ON GENERALIZED VECTOR QUASI-VARIATIONAL TYPE INEQUALITIES

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Abstract. In this paper, we consider and study a new class of generalized vector quasi-variational type inequalities and obtain some existence theorems for both under compact and noncompact assumptions in topological vector spaces without using monotonicity. For the noncompact case, we use the concept of escaping sequences.

1. Introduction

The vector variational inequality problem was initiated by Giannessi [11] in finite dimensional Euclidean spaces with applications. Later on many authors [6, 7, 12, 15, 24] generalized vector variational inequalities in abstract spaces in several ways. The vector variational-like inequality is one of the generalized forms of vector variational inequalities [11]. In 1989, Parida et al [19] studied the existence of solutions for variational-like inequalities in R^n space and have shown a relationship between variational like inequalities problems and convex programming as well as with complementarity problems. The vector variational like inequality and generalized vector variational like inequality are powerful tools to study non-convex vector optimization problems and convex and nondifferentiable vector optimization problems respectively, see [10, 18, 21].

In 1973, Bensoussan and Lions [3] introduced the quasi-variational inequality problem (QVIP). Since then, many generalizations of the QVIP have been appeared in the literature (see, for example, [1, 2]). In the last decade, because of applications in the optimization problems, mathematical programming and equilibrium problems, the QVIP has been intensively studied by many authors [5, 8, 16, 20, 22, 23, 25].

In this paper, we consider the generalized vector quasivariational type inequality problem (GVQVTIP) and obtain some existence theorems for solutions

of the GVQVTIP in the settings of compact convex subsets of Hausdorff topological vector spaces and noncompact convex subsets of locally convex Hausdorff topological vector spaces. We shall also obtain the existence theorem for the solutions of the GVQVTIP in the noncompact case by using the concept of escaping sequences introduced in [4].

Let Y be an ordered Hausdorff topological vector space and C be a closed convex pointed solid proper cone in Y. Then C defines an ordering on Y by means of

$$x \leq 0 \Leftrightarrow x \in -C, \; x < 0 \Leftrightarrow x \in -\operatorname{int} C,$$

which can be extended to an arbitrary set $P \subseteq Y$ by setting

$$P \leq 0 \Leftrightarrow P \subseteq -C, \; P < 0 \Leftrightarrow P \subseteq -\operatorname{int} C.$$

Let $A \subseteq Y$. Then a point $x_0 \in A$ is called a *vector maximum point* of A if the set

$$\{x_1 \in A : \; x_0 \leq x_1, \; x_1 \neq x_0\} = \emptyset,$$

which is equivalent to the following:

$$A \cap (x_0 + C) \;=\; \{x_0\}$$

(see Luc [17]). We denote by $\max(A)$ the set of all vector maximal points of A.

Note that, if A is a compact set in Y, then $\max(A) \neq \emptyset$.

Let 2^A denote the family of all subsets of A, $\operatorname{int}_Y A$ the interior of A in Y, $\operatorname{cl}_Y A$ the closure of A in Y, and $\operatorname{co}(A)$ the convex hull of A.

If K is a nonempty subset of a topological vector space X and $S, T : K \to 2^X$ are multivalued mappings, then $\operatorname{cl}(S), \operatorname{co}(S), S \cap T : K \to 2^X$ are multivalued mappings defined by

$$(clS)(x) = clS(x),$$
$$(coS)(x) = coS(x),$$
$$(S \cap T)(x) \;=\; S(x) \cap T(x), \quad \forall x \in K,$$

respectively.

Let X be a Hausdorff topological vector space and Y be an ordered Hausdorff topological vector space. Let K be a nonempty closed convex subset of X and $T : K \to 2^{L(X,Y)}$ be a multivalued mapping, where $L(X,Y)$ denotes the space of all continuous linear operators from X to Y. Let $C : K \to 2^Y$ be a multivalued mapping such that, for each $x \in K$, $C(x)$ is a closed convex pointed proper and solid cone in Y. Suppose that $\eta : K \times K \to X$ and $A : K \to 2^K$ are continuous mappings.

