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A LARGE-UPDATE INTERIOR POINT ALGORITHM FOR $P_*(\kappa)$ LCP BASED ON A NEW KERNEL FUNCTION

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Abstract. In this paper we generalize large-update primal-dual interior point methods for linear optimization problems in [2] to the $P_*(\kappa)$ linear complementarity problems based on a new kernel function which includes the kernel function in [2] as a special case. The kernel function is neither self-regular nor eligible. Furthermore, we improve the complexity result in [2] from $O(\sqrt{n}(\log n)^2 \log \frac{n\mu_0}{\epsilon})$ to $O(\sqrt{n}(\log n)\log(\log n)\log \frac{n\mu_0}{\epsilon})$.

1. Introduction

In this paper we propose a new large-update interior point algorithm for solving linear complementarity problem(LCP) as follows:

$$s = Mx + q, \ xs = 0, \ x \geq 0, \ s \geq 0, \tag{1}$$

where $x, s, q \in \mathbf{R}^n$, $M \in \mathbf{R}^{n\times n}$ is a $P_*(\kappa)$ matrix, and xs denotes the componentwise product of the vectors x and s.

Primal-dual interior point method(IPM) is one of the most efficient numerical methods for various optimization problems. Linear complementarity problems(LCPs) have many applications in science, economics, and engineering([5]).

It is generally agreed that the iteration complexity of the algorithm is an appropriate measure for its efficiency([6]). Most of polynomial-time interior point algorithms are based on the logarithmic barrier function. Peng et al.([11], [12], [13]) proposed a new variant of interior point methods(IPMs) based on self-regular barrier functions and achieved so far the best known complexity result for large-update methods with a specific self-regular barrier function. Roos et al.([1], [2]) proposed new primal-dual IPMs for linear optimization(LO) problems based on eligible barrier functions and proposed the unified scheme for analyzing the algorithm based on four conditions on the kernel function([2]). Cho et al.([3], [4]) extended the algorithm for LO to $P_*(\kappa)$ LCPs.

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Motivated by their works, we introduce a new class of kernel functions which is the generalized form of the ones in [2] and is not eligible. We obtained $\mathcal{O}\big(\frac{(1+2\kappa)}{r}n^{\frac{1}{1+p}}(\log n)^{1+r}\log\frac{n\mu_0}{\epsilon}\big)$ iteration complexity for large-update method. Taking $p=1$ and $r=\frac{1+\epsilon}{\log(\log n)}$, we have $\mathcal{O}\big((1+2\kappa)\sqrt{n}\log n\log(\log n)\log\frac{n\mu_0}{\epsilon}\big)$ iteration complexity for $P_*(\kappa)$ LCP which is better than the one in [2].

The paper is organized as follows. In Section 2 we recall the generic IPM and propose some basic concepts for LCP. In Section 3 we introduce a new class of kernel functions and its properties. In Section 4 we derive the complexity result for the algorithm based on a new kernel function.

We will make use of the following notations throughout the paper. $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of n-dimensional nonnegative vectors and positive vectors, respectively. For $x \in \mathbf{R}^n$, x_{min} denotes the smallest component of the vector x. We denote X and S the diagonal matrices from a vector x and s, respectively, i.e. $X = \mathrm{diag}(x)$ and $S = \mathrm{diag}(s)$. e and E denote the n-dimensional vector of ones and the identity matrix, respectively. For $f(t),\ g(t) : \mathbf{R}_{++} \to \mathbf{R}_{++}$, $f(t) = \mathcal{O}(g(t))$ if $f(t) \le c_1 g(t)$ for some positive constant c_1 and $f(t) = \Theta(g(t))$ if $c_2 g(t) \le f(t) \le c_3 g(t)$ for some positive constants c_2 and c_3. I denotes the index set, e.g. $I = \{1, 2, \cdots, n\}$. log denotes the natural logarithmic function.

2. Preliminaries

In this section, we recall the generic IPM and introduce basic concepts.

Definition 1. [8] Let $\kappa \ge 0$. $P_*(\kappa)$ is the class of matrices M satisfying

$$(1+4\kappa)\sum_{i\in I_+(\xi)}\xi_i[M\xi]_i + \sum_{i\in I_-(\xi)}\xi_i[M\xi]_i \ge 0,$$

where $\xi \in \mathbf{R}^n$, $[M\xi]_i$ denotes the i-th component of the vector $M\xi$ and

$$I_+(\xi) = \{i \in I\ :\ \xi_i[M\xi]_i \ge 0\},\ I_-(\xi) = \{i \in I\ :\ \xi_i[M\xi]_i < 0\}.$$

Lemma 2.1. [8] *If $M \in \mathbf{R}^{n\times n}$ is a $P_*(\kappa)$ matrix, then*

$$M' = \begin{pmatrix} -M & E \\ S & X \end{pmatrix}$$

is a nonsingular matrix for any positive diagonal matrices $X, S \in \mathbf{R}^{n\times n}$.

Corollary 2.2. *Let $M \in \mathbf{R}^{n\times n}$ be a $P_*(\kappa)$ matrix and $x, s \in \mathbf{R}^n_{++}$. Then for all $c \in \mathbf{R}^n$ the system*

$$-M\Delta x + \Delta s = 0,\ S\Delta x + X\Delta s = c$$

has a unique solution $(\Delta x, \Delta s)$.

