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COMMON FIXED POINTS AND INVARIANT APPROXIMATIONS FOR SUBCOMPATIBLE MAPPINGS IN CONVEX METRIC SPACE

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Abstract. Existence of common fixed points for generalized S-nonexpansive subcompatible mappings in convex metric spaces have been obtained. Invariant approximation results have also been derived by its application. These results extend and generalize various known results in the literature with the aid of more general class of noncommuting mappings.

1. Introduction and Preliminaries

Following definition is the notion of convex structure introduced by Takahashi [17].

Definition 1.1. [17] Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X, if for all $x, y \in X$ and $\lambda \in [0,1]$ the following condition is satisfied:

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y), \quad for\ \ all\ \ u \in X.$$

A metric space X with convex structure W is called a convex metric space. Every normed space is a convex metric space with $W(x, y, \lambda) = \lambda x + (1-\lambda)y$. But a *Fréchet* space is not necessary a convex metric space. There are many examples of convex metric spaces which are not imbedded in a normed space.

Example 1.2. Let I be the unit interval $[0,1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i], I_j = [a_j, b_j]$ and $\lambda (0 \leq \lambda \leq 1)$, we define a mapping W by $W(I_i, I_j, \lambda) = [\lambda a_i + (1-\lambda)a_j, \lambda b_i + (1-\lambda)b_j]$ and define a metric d in X by the Hausdorff distance, i.e.,

$$d(I_i, I_j) = \sup_{a \in I}\{|\inf_{b \in I_i}\{|a-b|\} - \inf_{c \in I_j}\{|a-c|\}|\}.$$

Then (X, d, W) is a convex metric space.

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Definition 1.3. [17] A subset M of a convex metric space X is said to be convex, if $W(x,y,\lambda) \in M$ for all $x,y \in M$ and $\lambda \in [0,1]$. The set M is said to be p-starshaped if there exists $p \in M$ such that $W(p,x,\lambda) \in M$ for all $x \in M$ and $\lambda \in [0,1]$. Clearly p-starshaped subsets of X contain all convex subsets of X as a proper subclass.

Definition 1.4. [17] A convex metric space X is said to satisfy property (I), if for all $x,y \in X$ and $\lambda \in [0,1]$,

$$d(W(p,x,\lambda), W(p,y,\lambda)) \leq \lambda d(x,y).$$

Definition 1.5. [17] A continuous function S from a closed convex subset M of a convex metric space X into itself is said to be affine if

$$S(W(x,y,\lambda)) = W(Sx,Sy,\lambda)$$

whenever $\lambda \in [0,1] \cap Q$ and $x,y \in M$, where Q denotes the set of all rational numbers.

Definition 1.6. [13] Let M be a subset of a metric space X. Let $x_0 \in X$. An element $y \in M$ is called a best approximant to $x_0 \in X$, if

$$d(x_0,y) = d(x_0,M) = \inf\{d(x_0,z) : z \in M\}.$$

Let $P_M(x_0)$ be the set of best $M-$approximants to x_0. That is,

$$P_M(x_0) = \{z \in M : d(x_0,z) = d(x_0,M)\}.$$

Definition 1.7. [5] A pair (S,T) of self-mappings of a metric space X is said to be compatible, if $d(TSx_n, STx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X such that $Tx_n, Sx_n \to t \in X$.

Every commuting pair of mappings is compatible but the converse is not true.

Jungck introduced the concept of weakly compatible maps as follows:

Definition 1.8. [6] A pair (S,T) of self-mappings of a metric space X is said to be weakly compatible, if they commute at there coincidence points, i.e., if $Tu = Su$ for some $u \in X$, then $TSu = STu$.

It is easy to see that compatible maps are weakly compatible.

