^{(1)}x_n+\gamma_n^{(1)}u_n^{(1)}-x^*\right\|\\
&\leq \alpha_n^{(1)}\left\|T(PT)^{n-1}x_n-x^*\right\|+\beta_n^{(1)}\|x_n-x^*\|+\gamma_n^{(1)}\left\|u_n^{(1)}-x^*\right\|\\
&\leq \alpha_n^{(1)}\|x_n-x^*\|+\varepsilon_n+\beta_n^{(1)}\|x_n-x^*\|+\gamma_n^{(1)}M \qquad (3.3)\\
&\leq (\alpha_n^{(1)}+\beta_n^{(1)})\|x_n-x^*\|+\varepsilon_n+\gamma_n^{(1)}M\\
&\leq \|x_n-x^*\|+t_n^{(1)}
\end{aligned}$$

where $t_n^{(1)}=\varepsilon_n+\gamma_n^{(1)}M$. Since $\sum_{n=1}^{\infty}\gamma_n^{(1)}<\infty$, we can see that $\sum_{n=1}^{\infty}t_n^{(1)}<\infty$. It follows from (3.3) that

$$\begin{aligned}
\left\|x_n^{(2)}-x^*\right\| &= \left\|P(\alpha_n^{(2)}T(PT)^{n-1}x_n^{(1)}+\beta_n^{(2)}x_n+\gamma_n^{(2)}u_n^{(2)})-Px^*\right\|\\
&= \left\|\alpha_n^{(2)}T(PT)^{n-1}x_n^{(1)}+\beta_n^{(2)}x_n+\gamma_n^{(2)}u_n^{(2)}-x^*\right\|\\
&\leq \alpha_n^{(2)}\left\|T(PT)^{n-1}x_n^{(1)}-x^*\right\|+\beta_n^{(2)}\|x_n-x^*\|+\gamma_n^{(2)}\left\|u_n^{(2)}-x^*\right\|\\
&\leq \alpha_n^{(2)}\left\|x_n^{(1)}-x^*\right\|+\varepsilon_n+\beta_n^{(2)}\|x_n-x^*\|+\gamma_n^{(2)}M \qquad (3.4)\\
&\leq \alpha_n^{(2)}(\|x_n-x^*\|+t_n^{(1)})+\varepsilon_n+\beta_n^{(2)}\|x_n-x^*\|\\
&\quad +\gamma_n^{(2)}M\\
&\leq (\alpha_n^{(2)}+\beta_n^{(2)})\|x_n-x^*\|+\varepsilon_n+\alpha_n^{(2)}t_n^{(1)}+\gamma_n^{(2)}M\\
&\leq \|x_n-x^*\|+t_n^{(2)}
\end{aligned}$$

where $t_n^{(2)} = \varepsilon_n + \alpha_n^{(2)} t_n^{(1)} + \gamma_n^{(2)} M$. Since $\sum_{n=1}^{\infty} \gamma_n^{(2)} < \infty$ and $\sum_{n=1}^{\infty} t_n^{(1)} < \infty$, we can see that $\sum_{n=1}^{\infty} t_n^{(2)} < \infty$. Similarly, we show that

$$\begin{aligned}
\left\| x_n^{(3)} - x^* \right\| &\leq \alpha_n^{(3)} \left\| T(PT)^{n-1} x_n^{(2)} - x^* \right\| + \beta_n^{(3)} \left\| x_n - x^* \right\| + \gamma_n^{(3)} \left\| u_n^{(3)} - x^* \right\| \\
&\leq \alpha_n^{(3)} \left\| x_n^{(2)} - x^* \right\| + \varepsilon_n + \beta_n^{(3)} \left\| x_n - x^* \right\| + \gamma_n^{(3)} M \\
&\leq \alpha_n^{(3)} (\left\| x_n - x^* \right\| + t_n^{(2)}) + \varepsilon_n + \beta_n^{(3)} \left\| x_n - x^* \right\| \\
&\quad + \gamma_n^{(3)} M \qquad (3.5) \\
&\leq (\alpha_n^{(3)} + \beta_n^{(3)}) \left\| x_n - x^* \right\| + \varepsilon_n + \alpha_n^{(3)} t_n^{(2)} + \gamma_n^{(3)} M \\
&\leq \left\| x_n - x^* \right\| + t_n^{(3)}
\end{aligned}$$