The basic idea of primal-dual IPMs is to replace the second equation in (1) by the parameterized equation $xs = \mu e$, $\mu > 0$. Now we consider the following system:

$$s = Mx + q,\ Xs = \mu e,\ x > 0,\ s > 0. \tag{2}$$

Without loss of generality, we assume that (1) has a strictly feasible point, i.e., there exists $(x^0, s^0) > 0$ such that $s^0 = Mx^0 + q$. For this, the reader refers to [8]. Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, the system (2) has a unique solution for each $\mu > 0$. We denote the solution $(x(\mu), s(\mu))$ for each $\mu > 0$. We call it the μ-center. The set of μ-centers ($\mu > 0$) is called the central path of (1). The limit of this central path (as μ goes to zero) exists and since the limit point satisfies (1), it yields an optimal solution for (1) ([8]). IPMs follow this central path approximately and approach the solution of (1) as μ goes to zero.

For given $(x, s) := (x^0, s^0)$ by applying Newton method to the system (2) we have the following Newton system:

$$-M\Delta x + \Delta s = 0, \;\; S\Delta x + X\Delta s = \mu e - xs. \tag{3}$$

By Corollary 2.2, the system (3) has a unique search direction $(\Delta x, \Delta s)$. By taking a step along the search direction $(\Delta x, \Delta s)$, one constructs a new positive iterate (x_+, s_+), where

$$x_+ = x + \alpha\Delta x, \;\; s_+ = s + \alpha\Delta s,$$

for some $\alpha \geq 0$. To have the motivation of new algorithm we define the following scaled vectors:

$$v := \sqrt{\frac{xs}{\mu}}, \;\; d := \sqrt{\frac{x}{s}}, \;\; d_x := \frac{v\Delta x}{x}, \;\; d_s := \frac{v\Delta s}{s}, \tag{4}$$

whose ith components are $\sqrt{x_i s_i/\mu}$, $\sqrt{x_i/s_i}$, $v_i[\Delta x]_i/x_i$, and $v_i[\Delta s]_i/s_i$, respectively. Using (4), we can rewrite the system (3) as follows:

$$-\bar{M}d_x + d_s = 0, \;\; d_x + d_s = v^{-1} - v, \tag{5}$$

where $\bar{M} := DMD$ and $D := \mathrm{diag}(d)$. Note that the right side of the second equation in (5) equals the negative gradient of the logarithmic barrier function $\Psi_l(v)$, i.e.,

$$d_x + d_s = -\nabla\Psi_l(v), \tag{6}$$

where

$$\Psi_l(v) := \sum_{i=1}^{n} \psi_l(v_i), \;\; \psi_l(t) = \frac{t^2 - 1}{2} - \log\, t, \;\; t > 0.$$

We call ψ_l the kernel function of the logarithmic barrier function $\Psi_l(v)$.

The generic interior point algorithm works as follows. Assume that we are given a strictly feasible point (x, s) which is in a τ-neighborhood of the given μ-center. Then we decrease μ to $\mu_+ := (1 - \theta)\mu$, for some fixed $\theta \in (0, 1)$ and solve the Newton system (3) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, s_+) that is in a τ-neighborhood of the μ_+-center and then we let $\mu := \mu_+$ and $(x, s) := (x_+, s_+)$. Then μ is again reduced by the factor $1-\theta$ and we solve the Newton system targeting at the new μ_+-center, and so on. This process is repeated until μ is small enough, say until $n\mu < \varepsilon$.

Generic Primal-Dual Algorithm

```
Input:
   a threshold parameter τ > 0;
   an accuracy parameter ε > 0;
   a fixed barrier update parameter θ, 0 < θ < 1;
   (x^0, s^0) and μ^0 > 0 such that Ψ_l(x^0, s^0, μ^0) ≤ τ.
begin
   x := x^0; s := s^0; μ := μ^0;
   while nμ ≥ ε do
   begin
      μ := (1 − θ)μ;
      while Ψ_l(v) > τ do
      begin
         solve the system (3) for Δx and Δs;
         determine a step size α;
         x := x + αΔx;
         s := s + αΔs;
         v := sqrt(xs/μ);
      end
   end
end
```

When the barrier update parameter θ is independent of n, we call the algorithm a large-update method.

3. New kernel function

In this section we define a new class of kernel functions and its properties.

Definition 2. The function $\psi : \mathbf{R}_{++} \to \mathbf{R}_+$ is called a kernel function if ψ is twice differentiable and satisfies the following conditions:

$$(a)\ \psi'(1) = \psi(1) = 0,\ (b)\ \psi''(t) > 0,\ t > 0,\ (c)\ \lim_{t\to 0}\psi(t) = \lim_{t\to\infty}\psi(t) = \infty.$$

Now we define a new class of kernel functions with parameters p and r as follows:

$$\psi(t) := \frac{t^{p+1}-1}{p+1} + r(e^{t^{-\frac{1}{r}}-1} - 1),\ 0 \le p \le 1,\ 0 < r \le 1,\ t > 0. \tag{7}$$