Definition 1.9. Suppose that M is p-starshaped with $p \in \mathcal{F}(S)$(set of fixed points of S) and is both T-invariant and S-invariant. Then T and S are called R-subcommuting on M, if for all $x \in M$ there exists a real number $R > 0$ such that

$$d(STx, TSx) \leq (\frac{R}{k}) d(seg[p,Tx], Sx)$$

for each $k \in (0, 1]$. If $R = 1$, then the maps are called 1-subcommuting. The S and T are called R-subweakly commuting on M, if for all $x \in M$ there exists a real number $R > 0$ such that

$$d(STx, TSx) \leq Rd(Sx, seg[p, Tx]),$$

where $seg[p, x] = W(p, x, k) : 0 \leq k \leq 1$.

Definition 1.10. Suppose that M is p-starshaped with $p \in \mathcal{F}(S)$, define $\bigwedge_p(S, T) = \{\bigwedge(S, T) : 0 \leq k \leq 1\}$, where $T_k x = seg[p, Tx]$ and $\bigwedge(S, T) = \{\{x_n\} \subset M : \lim_n Sx_n = \lim_n T_k x_n = t \in M \Rightarrow \lim_n d(ST_k x_n, T_k Sx_n) = 0\}$. Then S and T are called subcompatible [14, 15] if

$$\lim_n d(STx_n, TSx_n) = 0$$

for all sequences $x_n \in \bigwedge_p(S, T)$.

Obviously, subcompatible maps are compatible but the converse does not hold.

Example 1.11. Let $X = R$ with usual metric and $M = [0, \infty)$. Let $Sx = 2x-1$ and $Tx = x^2$, for all $x \in M$. Let $p = 1$. Then M is p-starshaped with $Sp = p$. Note that S and T are compatible. For any sequence $\{x_n\}$ in M with $\lim_n x_n = 2$, we have, $\lim_n Sx_n = \lim_n T_{\frac{2}{3}} x_n = 3 \in M \Rightarrow \lim_n d(ST_{\frac{2}{3}} x_n, T_{\frac{2}{3}} x_n) = 0$. However, $\lim_n d(STx_n, TSx_n) = 0$. Thus S and T are not subcompatible maps.

Note that R-subweakly commuting and R-subcommuting maps are subcompatible. The following simple example reveals that the converse is not true.

Example 1.12. Let $X = R$ with usual metric and $M = [0, \infty)$. Let $Sx = \frac{x}{2}$ if $0 \leq x < 1$ and $Sx = x$ if $x = 1$, and $Tx = \frac{1}{2}$ if $0 \leq x < 1$ and $Tx = x^2$ if $x = 1$. Then M is 1-starshaped with $S1 = 1$ and $\bigwedge_p(S, T) = \{\{x_n\} : 1 \leq x_n < \infty\}$. Note that S and T are subcompatible but not R-weakly commuting for all $R > 0$. Thus S and T are neither R-subweakly commuting nor R-subcommuting maps.

The weak commutativity of a pair of selfmaps on a metric space depends on the choice of the metric. This is true for compatibility, R-weak commutativity and other variants of commutativity of maps as well.

Example 1.13. Let $X = R$ with usual norm and $M = [1, \infty)$. Let $Sx = 1 + x$ and $Tx = 2+x^2$. Then $|STx-TSx| = 2x$ and $|Sx-Tx| = |x^2-x+1|$. Thus the pair (S, T) is not weakly commuting on M with respect to usual metric. But if X is endowed with the discrete metric d, then $d(STx, TSx) = 1 = d(Sx, Tx)$ for $x > 1$. Thus the pair (S, T) is weakly commuting on M with respect to discrete metric.

Existence of fixed point has been used at many fields in approximation theory. Number of results exist in the literature where fixed point theorems are used to prove the existence of best approximation(see in [1, 3, 4, 7, 8, 11, 12, 16]).

Meinardus [7] was the first to employ a fixed point theorem of Schauder to establish the existence of an invariant approximation. Further, Brosowski [3] obtained a celebrated result and generalized the Meinardus's result. Later, several results [4, 11, 16] have been proved in the direction of Brosowski [3]. In the year 1988, Sahab et al. [8] extended the result of Hicks and Humpheries [4] and Singh [11] by considering one linear and the other nonexpansive mappings. Al-Thagafi [1] generalized result of Sahab at el. [8] and proved some results on invariant approximations for commuting mappings. The introduction of non-commuting maps to this area, Shahzad [9, 10] further extended Al-Thagafi's results and obtained a number of results regarding invariant approximation.