We consider the following problem (GVQVTIP):
Find $x^* \in K$ such that $x^* \in \operatorname{cl}_K A(x^*)$ and

$$\max\langle T(x^*), \eta(y, x^*)\rangle - h(x^*) + h(y) \; \not\subseteq \; -\operatorname{int}_Y C(x^*), \quad \forall y \in A(x^*), \qquad (1)$$

where

$$\langle T(x^*),\, \eta(y,x^*)\rangle \;=\; \bigcup_{u\in T(x^*)} \langle u,\, \eta(y,x^*)\rangle,$$

$$\max\langle T(x^*),\, \eta(y,x^*)\rangle \;=\; \max_{u\in T(x^*)} \langle u,\, \eta(y,x^*)\rangle,$$

$h : K \to Y$ is continuous and convex functional and $\langle u, x\rangle$ denotes the evaluation of a linear operator u from X into Y for all $u \in T(x^*)$ and $x \in X$.

Special Cases:

(1) We note that, if $Y = R$, $L(X,Y) = X^*$ (the dual space of X), $C(x) = R^+$ for all $x \in K$ and T is a single-valued mapping, then (1) collapses to finding $x^* \in K$ such that $x^* \in \mathrm{cl}_K A(x^*)$ and

$$\langle T(x^*),\, \eta(y,x^*)\rangle + h(y) - h(x^*) \;\geq\; 0, \quad \forall y \in A(x^*). \qquad (2)$$

(2) If $A(x) = K$ for all $x \in K$ and $h \cong 0$, then (1.2) reduces to finding $x^* \in K$ such that

$$\langle T(x^*),\, \eta(y,x^*)\rangle \;\geq 0, \quad \forall y \in K, \qquad (3)$$

which is called variational-like inequality problems considered by Parida, Sahoo and Kumar [19].

(3) If $\eta(y,x^*) = y - x^*$ and T is a single valued mapping, then (1.3) reduces to finding $x^* \in K$ such that

$$\langle T(x^*),\, y - x^*\rangle \;\geq 0, \quad \forall y \in K, \qquad (4)$$

which is called classical variational inequality problem considered by Hartman and Stampacchia [13].

(4) If we take $h \cong 0$ and $A(x) = K$ for all $x \in K$, then (1.1) is reduced to the following:

   Find $x^* \in K$ such that

$$\max\langle T(x^*),\, \eta(y,x^*)\rangle \;\nsubseteq\; -\mathrm{int}_Y C(x^*), \quad \forall y \in K, \qquad (5)$$

which was considered and studied by Chang, Thompson and Yuan [5].

(5) We note that, if $T : K \to L(X,Y)$ is a single valued mapping and $\eta(y,x) = y - g(x)$, where $g : K \to K$ and $h \cong 0$, then (1.1) is equivalent to the following:

   Find $x^* \in K$ such that $x^* \in \mathrm{Cl}_K A(x^*)$ and

$$\langle T(x^*),\, y - g(x^*)\rangle \;\notin\; -\mathrm{int}_Y C(x^*), \quad \forall y \in A(x^*), \qquad (6)$$

which was due to Kim and Tan [14].

In the sequel, we shall use the following:

Definition 1. Let $T : X \to 2^Y$ be a multivalued mapping.

(i) T is said to be *upper semi-continuous* on X if, for each $x \in X$ and each open set U in Y containing $T(x)$, there exists an open neighborhood V of x in X such that $T(y) \subseteq U$ for each $y \in V$.

(ii) The *graph* of T denoted by $G(T)$ is defined by

$$G(T) = \{(x, y) \in X \times Y : x \in X, \ y \in T(x)\}.$$

(iii) The *inverse* T^{-1} of T is a multivalued mapping from $R(T)$ (the range of T) to X defined as follows:

$$x \in T^{-1}(y) \quad \Leftrightarrow \quad y \in T(x).$$

2. The existence result for compact sets

In this section, we prove an existence theorem for the GVQVTIP in compact sets by using the following lemma:

Lemma 2.1. [9] *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let $A : K \to 2^K$ be a mapping such that, for each $x \in K$, $A(x)$ is nonempty convex and, for each $y \in K$, $A^{-1}(y)$ is open in K. Let the mapping $cl\, A : K \to 2^K$ be upper semi-continuous. Suppose that the mapping $P : K \to 2^K$ is such that $P^{-1}(y)$ is open in K for each $y \in K$ and, for each $x \in K$, $x \notin co\, P(x)$. Then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and $A(x^*) \cap P(x^*) = \emptyset$.*

Theorem 2.2. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and Y an ordered Hausdorff topological vector space. Let $T : K \to 2^{L(X,Y)}$, $C : K \to 2^Y$, $A : K \to 2^K$, $\eta(\cdot, \cdot) : K \times K \to X$ be mappings and $h : K \to Y$ a continuous mapping satisfying the following assumptions:*