Note that $\psi(t)$ includes the kernel function defined in [2] as a special case. For $\psi(t)$ we have the following:

$$\psi'(t) = t^p - t^{-\frac{1}{r}-1} e^{t^{-\frac{1}{r}}-1},$$
$$\psi''(t) = pt^{p-1} + \Big(\frac{1}{r} + (\frac{1}{r}+1)t^{\frac{1}{r}}\Big) t^{-\frac{2}{r}-2} e^{t^{-\frac{1}{r}}-1},$$
$$\psi'''(t) = p(p-1)t^{p-2} - \Big(\frac{1}{r^2} + \frac{3}{r}(\frac{1}{r}+1)t^{\frac{1}{r}} + (\frac{1}{r}+1)(\frac{1}{r}+2)t^{\frac{2}{r}}\Big) t^{-\frac{3}{r}-3} e^{t^{-\frac{1}{r}}-1}. \tag{8}$$

In this paper, we replace the function $\Psi_l(v)$ in (6) with the function $\Psi(v)$ as follows:

$$d_x + d_s = -\nabla\Psi(v), \tag{9}$$

where $\Psi(v) = \sum_{i=1}^n \psi(v_i)$ and $\psi(t)$ is defined in (7). Hence the new search direction $(\Delta x, \Delta s)$ is obtained by solving the following modified Newton-system:

$$-M\Delta x + \Delta s = 0,\ S\Delta x + X\Delta s = -\mu v \nabla\Psi(v). \tag{10}$$

Since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have

$$\Psi(v) = 0 \ \Leftrightarrow\ v \ = \ e \ \Leftrightarrow x = x(\mu),\ s = s(\mu).$$

We use $\Psi(v)$ as the proximity function to measure the distance between the current iterate and the μ-center. Also, we define the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \frac{1}{2}||\nabla\Psi(v)|| = \frac{1}{2}||d_x + d_s||. \tag{11}$$

In the following we give properties of $\psi(t)$ which are essential to the complexity analysis.

Lemma 3.1. *Let $\psi(t)$ be as defined in* (7)*. Then we have the following:*
(i) $\psi(t)$ is exponentially convex, $t > 0$.
(ii) $\psi''(t)$ is monotonically decreasing, $t > 0$.

Proof. For (i), by Lemma 2.1.2 in [13], it suffices to show the function $\psi(t)$ satisfies $t\psi''(t) + \psi'(t) \geq 0$ for all $t > 0$. Using (8), we have

$$t\psi''(t) + \psi'(t) = (p+1)t^p + \frac{1}{r} t^{-\frac{2}{r}-1} e^{t^{-\frac{1}{r}}-1} + \frac{1}{r} t^{-\frac{1}{r}-1} e^{t^{-\frac{1}{r}}-1} \geq 0,\ t > 0.$$

For (ii), from (8), $\psi'''(t) < 0$. This completes the proof. □

Remark 1. Recall that the function $\psi : \mathbf{R}_{++} \to \mathbf{R}_+$ is eligible if ψ is three times differentiable and satisfies the following conditions([2]):
(a) $t\psi''(t) + \psi'(t) > 0,\ t < 1,$
(b) $t\psi''(t) - \psi'(t) > 0,\ t > 1,$
(c) $\psi'''(t) < 0,\ t > 0,$
(d) $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0,\ t < 1.$

Using (8), we have

$$t\psi''(t) - \psi'(t) = (p-1)t^p + \frac{1}{r}t^{-\frac{2}{r}-1}e^{t^{-\frac{1}{r}}-1} + \Big(\frac{1}{r}+2\Big)t^{-\frac{1}{r}-1}e^{t^{-\frac{1}{r}}-1}.$$

Since

$$t\psi''(t) - \psi'(t) < -7 < 0 \text{ for } p = \frac{1}{2},\ r = \frac{1}{4},\ t = 2^8,$$

condition (b) is not satisfied. Hence $\psi(t)$ is not eligible. Note that the kernel function in [2] is eligible.

Lemma 3.2. *For $\psi(t)$ and $p \in [0,1]$, we have*

$$\frac{1}{p+1}\sum_{i=1}^{n} {v_i}^{p+1} \le \Psi(v) + \frac{(pr+r+1)n}{p+1}.$$

Proof. Since $re^{t^{-\frac{1}{r}}-1} > 0$, we have

$$\psi(t) = \frac{t^{p+1}}{p+1} - \frac{1}{p+1} + re^{t^{-\frac{1}{r}}-1} - r \ge \frac{t^{p+1}}{p+1} - \frac{1}{p+1} - r.$$

So we have $\frac{t^{p+1}}{p+1} \le \psi(t) + \frac{pr+r+1}{p+1}$. Hence we have

$$\frac{1}{p+1}\sum_{i=1}^{n} {v_i}^{p+1} \le \Psi(v) + \frac{(pr+r+1)n}{p+1}.$$

This completes the proof. □

Define $\psi_b(t) := r(e^{t^{-\frac{1}{r}}-1} - 1)$. Then we have $\psi(t) := \frac{t^{p+1}-1}{p+1} + \psi_b(t)$. Since $\psi_b'(t) = -t^{-\frac{1}{r}-1}e^{t^{-\frac{1}{r}}-1} < 0$, $\psi_b(t)$ is monotonically decreasing in t.