Recently, using compatible maps Jungck and Hussain [6] unified, and generalized above results in normed spaces or Banach spaces. Here it is important to remark that Takahashi [17] introduced the notion of convex metric space and studied the fixed point theory for nonexpansive mappings in such a setting. This idea was utilized in 1992 by Beg et al. [2] to prove existence of fixed point and then to apply it for proving existence of best approximant for relatively contractive commuting mappings. In this way, they generalized the result of Sahab et al. [8] and others.

Attempt has been made to find existence results on common fixed point to generalize $\mathcal{S}$-nonexpansive subcompatible maps in the setup of convex metric space which is further applied to prove some invariant approximation results. In this way, results of Jungck and Hussain [6] are unified, and generalized with the aid of more general class of noncommuting mappings instead of compatible mappings in convex metric space and also results of Beg et al. [2] are generalized for generalize $\mathcal{S}$-contractive noncommuting mappings, incidently, results of Al-Thagafi [1], Brosowski [3], Meinardus [7], Sahab et al. [8] and Singh [11, 12] have also been extended by considering Ciric's contraction type condition and more general class of noncommuting mappings in convex metric spaces.

2. Main results

The following result would also be used in the sequel:

Theorem 2.1. [6, Theorem 2.1] *Let M be a subset of a metric space (X, d), and T and S be weakly compatible self mappings of M. Assume that $clT(M) \subset S(M)$, $clT(M)$ is complete, and T and S satisfy for all $x, y \in M$ and $0 \leq h < 1$,*

$$d(Tx, Ty) \leq h \max\{d(Sx, Sy), d(Tx, Sx), d(Ty, Sy), d(Ty, Sx), d(Tx, Sy)\}. \tag{2.1}$$

Then $M \cap \mathcal{F}(T) \cap \mathcal{F}(S)$ is a singleton.

First, a more general result in common fixed point theory for more general class of noncommuting mappings is presented below:

Theorem 2.2. *Let M be a nonempty p-starshaped subset of a convex metric space X satisfying property (I). Suppose T and S are subcompatible self-mappings of M such that $clT(M) \subset S(M)$, and S is affine with $p \in \mathcal{F}(S)$. If T is continuous and T and S satisfy, for all $x, y \in M$,*

$$d(Tx,Ty) \leq \max\{d(Sx,Sy), d(seg[Tx,p],Sx), d(seg[Ty,p],Sy), d(seg[Ty,p],Sx), d(seg[Tx,p],Sy)\}, \tag{2.2}$$

then $\mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$, provided one of the following conditions holds;

(1) *$clT(M)$ is compact and S is continuous,*
(2) *M is complete, $\mathcal{F}(S)$ is bounded and T is a compact map,*
(3) *X is complete, M is weakly compact, S is weakly continuous and $(S-T)$ is demiclosed at 0,*
(4) *X is complete, M is weakly compact, T is completely continuous and S is continuous.*

Proof. Choose a sequence $\{k_n\} \subset (0,1)$ with $k_n \to 1$ as $n \to \infty$. Define for each $n \geq 1$ and for all $x \in M$, a mapping T_n by

$$T_n x = W(p, Tx, k_n).$$

Then each T_n is a self-mapping of M and for each n, $clT_n(M) \subset S(M)$ from the affine of S, $p \in \mathcal{F}(S)$ and $clT(M) \subset S(M)$. The subcompatibility of the pair (S,T), affinity of S and property (I) imply that

$$\begin{aligned} 0 &\leq \lim_n d(T_n S x_m, S T_n x_m) \\ &= d(W(p, TSx, k_n), SW(p, Ty, k_n)) \\ &= d(W(p, TSx, k_n), W(p, STy, k_n)) \\ &\leq \lim_m k_n d(TSx_m, STx_m) \\ &= 0 \end{aligned}$$