where $t_n^{(3)} = \varepsilon_n + \alpha_n^{(3)} t_n^{(2)} + \gamma_n^{(3)} M$. Since $\sum_{n=1}^{\infty} \gamma_n^{(3)} < \infty$ and $\sum_{n=1}^{\infty} t_n^{(2)} < \infty$, we can see that $\sum_{n=1}^{\infty} t_n^{(3)} < \infty$. Continuing this process, we get

$$\begin{aligned}
\left\| x_{n+1} - x^* \right\| &= \left\| x_n^{(N)} - x^* \right\| \\
&\leq \left\| x_n - x^* \right\| + t_n^{(N)} \qquad (3.6)
\end{aligned}$$

where $\{t_n^{(N)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} t_n^{(N)} < \infty$. It follows from Lemma 2.4, we have $\lim_{n\to\infty} \left\| x_n - x^* \right\|$ exists. This completes the proof. □

Theorem 3.4. Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E and P a nonexpansive retraction of E onto K. Let $T\colon K \to E$ be a uniformly L-Lipschitzian asymptotically quasi-nonexpansive type nonself mapping with $F(T) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (1.3) and some $\alpha, \beta \in (0, 1)$ with the following restrictions:

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $1 \leq i \leq N$,

(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$,

(iii) $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ for some $n_0 \in N$.

Then $\lim_{n\to\infty} \left\| x_n - Tx_n \right\| = 0$.

Proof. For any $x^* \in F(T)$, it follows from Lemma 3.3 that $\lim_{n\to\infty} \left\| x_n - x^* \right\|$ exists. Let $\lim_{n\to\infty} \left\| x_n - x^* \right\| = a$ for some $a \geq 0$. we note that

$$\left\| x_n^{(N-1)} - x^* \right\| \leq \left\| x_n - x^* \right\| + t_n^{(N-1)}, \quad \forall n \geq 1,$$

where $\{t_n^{(N-1)}\}$ is nonnegative real sequence such that $\sum_{n=1}^{\infty} t_n^{(N-1)} < \infty$. It follows that

$$\begin{aligned}
\limsup_{n\to\infty} \left\| x_n^{(N-1)} - x^* \right\| &\leq \limsup_{n\to\infty} (\left\| x_n - x^* \right\| + t_n^{(N-1)}) \\
&= \lim_{n\to\infty} \left\| x_n - x^* \right\| \\
&= a
\end{aligned}$$

and so

$$\begin{aligned}\limsup_{n\to\infty}\left\|T(PT)^{n-1}x_n^{(N-1)}-x^*\right\| &\le \limsup_{n\to\infty}(\left\|x_n^{(N-1)}-x^*\right\|+\varepsilon_n)\\ &= \limsup_{n\to\infty}\left\|x_n^{(N-1)}-x^*\right\|\\ &\le a.\end{aligned}$$

Next, consider

$$\begin{aligned}\left\|T(PT)^{n-1}x_n^{(N-1)}-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n)\right\| &\le \left\|T(PT)^{n-1}x_n^{(N-1)}-x^*\right\|\\ &\quad+\gamma_n^{(N)}\left\|u_n^{(N)}-x_n\right\|.\end{aligned}$$

Thus,

$$\limsup_{n\to\infty}\left\|T(PT)^{n-1}x_n^{(N-1)}-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n)\right\| \le a. \tag{3.7}$$