Lemma 3.3. *Let $\beta \ge 1$. Then $\psi(\beta t) \le \psi(t) + \frac{t^{p+1}}{p+1}(\beta^{p+1} - 1)$.*

Proof. Since $\psi_b(t)$ is monotonically decreasing in t, $\psi_b(\beta t) - \psi_b(t) \le 0$ for $\beta \ge 1$. Hence we have

$$\begin{aligned}
\psi(\beta t) &= \frac{(\beta t)^{p+1} - 1}{p+1} + \psi_b(\beta t) \\
&= \frac{t^{p+1} - 1}{p+1} + \psi_b(t) + \frac{1}{p+1}(\beta^{p+1}t^{p+1} - t^{p+1}) + \psi_b(\beta t) - \psi_b(t) \\
&= \psi(t) + \frac{t^{p+1}}{p+1}(\beta^{p+1} - 1) + \psi_b(\beta t) - \psi_b(t) \\
&\le \psi(t) + \frac{t^{p+1}}{p+1}(\beta^{p+1} - 1).
\end{aligned}$$

This completes the proof. □

In the following we obtain an estimate for the effect of a μ-update on the value of $\Psi(v)$.

Theorem 3.4. *Let $0 \le \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. Then we have*

$$\Psi(v_+) \le \Psi(v) + \frac{\theta}{(1-\theta)^{\frac{1+p}{2}}} \left(\Psi(v) + \frac{(pr+r+1)n}{p+1} \right).$$

Proof. Using Lemma 3.3 with $\beta = \frac{1}{\sqrt{1-\theta}}$ and Lemma 3.2, we have

$$\begin{aligned}\Psi(v_+) = \Psi(\beta v) &= \sum_{i=1}^{n} \psi(\beta v_i) \le \sum_{i=1}^{n} \Big(\psi(v_i) + \frac{1}{p+1}(\beta^{p+1} - 1) v_i^{p+1} \Big) \\ &= \Psi(v) + \left(\frac{1}{(1-\theta)^{\frac{1+p}{2}}} - 1 \right) \frac{1}{p+1} \sum_{i=1}^{n} v_i^{p+1} \\ &\le \Psi(v) + \frac{1-(1-\theta)^{\frac{1+p}{2}}}{(1-\theta)^{\frac{1+p}{2}}} \left(\Psi(v) + \frac{(pr+r+1)n}{p+1} \right).\end{aligned}$$

Since $1-(1-\theta)^{\frac{1+p}{2}} \le \theta$ for $0 \le \theta < 1$,

$$\Psi(v_+) \le \Psi(v) + \frac{\theta}{(1-\theta)^{\frac{1+p}{2}}} \left(\Psi(v) + \frac{(pr+r+1)n}{p+1} \right).$$

This completes the proof. □

Note that at the start of outer iteration of the algorithm, i.e., just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \le \tau$. During the inner iteration we have

$$\begin{aligned}\Psi(v_+) &\le \Psi(v) + \frac{\theta}{(1-\theta)^{\frac{1+p}{2}}} \left(\Psi(v) + \frac{(pr+r+1)n}{p+1} \right) \\ &\le \tau + \frac{\theta}{(1-\theta)^{\frac{1+p}{2}}} \left(\tau + \frac{(pr+r+1)n}{p+1} \right).\end{aligned}$$

Each subsequent inner iteration will rise to a decrease of the value of $\Psi(v)$. Denote

$$\tilde{\Psi}_0 := \tau + \frac{\theta}{(1-\theta)^{\frac{1+p}{2}}} \left(\tau + \frac{(pr+r+1)n}{p+1} \right). \tag{12}$$

Define Ψ_0 the value of $\Psi(v)$ after the μ-update. Then $\Psi_0 \le \tilde{\Psi}_0$.

Lemma 3.5. *Define $\varrho : [0,\infty) \to [1,\infty)$ be the inverse function of $\psi(t)$ for $t \ge 1$. For $0 \le p \le 1$ and $u \ge 0$ we have*

$$\varrho(u) \ge \big(1 + (p+1)u\big)^{\frac{1}{1+p}}.$$

Proof. Let $u = \psi(t)$, $t \ge 1$. Since $\psi_b(t)$ is monotonically decreasing in t and $\psi_b(1) = 0$, $\psi_b(t) < 0$ for $t > 1$. Hence $u = \psi(t) = \frac{t^{p+1}-1}{p+1} + \psi_b(t) \le \frac{t^{p+1}-1}{p+1}$, $t \ge 1$. This implies $(p+1)u + 1 \le t^{p+1}$. By the definition of ϱ, $\varrho(u) = t \ge \big(1+(p+1)u\big)^{\frac{1}{1+p}}$. This completes the proof. □

From Lemma 3.1 (*ii*), we cite the following lemma in [2] without proof.

Lemma 3.6. (Theorem 4.9 in [2]) *Let $\delta(v)$ be as defined in* (11). *Then we have*

$$\delta(v) \geq \frac{1}{2}\psi'\big(\varrho\,(\Psi(v))\big).$$

For notational convenience we denote $\delta := \delta(v)$ and $\Psi := \Psi(v)$.