for any $\{x_m\} \subset M$ with $\lim_m T_n x_m = \lim_m S x_m = t \in M$. Thus (T_n, S) is compatible and hence weakly compatible on M for each n. Also, by the property (I),

$$\begin{aligned} d(T_n x, T_n y) &= d(W(p, Tx, k_n), W(p, Ty, k_n)) \\ &\leq k_n d(Tx, Ty) \\ &\leq k_n \max\{d(Sx,Sy), d(seg[Tx,p],Sx), d(seg[Ty,p],Sy), \\ &\qquad d(seg[Ty,p],Sx), d(seg[Tx,p],Sy)\} \\ &\leq k_n \max\{d(Sx,Sy), d(T_n x,Sx), d(T_n y,Sy), \\ &\qquad d(T_n y,Sx), d(T_n x,Sy)\}, \end{aligned}$$

that is,

$$d(T_n x, T_n y) \leq k_n \max\{d(Sx,Sy), d(T_n x,Sx), d(T_n y,Sy), d(T_n y,Sx), d(T_n x,Sy)\}$$

for all $x, y \in M$. Now, we prove the each case.

(1) Since $clT(M)$ is compact, $clT_n(M)$ is also compact. By Theorem 2.1, for each $n \geq 1$, there exists $y_n \in M$ such that $y_n = Sy_n = T_n y_n$. The compactness of $clT(M)$ implies that there exists a subsequence $\{Ty_m\}$ of $\{Ty_n\}$ such that $Ty_m \to y$ as $m \to \infty$. Then the definition of $T_m y_m$ implies $y_m \to y$, so by the continuity of T and S we have $y \in \mathcal{F}(T) \cap \mathcal{F}(S)$. Thus $\mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$.

(2) As in (1), there is a unique $y_n \in M$ such that $y_n = T_n y_n = Sy_n$. As T is compact and $\{y_n\}$ being in $\mathcal{F}(S)$ is bounded so $\{Ty_n\}$ has a subsequence $\{Ty_m\}$ such that $Ty_m \to y$ as $m \to \infty$. Then the definition of $T_m y_m$ implies $y_m \to y$, so by the continuity of T and S we have $y \in \mathcal{F}(T) \cap \mathcal{F}(S)$. Thus $\mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$.

(3) As in (1) there exists $y_n \in M$ such that $y_n = Sy_n = T_n y_n$. Since M is weakly compact, we can find a subsequence $\{y_m\}$ of $\{y_n\}$ in M converging weakly to $y \in M$ as $m \to \infty$ and as S is weakly continuous so $Sy = y$. By (3) $d(Sy_m, Ty_m) \to 0$ as $m \to \infty$. The demiclosedness of $(S - T)$ at 0 implies that $Sy = Ty$. Thus $\mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$.

(4) As in (3), we can find a subsequence $\{y_m\}$ of $\{y_n\}$ in M converging weakly to y as $m \to \infty$. Since T is completely continuous, $Ty_m \to Ty$ as $m \to \infty$. Since $k_n \to 1$, $y_m = T_m y_m = W(p, Ty_m, k_m) \to Ty$ as $m \to \infty$. Thus $Ty_m \to T^2 y$ as $m \to \infty$ and consequently $T^2 y = Ty$ implies that $Tw = w$, where $w = Ty$. Also, since $Sy_m = y_m \to Ty = w$, using the continuity of S and the uniqueness of the limit, we have $Sw = w$. Hence $\mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$.

□

Following are the immediately consequences from Theorem 2.2.