(i) *for each $x \in K$, $C(x)$ is a closed pointed proper and solid cone,*
(ii) *$\eta(\cdot, \cdot)$ is continuous affine in the first argument and $\eta(x, x) = 0$ for all $x \in K$,*
(iii) *the mapping $W : K \to 2^Y$ defined by $W(x) = Y \backslash (-int_Y C(x))$ for each $x \in K$ is upper semi-continuous on K,*
(iv) *h is upper semi-continuous and convex,*
(v) *$\max\langle T(x_\lambda), \eta(\cdot, x_\lambda)\rangle + h(\cdot) - h(x_\lambda) \to \max\langle T(x), \eta(\cdot, x)\rangle + h(\cdot) - h(x)$ whenever $x_\lambda \to x \in K$,*
(vi) *for each $x \in K$, $A(x)$ is nonempty convex and, for each $y \in K$, $A^{-1}(y)$ is open in K. Also, $cl_K A : K \to 2^K$ is upper semi-continuous.*

Then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and

$$\max\langle T(x^*), \eta(y, x^*)\rangle + h(y) - h(x^*) \not\subseteq -int_Y C(x^*), \quad \forall y \in A(x^*).$$

Proof. Define a mapping $P : K \to 2^K$ by

$$P(x) = \{y \in K : \max\langle T(x), \eta(y, x)\rangle + h(y) - h(x) \subseteq -int_Y C(x)\}, \quad \forall x \in K.$$

We prove that $x \notin \operatorname{co} P(x)$ for all $x \in K$. Suppose that $x_0 \in \operatorname{co} P(x_0)$ for some $x_0 \in K$. This implies that x_0 can be expressed as

$$x_0 = \sum_{i=1}^{n} \lambda_i y_i, \ \sum_{i=1}^{n} \lambda_i = 1, \ \lambda_i \geq 0, \quad \forall i = 0, 1, 2, \cdots,$$

where n is positive integer and $y_1, y_2, \ldots, y_n \in P(x_0)$. Thus we have

$$\max \langle T(x_0), \eta(y_i, x_0) \rangle + h(y_i) - h(x_0) \subseteq -\operatorname{int}_Y C(x_0) \}, \quad \forall i = 1, 2, \cdots, n,$$

which implies that there exists $u_i \in T(x_0)$ $(i = 1, 2, \cdots, n)$ such that

$$\langle u_i, \eta(y_i, x_0) \rangle + h(y_i) - h(x_0) \ \in -\operatorname{int}_Y C(x_0) \}, \quad \forall i = 1, 2, \cdots, n.$$

Since $C(x_0)$ is a cone, $-\operatorname{int}_Y C(x_0)$ is convex and hence

$$\sum_{i=1}^{n} \lambda_i \Big[\langle u_i, \eta(y_i, x_0) \rangle + h(y_i) - h(x_0) \Big] \ \in -\operatorname{int}_Y C(x_0).$$

Since $\eta(\cdot, \cdot)$ is affine in the first argument and h is affine, we have

$$\sum_{i=1}^{n} \lambda_i \langle u_i, \eta(y_i, x_0) \rangle + h(y_i) - h(x_0)$$

$$\in \sum_{i=1}^{n} \lambda_i \langle T(x_0), \eta(y_i, x_0) \rangle + \sum_{i=1}^{n} \lambda_i h(y_i) - h(x_0)$$

$$= \langle T(x_0), \eta(\sum_{i=1}^{n} \lambda_i y_i, x_0) \rangle + h \Big(\sum_{i=1}^{n} \lambda_i y_i \Big) - h(x_0)$$

$$= \langle T(x_0), \eta(x_0, x_0) \rangle + h(x_0) - h(x_0)$$

$$= 0,$$

which contradicts the fact that $C(x_0)$ is a pointed convex cone.