Lemma 3.7. *Let δ be as defined in* (11). *Then for all $\Psi \geq 1$ and $0 \leq p \leq 1$ we have*

$$\delta \geq \frac{1}{4}\big((p+1)\Psi\big)^{\frac{p}{1+p}}.$$

Proof. By Lemma 3.6, Lemma 3.5, and $\psi''(t) > 0$,

$$\begin{aligned}
\delta \;&\geq\; \frac{1}{2}\psi'\Big(\varrho(\Psi)\Big) \geq \frac{1}{2}\psi'\Big(\big(1+(p+1)\Psi\big)^{\frac{1}{1+p}}\Big)\\
&=\; \frac{1}{2}\left(\big(1+(p+1)\Psi\big)^{\frac{p}{1+p}} - e^{\big(1+(p+1)\Psi\big)^{-\frac{1}{r(1+p)}}-1}\frac{1}{\big(1+(p+1)\Psi\big)^{\frac{1+r}{r(1+p)}}}\right)\\
&\geq\; \frac{1}{2}\left(\big(1+(p+1)\Psi\big)^{\frac{p}{1+p}} - \frac{1}{\big(1+(p+1)\Psi\big)^{\frac{1+r}{r(1+p)}}}\right)\\
&\geq\; \frac{1}{2}\left(\big(1+(p+1)\Psi\big)^{\frac{p}{1+p}} - \frac{1}{\big(1+(p+1)\Psi\big)^{\frac{1}{1+p}}}\right)\\
&=\; \frac{1}{2}\frac{(p+1)\Psi}{\big(1+(p+1)\Psi\big)^{\frac{1}{1+p}}} \geq \frac{1}{4}\big((p+1)\Psi\big)^{\frac{p}{1+p}},
\end{aligned}$$

where the third inequality is satisfied from $e^{(1+(p+1)\Psi)^{-\frac{1}{r(1+p)}}-1} \leq 1$ and the last inequality from the fact $1 \leq (p+1)\Psi$. This completes the proof. □

4. Complexity result

In this section we compute a feasible step size and the decrease of the proximity function during an inner iteration and give the complexity result of the algorithm. For fixed μ if we take a step size α, then we have new iterates $x_+ = x + \alpha\Delta x,\ s_+ = s + \alpha\Delta s$. Using (4), we have

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}(v + \alpha d_x)$$

and

$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}(v + \alpha d_s).$$

Thus we have

$$v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}.$$

Define for $\alpha > 0$, $f(\alpha) = \Psi(v_+) - \Psi(v)$. Then $f(\alpha)$ is the difference between proximities of a new iterate and a current iterate for fixed μ. Using Lemma 3.1 (i), we have

$$\Psi(v_+) = \Psi(\sqrt{(v+\alpha d_x)(v+\alpha d_s)}\ ) \leq \frac{1}{2}\big(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\big).$$

Hence we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}\big(\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\big) - \Psi(v).$$

We have $f(0) = f_1(0) = 0$. Taking the derivative of $f_1(\alpha)$ with respect to α, we have

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\big(\psi'(v_i + \alpha[d_x]_i)[d_x]_i + \psi'(v_i + \alpha[d_s]_i)[d_s]_i\big),$$

where $[d_x]_i$ and $[d_s]_i$ denote the i-th components of the vectors d_x and d_s, respectively. Using (9) and (11), we have

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2.$$

Differentiating $f_1'(\alpha)$ with respect to α, we have

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\big(\psi''(v_i + \alpha[d_x]_i)[d_x]_i^2 + \psi''(v_i + \alpha[d_s]_i)[d_s]_i^2\big).$$

Since $f_1''(\alpha) > 0$, $f_1(\alpha)$ is strictly convex in α unless $d_x = d_s = 0$. Since M is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (10), for $\Delta x \in \mathbf{R}^n$,

$$(1+4\kappa)\sum_{i\in I_+}[\Delta x]_i[\Delta s]_i + \sum_{i\in I_-}[\Delta x]_i[\Delta s]_i \geq 0,$$

where $I_+ = \{i \in I \ : \ [\Delta x]_i[\Delta s]_i \geq 0\}$, $I_- = I - I_+$. Since $d_x d_s = \frac{v^2\Delta x\Delta s}{xs} = \frac{\Delta x\Delta s}{\mu}$ and $\mu > 0$, we have

$$(1+4\kappa)\sum_{i\in I_+}[d_x]_i[d_s]_i + \sum_{i\in I_-}[d_x]_i[d_s]_i \geq 0.$$

For convenience we denote $\sigma_+ := \sum_{i\in I_+}[d_x]_i[d_s]_i$ and $\sigma_- := -\sum_{i\in I_-}[d_x]_i[d_s]_i$. In the following we cite some lemmas in [4] without proof.

Lemma 4.1. (Modification of Lemma 4.1 in [4]) $\sigma_+ \leq \delta^2$ and $\sigma_- \leq (1+4\kappa)\delta^2$.

Lemma 4.2. (Modification of Lemma 4.2 in [4]) $\sum_{i=1}^n([d_x]_i^2 + [d_s]_i^2) \leq 4(1+2\kappa)\delta^2$, $\|d_x\| \leq 2\delta\sqrt{1+2\kappa}$, and $\|d_s\| \leq 2\delta\sqrt{1+2\kappa}$.