Also,

$$\left\|x_n-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n)\right\|\le\|x_n-x^*\|+\gamma_n^{(N)}\left\|u_n^{(N)}-x_n\right\|$$

implies that

$$\limsup_{n\to\infty}\left\|x_n-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n)\right\| \le a, \tag{3.8}$$

and we observe that

$$\begin{aligned}&x_n^{(N)}-x^*\\ &=\alpha_n^{(N)}T(PT)^{n-1}x_n^{(N-1)}-\alpha_n^{(N)}x^*+\alpha_n^{(N)}\gamma_n^{(N)}u_n^{(N)}-\alpha_n^{(N)}\gamma_n^{(N)}x_n\\ &\quad+(1-\alpha_n^{(N)})x_n-(1-\alpha_n^{(N)})x^*-\gamma_n^{(N)}x_n+\gamma_n^{(N)}u_n^{(N)}-\alpha_n^{(N)}\gamma_n^{(N)}u_n^{(N)}\\ &\quad+\alpha_n^{(N)}\gamma_n^{(N)}x_n\\ &=\alpha_n^{(N)}(T(PT)^{n-1}x_n^{(N-1)}-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n)+(1-\alpha_n^{(N)})(x_n-x^*)\\ &\quad-(1-\alpha_n^{(N)})\gamma_n^{(N)}x_n+(1-\alpha_n^{(N)})\gamma_n^{(N)}u_n^{(N)}\\ &=\alpha_n^{(N)}(T(PT)^{n-1}x_n^{(N-1)}-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n)\\ &\quad+(1-\alpha_n^{(N)})(x_n-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n)).\end{aligned}$$

Therefore,

$$\begin{aligned}a&=\lim_{n\to\infty}\|x_n^{(N)}-x^*\|\\ &=\lim_{n\to\infty}\|\alpha_n^{(N)}(T(PT)^{n-1}x_n^{(N-1)}-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n))\\ &\quad+(1-\alpha_n^{(N)})(x_n-x^*+\gamma_n^{(N)}(u_n^{(N)}-x_n))\|.\end{aligned}$$

By (3.7), (3.8) and Lemma 2.5, we have

$$\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n^{(N-1)}-x_n\right\|=0. \tag{3.9}$$

Now, we shall show that $\lim_{n\to\infty}\left\|T(PT)^{n-1}x_n^{(N-2)}-x_n\right\|=0$. For each $n\geq 1$,

$$\|x_n-x^*\|\leq\left\|T(PT)^{n-1}x_n^{(N-1)}-x_n\right\|+\left\|T(PT)^{n-1}x_n^{(N-1)}-x^*\right\|$$
$$\leq\left\|T(PT)^{n-1}x_n^{(N-1)}-x_n\right\|+\left\|x_n^{(N-1)}-x^*\right\|+\varepsilon_n.$$

Using (3.9), we have

$$a=\lim_{n\to\infty}\|x_n-x^*\|\leq\liminf_{n\to\infty}\left\|x_n^{(N-1)}-x^*\right\|.$$

It follows that

$$a\leq\liminf_{n\to\infty}\left\|x_n^{(N-1)}-x^*\right\|\leq\limsup_{n\to\infty}\left\|x_n^{(N-1)}-x^*\right\|\leq a.$$

This implies that

$$\lim_{n\to\infty}\left\|x_n^{(N-1)}-x^*\right\|=a.$$

On the other hand, we have

$$\left\|x_n^{(N-2)}-x^*\right\|\leq\|x_n-x^*\|+t_n^{(N-2)},\quad\forall n\geq 1,$$

where $\sum_{n=1}^{\infty}t_n^{(N-2)}<\infty$. Therefore,

$$\limsup_{n\to\infty}\left\|x_n^{(N-2)}-x^*\right\|\leq\limsup_{n\to\infty}(\|x_n-x^*\|+\varepsilon_n)\leq a.$$

Next, consider

$$\left\|T(PT)^{n-1}x_n^{(N-2)}-x^*+\gamma_n^{(N-1)}(u_n^{(N-1)}-x_n)\right\|$$
$$\leq\left\|T(PT)^{n-1}x_n^{(N-2)}-x^*\right\|+\gamma_n^{(N-1)}\left\|u_n^{(N-1)}-x_n\right\|.$$

Thus,

$$\limsup_{n\to\infty}\left\|T(PT)^{n-1}x_n^{(N-2)}-x^*+\gamma_n^{(N-1)}(u_n^{(N-1)}-x_n)\right\|\quad\leq\quad a.\tag{3.10}$$