Now, to prove that $P^{-1}(y)$ for each $y \in K$ is open in K, it is sufficient to prove that $[P^{-1}(y)]^c = K \backslash P^{-1}(y)$ is closed. In fact, let $\{x_\lambda\}$ be a net in $K \backslash P^{-1}(y)$ such that $\{x_\lambda\}$ converges to a point $u \in K$. Then we have

$$\max \langle T(x_\lambda), \eta(y, x_\lambda) \rangle + h(y) - h(x_\lambda) \not\subseteq -\operatorname{int}_Y C(x_\lambda).$$

That is,

$$\max \langle T(x_\lambda), \eta(y, x_\lambda) \rangle + h(y) - h(x_\lambda) \ \subseteq W(x_\lambda).$$

But, from the assumption (v), we have

$$\max \langle T(x_\lambda), \eta(y, x_\lambda) \rangle + h(y) - h(x_\lambda) \rightarrow \max \langle T(u), \eta(y, x) \rangle + h(y) - h(u).$$

By the upper semi-continuity of W, we have

$$\max \langle T(u), \eta(y, u) \rangle + h(y) - h(u) \ \subseteq W(u).$$

Therefore, it follows that

$$\max \langle T(u), \eta(y, u) \rangle + h(y) - h(u) \not\subseteq -\operatorname{int}_Y C(u).$$

Hence, from the assumption (v), it follows that the hypothesis of Lemma 2.1 are satisfied, which implies that there exists $x^* \in K$ such that

$$x^* \in \mathrm{cl}_K A(x^*), \quad A(x^*) \cap P(x^*) = \emptyset.$$

This implies that $x^* \in \mathrm{cl}_K A(x^*)$ and

$$\max \langle T(x^*), \eta(y, x^*) \rangle + h(y) - h(x^*) \not\subseteq -\mathrm{int}_Y C(x^*), \quad \forall y \in A(x^*).$$

This completes the proof.

$\square$

Corollary 2.3. *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X and Y an ordered Hausdorff topological vector space. Let $T : K \to 2^{L(X,Y)}$, $C : K \to 2^Y$, $A : K \to 2^K$, $\eta(\cdot, \cdot) : K \times K \to X$ be mappings and $h \cong 0$, satisfying the following assumptions:*

(i) *for each $x \in K$, $C(x)$ is a closed pointed proper and solid cone,*

(ii) *$\eta(\cdot, \cdot)$ is continuous affine in the first argument and $\eta(x, x) = 0$ for all $x \in K$,*

(iii) *the mapping $W : K \to 2^Y$ defined by $W(x) = Y \backslash (-\mathrm{int}_Y C(x))$ for each $x \in K$ is upper semi-continuous on K,*

(iv) *$\max \langle T(x_\lambda), \eta(\cdot, x_\lambda) \rangle \to \max \langle T(x), \eta(\cdot, x) \rangle$, whenever $x_\lambda \to x \in K$,*

(vi) *for each $x \in K$, $A(x)$ is nonempty convex and for each $y \in K$, $A^{-1}(y)$ is open in K. Also, $\mathrm{cl}_K A : K \to 2^K$ is upper semi-continuous.*

Then there exists $x^* \in K$ such that $x^* \in \mathrm{cl}_K A(x^*)$ and

$$\max \langle T(x^*), \eta(y, x^*) \rangle \not\subseteq -\mathrm{int}_Y C(x^*), \quad \forall y \in A(x^*).$$

Corollary 2.4. *In corollary 2.1, if we take $Y = R, L(X, Y) = X^*, C(x) = R^+$ for all $x \in K$ and T a single valued mapping, then there exists $x^* \in K$ such that $x^* \in \mathrm{cl}_K A(x^*)$ and $\langle T(x^*), \eta(y, x^*) \rangle \geq 0, \quad \forall y \in A(x^*)$, is solvable.*

Corollary 2.5. *In corollary 2.1, if we take T a single valued mapping and all other assumptions are satisfied, then there exists $x^* \in K$ such that $x^* \in \mathrm{cl}_K A(x^*)$ and $\langle T(x^*), \eta(y, x^*) \rangle \notin -\mathrm{int}_Y C(x^*), \quad \forall y \in A(x^*)$. This corollary generalizes theorem 1 in [14].*

Corollary 2.6. *In corollary 2.3, if we take $A(x) = K$ for all $x \in K$ with all other assumptions, then there exists $x^* \in K$ such that*

$$\max \langle T(x^*), \eta(y, x^*) \rangle \not\subseteq -\mathrm{int}_Y C(x^*), \quad \forall y \in K.$$

This corollary is a different version of theorem 2.1 in [5].

3. The existence result for non-compact sets

In this section, we prove the existence results for the GVQVTIP in noncompact sets by using the following lemma, which is a special case of Theorem 2 of [8, 9].