Lemma 4.3. (Modification of lemma 4.3 in [4]) $f_1^{''}(\alpha) \leq 2(1+2\kappa)\ \delta^2\psi^{''}(v_{min} - 2\alpha\delta\sqrt{1+2\kappa})$.

Lemma 4.4. (Modification of lemma 4.4 in [4]) *$f_1^{'}(\alpha) \le 0$ if α is satisfying*

$$-\psi'(v_{min} - 2\alpha\delta\sqrt{1+2\kappa}) + \psi^{'}(v_{min}) \le \frac{2\delta}{\sqrt{1+2\kappa}}. \tag{13}$$

Lemma 4.5. (Modification of lemma 4.5 in [4]) *Define $\rho : [0,\infty) \to (0,1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$ for $0 < t \le 1$ and $a := 1 + \frac{1}{\sqrt{1+2\kappa}}$. Then the largest step size α satisfying (13) is given by*

$$\hat{\alpha} := \frac{1}{2\delta\sqrt{1+2\kappa}}\Big(\rho(\delta) - \rho(\,a\delta)\Big).$$

Lemma 4.6. (Modification of lemma 4.6 in [4]) *Let ρ and $\hat{\alpha}$ be as defined in Lemma 4.5. Then*

$$\hat{\alpha} \ge \frac{1}{(1+2\kappa)\psi''\big(\rho(a\delta)\big)}.$$

Define

$$\bar{\alpha} := \frac{1}{(1+2\kappa)\psi''\big(\rho(\,a\delta)\big)}. \tag{14}$$

Then we have $\bar{\alpha} \le \hat{\alpha}$.

Lemma 4.7. *Let $\bar{\alpha}$ be as defined in (14). Then for $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$ and $\kappa \ge 0$, we have*

$$\bar{\alpha} \ge \frac{1}{(1+2\kappa)\Big(p + (2a\delta+1)\big(\frac{2}{r}+1\big)\big(1+\log(2a\delta+1)\big)^{1+r}\Big)}.$$

Proof. Using the definition of ρ, we have $-\frac{1}{2}\psi'(\rho(a\delta)) = a\delta$. Let $z = \rho(a\delta)$. Then $-\psi'(z) = 2a\delta$ and $0 < z \le 1$. From (8), we have $-z^p + z^{-\frac{1}{r}-1}e^{z^{-\frac{1}{r}}-1} = 2a\delta$. Then for $0 < z \le 1$,

$$z^{-\frac{1}{r}-1}e^{z^{-\frac{1}{r}}-1} = 2a\delta + z^p \le 2a\delta + 1. \tag{15}$$

By taking the natural logarithmic function on both sides of (15), we have

$$z^{-\frac{1}{r}} - 1 - \Big(\frac{1}{r}+1\Big)\log z \le \log(2a\delta+1). \tag{16}$$

Using (16) and $0 < z \le 1$, we obtain

$$z^{-\frac{1}{r}} \le 1 + \log(2a\delta+1) + \Big(\frac{1}{r}+1\Big)\log z \le 1 + \log(2a\delta+1).$$

This implies

$$\begin{aligned} z^{p-1} &\le \big(1+\log(2a\delta+1)\big)^{r(1-p)} \le \big(1+\log(2a\delta+1)\big)^{1+r}, \\ z^{-1} &\le \big(1+\log(2a\delta+1)\big)^{r} \le \big(1+\log(2a\delta+1)\big)^{1+r}, \\ z^{-\frac{1}{r}-1} &\le \big(1+\log(2a\delta+1)\big)^{1+r}. \end{aligned} \tag{17}$$

From (14), we have for $0 < z \le 1$ and $0 \le p \le 1$,

$$\begin{aligned}
\bar{\alpha} &= \frac{1}{(1+2\kappa)\psi''\big(\rho(a\delta)\big)} = \frac{1}{(1+2\kappa)\psi''(z)} \\
&= \frac{1}{(1+2\kappa)\Big(pz^{p-1} + \frac{1}{r}z^{-\frac{2}{r}-2}e^{z^{-\frac{1}{r}}-1} + (\frac{1}{r}+1)z^{-\frac{1}{r}-2}e^{z^{-\frac{1}{r}}-1}\Big)} \\
&\ge \frac{1}{(1+2\kappa)\Big(pz^{p-1} + \frac{1}{r}(2a\delta+1)z^{-\frac{1}{r}-1} + (\frac{1}{r}+1)(2a\delta+1)z^{-1}\Big)} \\
&= \frac{1}{(1+2\kappa)\Big(pz^{p-1} + (2a\delta+1)\big(\frac{1}{r}z^{-\frac{1}{r}-1} + (\frac{1}{r}+1)z^{-1}\big)\Big)} \\
&\ge \frac{1}{(1+2\kappa)\big(p + (2a\delta+1)(\frac{2}{r}+1)\big)\big(1+\log(2a\delta+1)\big)^{1+r}},
\end{aligned}$$

where the first inequality follows from (15) and the last inequality from (17). This proves the lemma. □

Define

$$\tilde{\alpha} = \frac{1}{(1+2\kappa)\big(p + (2a\delta+1)(\frac{2}{r}+1)\big)\big(1+\log(2a\delta+1)\big)^{1+r}}. \tag{18}$$

Note that $\tilde{\alpha} \le \bar{\alpha}$. We will use $\tilde{\alpha}$ as the default step size.