Also,

$$\left\|x_n-x^*+\gamma_n^{(N-1)}(u_n^{(N-1)}-x_n)\right\|\leq\|x_n-x^*\|+\gamma_n^{(N-1)}\left\|u_n^{(N-1)}-x_n\right\|$$

implies that

$$\limsup_{n\to\infty}\left\|x_n-x^*+\gamma_n^{(N-1)}(u_n^{(N-1)}-x_n)\right\|\quad\leq\quad a,\tag{3.11}$$

and we observe that

$$x_n^{(N-1)}-x^*=\alpha_n^{(N-1)}T(PT)^{n-1}x_n^{(N-2)}+(1-\alpha_n^{(N-1)})x_n$$
$$-\gamma_n^{(N-1)}x_n+\gamma_n^{(N-1)}u_n^{(N-1)}-(1-\alpha_n^{(N-1)})x^*-\alpha_n^{(N-1)}x^*$$
$$=\alpha_n^{(N-1)}(T(PT)^{n-1}x_n^{(N-2)}-x^*+\gamma_n^{(N-1)}(u_n^{(N-1)}-x_n)$$
$$+(1-\alpha_n^{(N-1)})(x_n-x^*+\gamma_n^{(N-1)}(u_n^{(N-1)}-x_n)),$$

and hence

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_n^{(N-1)} - x^*\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(N-1)}(T(PT)^{n-1}x_n^{(N-2)} - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n)) \\ &\quad + (1-\alpha_n^{(N-1)})(x_n - x^* + \gamma_n^{(N-1)}(u_n^{(N-1)} - x_n))\|. \end{aligned}$$

By (3.10), (3.11) and Lemma 2.5, we have

$$\lim_{n\to\infty} \left\| T(PT)^{n-1}x_n^{(N-2)} - x_n \right\| = 0. \tag{3.12}$$

Similarly, we have

$$\lim_{n\to\infty} \left\| T(PT)^{n-1}x_n^{(N-2)} - x^* \right\| = 0.$$

Continuing this process, we have

$$\lim_{n\to\infty} \left\| T(PT)^{n-1}x_n^{(1)} - x_n \right\| = 0.$$

Now,

$$\begin{aligned} &\left\| T(PT)^{n-1}x_n - x^* + \gamma_n^{(1)}(u_n^{(1)} - x_n)) \right\| \\ &\le \left\| T(PT)^{n-1}x_n - x^* \right\| + \gamma_n^{(1)} \left\| u_n^{(1)} - x_n \right\|. \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} \left\| T(PT)^{n-1}x_n - x^* + \gamma_n^{(1)}(u_n^{(1)} - x_n)) \right\| \quad \le \quad a. \tag{3.13}$$

Also,

$$\left\| x_n - x^* + \gamma_n^{(1)}(u_n^{(1)} - x_n)) \right\| \le \|x_n - x^*\| + \gamma_n^{(1)} \left\| u_n^{(1)} - x_n \right\|$$

implies that

$$\limsup_{n\to\infty} \left\| x_n - x^* + \gamma_n^{(1)}(u_n^{(1)} - x_n)) \right\| \quad \le \quad a, \tag{3.14}$$

and hence

$$\begin{aligned} a &= \lim_{n\to\infty} \|x_n^{(1)} - x^*\| \\ &= \lim_{n\to\infty} \|\alpha_n^{(1)}(T(PT)^{n-1}x_n - x^* + \gamma_n^{(1)}(u_n^{(1)} - x_n)) \\ &\quad + (1-\alpha_n^{(1)})(x_n - x^* + \gamma_n^{(1)}(u_n^{(1)} - x_n))\|. \end{aligned}$$