Corollary 2.3. *Let M be a p-starshaped subset of a convex metric space X satisfying property (I), and T and S continuous self-maps of M. Suppose that S is affine with $p \in \mathcal{F}(S)$, $clT(M) \subset S(M)$ and $clT(M)$ is compact. If the pair (T, S) is R-subweakly commuting and satisfies (2.2) for all $x, y \in M$, then $\mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$.*

Corollary 2.4. *Let M be a p-starshaped subset of a convex metric space X satisfying property (I), and T and S continuous self-maps of M. Suppose that S is affine with $p \in \mathcal{F}(S)$, $clT(M) \subset S(M)$ and $clT(M)$ is compact. If the pair (T, S) is R-subcompatible, and T is S-nonexpansive for all $x, y \in M$, then $\mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$.*

As an application of Theorem 2.2, the following are more general results in invariant approximations theory with the aid of more general class of noncommuting, say, subcompatible mappings in the frame work of convex metric space:

Theorem 2.5. *Let X be a convex metric space satisfying property (I) and $T, S : X \to X$. Let M be a subset of X such that $T(\partial M) \subseteq M$ and $x_0 \in$*

$\mathcal{F}(T)\cap\mathcal{F}(S)$. Suppose S is affine on $P_M(x_0)$, $p\in\mathcal{F}(S)$, $P_M(x_0)$ is closed and p-starshaped, $S(P_M(x_0))=P_M(x_0)$, and $clT(P_M(x_0))$ is compact. If the pair (T,S) is continuous, subcompatible and satisfies for all $x\in P_M(x_0)\cup\{x_0\}$

$$d(Tx,Ty)\le\begin{cases} d(Sx,Sx_0) \ \ \text{if} \ \ y=x_0,\\ \max\{d(Sx,Sy),d(seg[Tx,p],Sx),d(seg[Ty,p],Sy),\\ d(seg[Ty,p],Sx),d(seg[Tx,p],Sy)\}, \ \ \text{if } y\in P_M(x_0),\end{cases} \tag{2.3}$$

then $P_M(x_0)\cap\mathcal{F}(T)\cap\mathcal{F}(S)\neq\emptyset$.

Proof. Let $y\in P_M(x_0)$. Then $y\in\partial M$ and so $Ty\in M$, because $T(\partial M)\subseteq M$. Now since $Tx_0=x_0=Sx_0$, we have

$$d(Ty,x_0)=d(Ty,Tx_0)\le d(Sy,Sx_0)=d(Sy,x_0)=d(x_0,M).$$

This shows that $Ty\in P_M(x_0)$. Consequently, $T(P_M(x_0))\subseteq P_M(x_0)=S(P_M(x_0))$. Now Theorem 2.2 guarantees that

$$P_M(x_0)\cap\mathcal{F}(T)\cap\mathcal{F}(S)\neq\emptyset.$$

□

Define

$$C_M^S(x_0)=\{x\in M:Sx\in P_M(x_0)\}$$

and

$$D_M^S(x_0)=P_M(x_0)\cap C_M^S(x_0).$$

Theorem 2.6. *Let X be a convex metric space satisfying property (I) and $T,S:X\to X$. Let M be a subset of X such that $T(\partial M)\subseteq M$ and $x_0\in\mathcal{F}(T)\cap\mathcal{F}(S)$. Suppose S is affine on $D^*=D_M^S(x_0)$, $p\in\mathcal{F}(S)$, D^* is compact and p-starshaped, $S(D^*)=D^*$, S is nonexpansive on $P_M(x_0)\cup\{x_0\}$ and $clT(D^*)$ is compact. If the pair (T,S) is continuous, subcompatible on D^* and T and S satisfy for all $x\in D^*\cup\{x_0\}$*

$$d(Tx,Ty)\le\begin{cases} d(Sx,Sx_0) \ \ \text{if} \ \ y=x_0,\\ \max\{d(Sx,Sy),d(seg[Tx,p],Sx),d(seg[Ty,p],Sy),\\ d(seg[Ty,p],Sx),d(seg[Tx,p],Sy)\}, \ \ \text{if } y\in D^*,\end{cases} \tag{2.4}$$

then $P_M(x_0)\cap\mathcal{F}(T)\cap\mathcal{F}(S)\neq\emptyset$.