Lemma 4.8. (Lemma 1.3.3 in [13]) *Suppose that $h(t)$ is a twice differentiable convex function with $h(0) = 0$ and $h'(0) < 0$ and $h(t)$ attains its global minimum at $t^* > 0$ and $h''(t)$ is increasing with respect to t. Then for any $t \in [0, t^*]$,*

$$h(t) \le \frac{th'(0)}{2}.$$

Lemma 4.9. (Modification of lemma 4.8 in [4]) *If the step size α is such that $\alpha \le \bar{\alpha}$, then*

$$f(\alpha) \le -\alpha\delta^2.$$

In our algorithm we assume that $\tau \ge 1$. Using Lemma 3.7 and the fact $\Psi \ge \tau$, we have

$$\delta \ge \frac{1}{4}\big((p+1)\Psi\big)^{\frac{p}{1+p}} \ge \frac{1}{4}. \tag{19}$$

Lemma 4.10. *For $0 < r \le 1$ the function*

$$g(\delta) = -\frac{\delta}{\big(1+\log(2a\delta+1)\big)^{1+r}}$$

is monotonically decreasing in δ.

Proof. It suffices to show that the function $-g(\delta)$ is monotonically increasing in δ. If we differentiate $-g(\delta)$ with respect to δ, we have

$$-g'(\delta) = \frac{(2a\delta+1)\big(1+\log(2a\delta+1)\big)^{1+r} - 2a\delta(1+r)\big(1+\log(2a\delta+1)\big)^{r}}{(2a\delta+1)\big(1+\log(2a\delta+1)\big)^{2(1+r)}}.$$

Since the denominator is strictly positive, it is enough to show that numerator is positive. The numerator is

$$\begin{aligned} & (2a\delta+1)(1+\log(2a\delta+1))^{1+r} - 2a\delta(1+r)\big(1+\log(2a\delta+1)\big)^{r} \\ = \; & \big(1+\log(2a\delta+1)\big)^{r}\big((1+\log(2a\delta+1))(2a\delta+1) - 2a\delta(1+r)\big) \\ = \; & \big(1+\log(2a\delta+1)\big)^{r}\big(1+(2a\delta+1)\log(2a\delta+1) - 2a\delta r\big). \end{aligned}$$

Let $\tilde{g}(\delta) := 1+(2a\delta+1)\log(2a\delta+1) - 2a\delta r$. Then $\tilde{g}'(\delta) = 2a\log(2a\delta+1) + 2a - 2ar$. From $1 < a \leq 2$ and (19), we have $\tilde{g}'(\delta) > 0$. Since $\tilde{g}(\frac{1}{4}) > 0$, $\tilde{g}(\delta) > 0$ for $\delta \geq \frac{1}{4}$. Hence, $-g(\delta)$ is monotonically increasing in δ. This completes the proof. □

Theorem 4.11. *Let $\tilde{\alpha}$ be as defined in* (18)*. Then*

$$f(\tilde{\alpha}) \leq -\frac{\big((p+1)\Psi\big)^{\frac{p}{1+p}}}{16(1+2\kappa)\big(p+\frac{4}{r}+2\big)\Big(1+\log\Big(\frac{a}{2}\big((p+1)\Psi_0\big)^{\frac{p}{1+p}}+1\Big)\Big)^{1+r}}.$$

Proof. Using (18) and Lemma 4.9, we have

$$\begin{aligned} f(\tilde{\alpha}) \;\; & \leq \;\; -\frac{\delta^2}{(1+2\kappa)\big(p+(2a\delta+1)(\frac{2}{r}+1)\big)\big(1+\log(2a\delta+1)\big)^{1+r}} \\ & \leq \;\; -\frac{\delta^2}{(1+2\kappa)\big(4p\delta+\delta(2a+4)(\frac{2}{r}+1)\big)\big(1+\log(2a\delta+1)\big)^{1+r}} \\ & = \;\; -\frac{\delta}{2(1+2\kappa)\big(2p+\frac{2a}{r}+\frac{4}{r}+a+2\big)\big(1+\log(2a\delta+1)\big)^{1+r}} \\ & \leq \;\; -\frac{\delta}{4(1+2\kappa)\big(p+\frac{4}{r}+2\big)\big(1+\log(2a\delta+1)\big)^{1+r}} \\ & \leq \;\; -\frac{\frac{1}{4}\big((p+1)\Psi\big)^{\frac{p}{1+p}}}{4(1+2\kappa)\big(p+\frac{4}{r}+2\big)\Big(1+\log\big(\frac{a}{2}\,((p+1)\Psi\,)^{\frac{p}{1+p}}+1\big)\Big)^{1+r}} \\ & = \;\; -\frac{\big((p+1)\Psi\big)^{\frac{p}{1+p}}}{16(1+2\kappa)\big(p+\frac{4}{r}+2\big)\Big(1+\log\big(\frac{a}{2}\,((p+1)\Psi\,)^{\frac{p}{1+p}}+1\big)\Big)^{1+r}} \\ & \leq \;\; -\frac{\big((p+1)\Psi\big)^{\frac{p}{1+p}}}{16(1+2\kappa)\big(p+\frac{4}{r}+2\big)\Big(1+\log\big(\frac{a}{2}\,((p+1)\Psi_0\,)^{\frac{p}{1+p}}+1\big)\Big)^{1+r}}, \end{aligned}$$

where the second inequality is satisfied from (19), third inequality from $1 < a \leq 2$, the fourth inequality from Lemma 3.7 and Lemma 4.10, and the last inequality from the definition of Ψ_0. This completes the proof. □