Lemma 3.1. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X and D be a nonempty compact subset of K. Let $A : K \to 2^D$ be a mapping such that, for each $x \in K$, $A(x)$ is nonempty convex and, for each $y \in D$, $A^{-1}(y)$ is open in K. Also, the mapping $cl\, A : K \to 2^K$ be upper semi-continuous. Suppose that the mapping $P : K \to 2^D$ is such that $P^{-1}(y)$ is open in K for each $y \in D$ and, for each $x \in K$, $x \notin co\, P(x)$. Then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and $A(x^*) \cap P(x^*) = \emptyset$.*

Theorem 3.2. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X and D a nonempty compact subset of K. Let Y be an ordered Hausdorff topological vector space. Let $T : K \to 2^{L(X,Y)}$, $C : K \to 2^Y$, $A : K \to 2^K$, $\eta : K \times K \to X$ and $h : K \to Y$ be the mappings satisfying the following assumptions:*

(i) *for each $x \in K$, $C(x)$ is a closed convex pointed proper and solid cone,*

(ii) *η is continuous affine in the first argument and $\eta(x,x) = 0$ for all $x \in K$,*

(iii) *the mapping $W : K \to 2^Y$ defined by $W(x) = Y \backslash (-int_Y C(x))$ for each $x \in K$ is upper semi-continuous on K,*

(iv) *h is upper semi-continuous and convex,*

(v) *$\max\langle T(x_\lambda), \eta(\cdot, x_\lambda)\rangle + h(\cdot) - h(x_\lambda) \to \max\langle T(x), \eta(\cdot, x)\rangle + h(\cdot) - h(x)$ whenever $x_\lambda \to x \in K$,*

(vi) *for each $x \in K$, $A(x)$ is nonempty convex and, for each $y \in K$, $A^{-1}(y)$ is open in K. Also, $cl_K A : K \to 2^D$ is upper semi-continuous and compact valued.*

Then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and

$$\max\langle T(x^*), \eta(y, x^*)\rangle + h(y) - h(x^*) \nsubseteq -int_Y C(x^*), \quad \forall y \in A(x^*).$$

Proof. Define a mapping $P : K \to 2^D$ by

$$P(x) = \{y \in D : \max\langle T(x), \eta(y, x)\rangle + h(y) - h(x) \subseteq -int_Y C(x)\}, \quad \forall x \in K.$$

Then, by using the same proof of Theorem 2.2, we have $x \notin co\, P(x)$ for all $x \in K$ and $P^{-1}(y)$ is open for each $y \in D$. Thus all the conditions of Lemma 3.1 are satisfied. Hence, by Lemma 3.1, there exists $x^* \in K$ such that

$$x^* \in cl_K A(x^*), \quad A(x^*) \cap P(x^*) = \emptyset,$$

which implies that

$$\max\langle T(x^*), \eta(y, x^*)\rangle + h(y) - h(x^*) \nsubseteq -int_Y C(x^*)\}, \quad \forall y \in A(x^*).$$

This completes the proof. □

Definition 2. ([4, 25]) Let X be a topological vector space and K be a subset of X such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of nonempty compact sets in the sense that $K_n \subseteq K_{n+1}$ for all $n \in N$. A sequence $\{x_n\}$ in K is said to be an *escaping sequence* from K (relative to $\{K_n\}$) if, for each $n = 1, 2, \cdots$, there exists $M > 0$ such that $x_k \notin K_n$ for all $k \geq M$.

Theorem 3.3. *Let K be a nonempty subset of a Hausdorff topological vector space X and Y be an ordered Hausdorff topological vector space. Let $K = \bigcup\limits_{n=1}^{\infty} K_n$, where $\{K_n\}$ is an increasing sequence of nonempty compact convex subsets of K. Let $T : K \to 2^{L(X,Y)}$, $C : K \to 2^Y$, $A : K \to 2^Y$, $\eta : K \times K \to X$ and $h : K \to Y$ be the mappings satisfying the following assumptions:*

   (i) *for each $x \in K$, $C(x)$ is a closed convex pointed proper and solid cone,*
   (ii) *η is continuous affine in the first argument and $\eta(x, x) = 0$ for all $x \in K$,*
   (iii) *the mapping $W : K \to 2^Y$ defined by $W(x) = Y \backslash (-int_Y C(x))$ for all $x \in K$ is upper semi-continuous on K,*
   (iv) *h is upper semi-continuous and convex,*
   (v) *$\max\langle T(x_\lambda), \eta(\cdot, x_\lambda)\rangle + h(\cdot) - h(x_\lambda) \to \max\langle T(x), \eta(\cdot, x)\rangle + h(\cdot) - h(x)$ whenever $x_\lambda \to x \in K$,*
   (vi) *for each $x \in K$, $A(x)$ is nonempty convex and, for each $y \in K$, $A^{-1}(y)$ is open in K. Also $cl_K A : K \to 2^K$ is upper semicontinuous with compact values,*
   (vii) *for each sequence $\{x_n\}_{n=1}^{\infty}$ in K with $x_n \in K_n$ for $n = 1, 2, \cdots$ which is escaping sequence from K (relative to $\{K_n\}_{n=1}^{\infty}$), there exist $m \in N$ and $y_m \in K_m$ such that*
$$\max\langle T(x_m), \eta(y_m, x_m)\rangle + h(y_m) - h(x_m) \subseteq -int_Y C(x_m).$$