Proof. First, we show that T is a selfmap on D^*, i.e., $T:D^*\to D^*$. Let $y\in D^*$. Then $Sy\in D^*$, since $S(D^*)=D^*$. By the definition of D^*, $y\in\partial M$. Also $Ty\in M$, since $T(\partial M)\subseteq M$. Now since $Tx_0=x_0=Sx_0$,

$$d(Ty,x_0)=d(Ty,Tx_0)\le d(Sy,Sx_0).$$

As $Sx_0=x_0$,

$$d(Ty,Tx_0)\le d(Sy,x_0)=d(x_0,M),$$

since $Sy\in P_M(x_0)$. This implies that Ty is also closest to x_0, so $Ty\in P_M(x_0)$. Since S is nonexpansive on $P_M(x_0)\cup\{x_0\}$,

$$d(STy,x_0)=d(STy,Sx_0)\le d(Ty,x_0)=d(Ty,Tx_0)\le d(Sy,Sx_0)=d(Sy,x_0).$$

Thus, $STy \in P_M(x_0)$. This implies that $Ty \in C_M^S(x_0)$ and hence $Ty \in D^*$. So T and S are selfmaps on D^*. Hence, all the condition of the Theorem 2.2 are satisfied. Thus, there exists $z \in P_M(x_0)$ such that $z = Sz = Tz$. □

Theorem 2.7. *Let X be a convex metric space satisfying property (I) and $T, S : X \to X$. Let M be a subset of X such that $T(\partial M \cap M) \subseteq M$ and $x_0 \in \mathcal{F}(T) \cap \mathcal{F}(S)$. Suppose S is linear on $D^* = D_M^S(x_0)$, $p \in \mathcal{F}(S)$, D^* is compact and p-starshaped, $S(D^*) = D^*$, S is nonexpansive on $P_M(x_0) \cup \{x_0\}$, and cl$T(D^*)$ is compact. If the pair (T, S) is continuous, subcompatible on D^* and T and S satisfy (2.4) for all $x \in D^* \cup \{x_0\}$, then $P_M(x_0) \cap \mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$.*

Proof. Let $x \in D^*$. Then, $x \in P_M(x_0)$ and hence $d(x, x_0) = d(x_0, M)$. Note that for any $k \in (0, 1)$,

$$d(W(x_0, x, k), x_0) = d(W(x_0, x, k), W(x_0, x_0, k)) \leq kd(x, x_0) < d(x_0, M).$$

It follows that the line segment $\{W(x_0, x, k) : 0 < k < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$. Since $T(\partial M \cap M) \subset M$, Tx must be in M. Along with the lines of the proof of Theorem 2.6, we have the result. □

Remark 2.8. It is observed that $S(P_M(x_0)) \subset P_M(x_0)$ implies $P_M(x_0) \subset D^*$ and hence $D^* = P_M(x_0)$. Consequently, Theorem 2.6, 2.7 remain valid when $D^* = P_M(x_0)$.

Remark 2.9. Theorem 2.2 - Theorem 2.7 generalize the results of Jungck and Hussain [6, Theorem 2.3 - Threom 2.5] in the sense that the more generalized noncommuting mappings, that is, subcompatible mappings have been used in the frame work of convex metric space in place of compatible mappings in normed spaces or Banach spaces.

Remark 2.10. Similarly, all other results of Jungck and Hussain [6, Theorem 2.9 - Theorem 2.12] hold by using subcompatible mappings instead of compatible mappings.

Remark 2.11. Theorem 2.5 - Theorem 2.7 contain Theorem 6 of Beg [2] in the sense that the more generalized contractive noncommuting mappings (i.e., subcompatible mappings) and generalized relatively nonexpansive maps have been used in place of relatively contractive commuting maps.

Remark 2.12. Theorem 2.2 contains [1, Theorem 2.2] and [10, Theorem 2.2].

Remark 2.13. Theorem 2.5 - Theorem 2.7 contain Theorem 3.2 of Al - Thagafi [1], Theorem 3 of Sahab, Khan and Sessa [8] and Singh [11, 12] in the sense that the more generalized noncommuting mappings(subcompatible mappings) and generalized relatively nonexpansive maps have been used in the frame work of convex metric space in place of relatively nonexpansive commuting maps.

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