Lemma 4.12. (Lemma 1.3.2 in [13]) *Let $t_0, t_1, \cdots, t_J$ be a sequence of positive numbers such that*

$$t_{j+1} \leq t_j - \gamma t_j^{1-\lambda},\ j = 0, 1, \cdots, J-1,$$

where $\gamma > 0$ and $0 < \lambda \leq 1$. Then $J \leq \lfloor \frac{t_0^\lambda}{\gamma\lambda} \rfloor$.

We define the value of $\Psi(v)$ after the μ-update as Ψ_0 and the subsequent values in the same outer iteration $\Psi_k,\ k = 1, 2, \cdots$. Let K denote the total number of inner iterations in the outer iteration. Then we have

$$\Psi_{K-1} > \tau,\ 0 \leq \Psi_K \leq \tau.$$

Lemma 4.13. *Let $\tilde{\Psi}_0$ be as defined in (12) and K be the total number of inner iterations in the outer iteration. Then we have*

$$K \leq 16(1+2\kappa)\Big(p + \frac{4}{r} + 2\Big)(p+1)^{\frac{1}{1+p}} \Big(1 + \log\big((p+1)\tilde{\Psi}_0\big)\Big)^{1+r} \tilde{\Psi}_0^{\frac{1}{1+p}}.$$

Proof. By Theorem 4.11 with $\gamma = \frac{(p+1)^{\frac{p}{1+p}}}{16(1+2\kappa)\left(p+\frac{4}{r}+2\right)\left(1+\log\left(\frac{a}{2}\left((p+1)\Psi_0\right)^{\frac{p}{1+p}}+1\right)\right)^{1+r}}$ and $\lambda = \frac{1}{1+p}$, we have

$$K \leq \frac{16(1+2\kappa)\big(p + \frac{4}{r} + 2\big)\Big(1 + \log\big(\frac{a}{2}\left((p+1)\Psi_0\right)^{\frac{p}{1+p}} + 1\big)\Big)^{1+r}}{(p+1)^{\frac{p}{1+p}}} (p+1)\Psi_0^{\frac{1}{1+p}}.$$

Since $\Psi_0 \leq \tilde{\Psi}_0$ and $1 < a \leq 2$, we have the result. □

Theorem 4.14. *Let a $P_*(\kappa)$ LCP be given and $\tau \geq 1$. Then the total number of iterations to have an approximate solution with $n\mu < \epsilon$ is bounded by*

$$\left\lceil 16(1+2\kappa)\Big(p + \frac{4}{r} + 2\Big)(p+1)^{\frac{1}{1+p}} (1 + \log((p+1)\tilde{\Psi}_0))^{1+r} \tilde{\Psi}_0^{\frac{1}{1+p}} \right\rceil \cdot \left\lceil \frac{1}{\theta} \log \frac{n\mu_0}{\epsilon} \right\rceil,$$

where $\epsilon > 0$ is the desired accuracy, $\mu_0 > 0$ is given, and θ, $0 < \theta < 1$, is the given barrier update parameter.

Proof. If the central path parameter μ has the initial value $\mu_0 > 0$ and is updated by multiplying $1 - \theta$ with $0 < \theta < 1$, then after at most

$$\left\lceil \frac{1}{\theta} \log \frac{n\mu_0}{\epsilon} \right\rceil$$

iterations we have $n\mu < \epsilon$([14]). For the total number of iterations, we multiply the number of inner iterations by that of outer iterations. i.e.,

$$\left\lceil 16(1+2\kappa)\Big(p+\frac{4}{r}+2\Big)(p+1)^{\frac{1}{1+p}}(1+\log((p+1)\tilde{\Psi}_0))^{1+r}\tilde{\Psi}_0^{\frac{1}{1+p}}\right\rceil\cdot\left\lceil\frac{1}{\theta}\log\frac{n\mu_0}{\epsilon}\right\rceil.$$

This completes the proof. □

Remark 2. Taking $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, the large-update algorithm has

$$\mathcal{O}\left(\frac{(1+2\kappa)}{r}n^{\frac{1}{1+p}}(\log n)^{1+r}\log\frac{n\mu_0}{\epsilon}\right)$$

iteration complexity. In particular, for $r = \frac{1+\epsilon}{\log(\log n)}$ with a sufficiently small $\epsilon > 0$, we have $\frac{1}{r}(\log n)^{1+r} = \frac{e^{1+\epsilon}}{1+\epsilon}(\log n)\log(\log n)$. So we have $\mathcal{O}\big((1+2\kappa)\sqrt{n}(\log n)$ $\log(\log n)\log\frac{n\mu_0}{\epsilon}\big)$ iteration complexity with $p = 1$ and $r = \frac{1+\epsilon}{\log(\log n)}$. This complexity result improves the one in [2].

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