By (3.13), (3.14) and Lemma 2.5, we have

$$\lim_{n\to\infty} \left\| T(PT)^{n-1}x_n - x_n \right\| = 0, \tag{3.15}$$

and this implies that

$$\begin{aligned} & \|x_{n+1}-x_n\| \\ \leq\ & \left\|\alpha_n^{(N)}(T(PT)^{n-1}x_n^{(N-1)}+(1-\alpha_n^{(N)}-\gamma_n^{(N)})x_n+\gamma_n^{(N)}u_n^{(N)}-x_n\right\| \\ \leq\ & \alpha_n^{(N)}\left\|T(PT)^{n-1}x_n^{(N-1)}-x_n\right\|+\gamma_n^{(N)}\left\|u_n^{(N)}-x_n\right\| \qquad (3.16) \\ & \to 0 \ \text{ as } \ n\to\infty. \end{aligned}$$

Thus, we have

$$\begin{aligned} & \left\|T(PT)^{n-1}x_n-x_n\right\| \\ \leq\ & \left\|T(PT)^{n-1}x_n-T(PT)^{n-1}x_n^{(N-1)}\right\| \\ & +\left\|T(PT)^{n-1}x_n^{(N-1)}-x_n\right\| \\ \leq\ & \left\|x_n-x_n^{(N-1)}\right\|+\varepsilon_n+\left\|T(PT)^{n-1}x_n^{(N-1)}-x_n\right\| \\ \leq\ & \alpha_n^{(N-1)}\left\|x_n-T(PT)^{n-1}x_n^{(N-2)}\right\|+\gamma_n^{(N-1)}\left\|u_n^{(N-1)}-x_n\right\| \qquad (3.17) \\ & +\varepsilon_n+\left\|T(PT)^{n-1}x_n^{(N-1)}-x_n\right\| \\ & \to 0 \ \text{ as } \ n\to\infty. \end{aligned}$$

We now show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. By (3.16) and (3.17), we show that

$$\begin{aligned} & \left\|T(PT)^{n-2}x_n-x_n\right\| \\ \leq\ & \|x_n-x_{n-1}\|+\left\|x_{n-1}-T(PT)^{n-2}x_{n-1}\right\| \\ & +\left\|T(PT)^{n-2}x_{n-1}-T(PT)^{n-2}x_n\right\| \qquad (3.18) \\ \leq\ & \|x_n-x_{n-1}\|+\left\|x_{n-1}-T(PT)^{n-2}x_{n-1}\right\|+L\|x_{n-1}-x_n\| \\ & \to 0 \ \ as \ \ n\to\infty. \end{aligned}$$

Thus from (3.17) and (3.18), we have

$$\begin{aligned} \|x_n-Tx_n\| \ \leq\ & \left\|x_n-T(PT)^{n-1}x_n\right\|+\left\|T(PT)^{n-1}x_n-Tx_n\right\| \\ \leq\ & \left\|x_n-T(PT)^{n-1}x_n\right\|+L\left\|T(PT)^{n-2}x_n-x_n\right\| \\ & \to 0 \ \ as \ \ n\to\infty. \end{aligned}$$

It implies that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

This completes the proof.

□

Theorem 3.5. *Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E and P a nonexpansive retraction of E onto K. Let $T\colon K\to E$ be a uniformly L-Lipschitzian completely continuous asymptotically* *quasi-nonexpansive type nonself mapping with* $F(T) \neq \emptyset$*. Let the sequence* $\{x_n\}$ *be defined by (1.3) and some* $\alpha, \beta \in (0,1)$ *with the following restrictions:*

(i) $\alpha_n^{(i)} + \beta_n^{(i)} + \gamma_n^{(i)} = 1$, $1 \leq i \leq N$,
(ii) $\sum_{n=1}^{\infty} \gamma_n^{(i)} < \infty$, $1 \leq i \leq N$,
(iii) $0 < \alpha \leq \alpha_n^{(i)} \leq \beta < 1$, $1 \leq i \leq N$, $\forall n \geq n_0$ *for some* $n_0 \in N$.