Then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and
$$\max\langle T(x^*), \eta(y, x^*)\rangle + h(y) - h(x^*) \not\subseteq -int_Y C(x^*), \quad \forall y \in A(x^*).$$

Proof. Since, for each $n \in N$, K_n is compact and convex set in X, hence applying theorem 2.2 we have, for all $n \in N$, there exists $x_n \in K_n$ such that $x_n \in \mathrm{cl}_K A(x_n)$ and
$$\max\langle T(x_n), \eta(z, x_n)\rangle + h(z) - h(x_n) \not\subseteq -\mathrm{int}_Y C(x_n), \quad \forall z \in A(x_n). \quad (7)$$

Suppose that the sequence $\{x_n\}$ is escaping from K relative to $\{K_n\}$. By the assumption (vii), there exist $m \in N$ and $z_m \in K_m$ such that
$$\max\langle T(x_m), \eta(z_m, x_m)\rangle + h(z_m) - h(x_m) \subseteq -\mathrm{int}_Y C(x_m),$$

which is a contradiction of (7). Hence $\{x_n\}$ is not an escaping sequence from K relative to $\{K_n\}$. Therefore, there exist $r \in N$ and a subsequence $\{x_{j_n}\}$ of $\{x_n\}_{n=1}^{\infty}$ which must lie entirely in K_r. Since K_r is compact, there exist a subsequence $\{x_{i_n}\}_{i_n \in \Lambda}$ of $\{x_{j_n}\}$ in K_r and $x^* \in K_r$ such that $x_{i_n} \to x^*$, where $i_n \to \infty$. Since $\{K_n\}$ is an increasing sequence, it follows that, for all $y \in K$, there exists $i_0 \in \Lambda$ with $i_0 > r$ such that $y \in K_{i_0}$ for all $i_n \in \Lambda$ and $i_n > i_0$. Thus there exist $y \in K_{i_0} \subseteq K_{i_n}$ and $T(x_{i_n}) \subseteq T(K_r)$ such that
$$\max\langle T(x_{i_n}), \eta(y, x_{i_n})\rangle + h(y) - h(x_{i_n}) \not\subseteq -\mathrm{int}_Y C(x_{i_n}),$$

which implies that
$$\max\langle T(x_{i_n}), \eta(y, x_{i_n})\rangle + h(y) - h(x_{i_n}) \subseteq W(x_{i_n}).$$

But, from the assumption (v), we have

$$\max\langle T(x_{i_n}),\, \eta(y, x_{i_n})\rangle + h(y) - h(x_{i_n}) \to \max\langle T(x^*),\, \eta(x, x^*)\rangle + h(x) - h(x^*).$$

From the upper semi-continuity of W, we have

$$\max\langle T(x^*),\, \eta(y, x^*)\rangle + h(y) - h(x^*) \ \subseteq W(x^*).$$

That is,

$$\max\langle T(x^*),\, \eta(y, x^*)\rangle + h(y) - h(x^*) \ \not\subseteq -\mathrm{int}_Y C(x^*).$$

Since $\mathrm{cl}_K A : K \to 2^K$ is compact valued, the required assertion follows. This completes the proof. $\qquad\square$

Corollary 3.4. *In theorem 3.2, if we take $h \cong 0$ with all other assumptions, then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and*

$$\max\langle T(x^*), \eta(y, x^*)\rangle \not\subseteq -int_Y C(x^*), \quad \forall y \in A(x^*).$$

Corollary 3.5. *In theorem 3.3, if we take $h \cong 0$ with all other assumptions, then there exists $x^* \in K$ such that $x^* \in cl_K A(x^*)$ and*

$$\max\langle T(x^*), \eta(y, x^*)\rangle \not\subseteq -int_Y C(x^*), \quad \forall y \in A(x^*).$$

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