Then $\{x_n\}$ *converges strongly to a fixed point of* T*.*

Proof. It follows from Theorem 3.4 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| \quad = \quad 0. \tag{3.19}$$

By Lemma 3.3, $\{x_n\}$ is bounded. It follows by our assumption that T is completely continuous, there exists a subsequence $\{Tx_{n_j}\}$ of $\{Tx_n\}$ such that $Tx_{n_j} \to p \in K$ as $j \to \infty$. Moreover, by (3.19), we have $\left\|Tx_{n_j} - x_{n_j}\right\| \to 0$ as $j \to \infty$ which implies that $x_{n_j} \to p$ as $j \to \infty$. By (3.19) again, we have

$$\|p - Tp\| = \lim_{j\to\infty} \left\|x_{n_j} - Tx_{n_j}\right\| = 0.$$

It shows that $p \in F(T)$. Furthermore, since $\lim_{n\to\infty} \|x_n - p\|$ exists, $\lim_{n\to\infty} \|x_n - p\| = 0$. That is, $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof. □

Remark 3.6. Our results generalize and extend the corresponding results of Xu and Noor [13] to the case of multi-step iterative sequences with errors for more general class of asymptotically nonexpansive nonself mappings.

Remark 3.7. Our results also extend the corresponding results of Wangkeeree [12] to the case of more general class of asymptotically quasi-nonexpansive non-self mapping and multi-step iterative sequences. The iterative scheme (1.2) for asymptotically quasi-nonexpansive type for nonself mapping is also extended.

Remark 3.8. Our results also extend the corresponding results of [10] to the case of multi step iterative sequence.

References

[1] S. S. Chang, J. K. Kim and S. M. Kang, *Approximating fixed points of asymptotically quasi-nonexpansive type mappings by the Ishikawa iterative sequences with mixed errors,* Dynamic Systems and Appl., **13(2)** (2004), 179–186.
[2] C. E. Chidume, E. U. Ofoedu and H. Zegeye, *Strong and weak convergence theorem for asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **280** (2003), 364–374.
[3] R. Glowinski and P. Le Tallec, *Augemented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, Siam Philadelphia, 1989.
[4] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35** (1972), 171–174.
[5] S. Haubruge, V. H. Nguyen and J. J. Strodiot, *Convergence analysis and applications of the Glowinski Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators*, J. Optim. Theory Appl., **97** (1998), 645–673.

[6] J. K. Kim, K. H. Kim and K. S. Kim, *Convergence theorems of modified three-step iterative sequences with mixed errors for asymptotically quasi-nonexpansive mappings in Banach spaces*, PanAmerican Math. Jour., **14(1)** (2004), 45–54.

[7] J. K. Kim, Y. M. Nam and J. Y. Sim, *Convergence theorems of implicit iterative sequences for a finite family of asymptotically quasi-nonexpansive type mappings*, Nonlinear Analysis TMA, **71** (2009), 2839–2848.

[8] M. A. Noor, *New approximation scheme for general variational inequalities*, J. Math. Anal. Appl., **251** (2000), 217–229.

[9] M. A. Noor, *Three-step iterative algorithms for multivalued quasi variational inclusions*, J. Math. Anal. Appl., **255** (2001).

[10] G. S. Saluja, *Strong convergence theorems of three step iterative sequences with errors for asymptotically quasi-nonexpansive type nonself mappings*, J. Advances Appl. Math., Anal. **2** (2007), No.1, 57–69.

[11] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings*, J. Math. Anal. Appl., **178** (1993), 301–308.

[12] R. Wangkeeree, *Strong convergence theorems of three step iterations with errors for asymptotically quasi-nonexpansive nonself-mappings in Banach spaces*, Thai J. Math., **2** (2004), 283–294.

[13] B. L. Xu and M. A. Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **267** (2002), No.2, 444–453.

Jong Kyu Kim
Department Mathematics Education
Kyungnam University
Masan, Kyungnam 631-730, Korea
E-mail address: jongkyuk@kyungnam.ac.kr

G. S. Saluja
Department of Mathematics & Information Technology
Govt. College of Science
Raipur (Chhattisgarh), India
E-mail address: saluja_1963@rediffmail.com

H. K. Nashine
Department of Mathematics
Disha Institute of Management and Technology
Satya Vihar, Vidhansabha-Chandrakhuri Marg (Baloda Bazar Road)
Mandir Hasaud, Raipur-492101 (Chhattisgarh), India
E-mail address: hemantnashine@rediffmail.com, nashine_09@rediffmail.com