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CONGRUENCES OF L-VALUES FOR CYCLIC EXTENSIONS

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Abstract. We study the consequences of Gross's conjecture for cyclic extensions of degree l^2 where l is prime, and deduce that the L-values at $s = 0$ satisfy certain congruence relations.

1. Introduction

We first review Gross's conjecture briefly. Let K/k be an abelian extension of global fields with Galois group G. Let S be a finite non-empty set of places of k which contains all archimedean places and all places ramified in K, and let T be a finite non-empty set of places of k which is disjoint from S. We choose T so that $U_{S,T}$, the group of S-units in k which are congruent to 1 (mod v) for all $v \in T$, is a free abelian group of rank $n = |S| - 1$.

For a complex character $\chi \in \widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$, the associated modified L-function is defined as

$$L_{S,T}(\chi, s) = \prod_{v \in T}(1 - \chi(g_v)\boldsymbol{N}v^{1-s}) \prod_{v \notin S}(1 - \chi(g_v)\boldsymbol{N}v^{-s})^{-1},$$

where $g_v \in G$ is the Frobenius element for v. The Stickelberger element $\theta_G \in \mathbb{C}[G]$ is the unique element that satisfies

$$\chi(\theta_G) = L_{S,T}(\chi, 0)$$

for all $\chi \in \widehat{G}$. In fact, $\theta_G \in \mathbb{Z}[G]$ which is a deep theorem of Deligne-Ribet(cf. [2]).

Let I_G be the augmentation ideal of $\mathbb{Z}[G]$, i.e. the kernel of the map from $\mathbb{Z}[G]$ to $\mathbb{Z}$ sending each group element to 1. Choose an ordered basis

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$\{u_1, \ldots, u_n\}$ of $U_{S,T}$. Pick a place $v_0 \in S$, and for each $v_i \in S \setminus \{v_0\}$, we let $f_i : k^* \to G$ denote the homomorphism induced from local reciprocity map for v_i. We set

$$R_G := \det_{1 \le i,j \le n} (f_i(u_j) - 1).$$

Gross has conjectured (cf. [3])

Conjecture 1.

$$\theta_G \equiv m \cdot R_G \pmod{I_G^{n+1}}.$$

Here, the integer m is defined by

$$m = \pm h_S \cdot \frac{\prod_{v \in T}(Nv - 1)}{(U_S : U_{S,T})},$$

where h_S is the S-class number of k and U_S is the set of S-units. The $\pm$ sign is determined by the (S, T)-version of the analytic class number formula.

Conjecture 1 is known to be true when G is a cyclic group (cf. [1]). The goal of this paper is to understand the meaning of Conjecture 1 in more concrete terms. We consider the case where G is a cyclic group of order l^2 for a prime number l. Our main result is Theorem 4, which states that there exist certain congruence relation among L-values.

2. Structure of $\mathbb{Z}[G]$

Let l be a prime and G be a cyclic group of order l^2 with generator σ. We note that

$$\mathbb{Z}[G] \cong \mathbb{Z}[x]/(x^{l^2} - 1),$$

where σ is identified with x. We have

$$x^{l^2} - 1 = f_0(x) f_1(x) f_2(x),$$

where

$$\begin{aligned} f_0(x) &= x - 1, \\ f_1(x) &= x^{l-1} + x^{l-2} + \cdots + x + 1, \\ f_2(x) &= x^{(l-1)l} + x^{(l-2)l} + \cdots + x^l + 1 = f_1(x^l). \end{aligned}$$

It is well-known that $f_i(x)$ is the l^i-th cyclotomic polynomial which is irreducible over $\mathbb{Z}$.

Choose a primitive l^2-th root of unity ζ_2 in $\mathbb{C}$, and set $\zeta_0 = 1, \zeta_1 = \zeta_2^l$. We have a ring homomorphism

$$\chi : \mathbb{Z}[x] \longrightarrow \prod_{i=0}^{2} \mathbb{Z}[\zeta_i]$$

that sends x to $(1, \zeta_1, \zeta_2)$. We note that $\ker \chi = (x^{l^2} - 1)$, hence it induces an injective ring homomorphism

$$\chi : \mathbb{Z}[G] \longrightarrow \prod_{i=0}^{2} \mathbb{Z}[\zeta_i].$$

We also note that each component function

$$\chi_i : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[\zeta_i]$$

of χ is surjective with kernel $(f_i(x))$, and it induces a ring homomorphism

$$\chi_i : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[\zeta_i].$$

Clearly, $I_G = \ker \chi_0$ is generated by $\sigma - 1$. We set $\lambda_i = \zeta_i - 1$ for $i = 1, 2$, so that $\chi(\sigma - 1) = (0, \lambda_1, \lambda_2)$. We also set $\eta = \lambda_1/\lambda_2$.

We now determine $\chi(I_G^n)$ where n is a positive integer. Suppose $a \in I_G^n$. Then

$$a = b(\sigma - 1)^n$$

for some $b \in \mathbb{Z}[G]$. If $\chi(b) = (\beta_0, \beta_1, \beta_2)$, then

$$\chi(a) = (0, \beta_1\lambda_1^n, \beta_2\lambda_2^n).$$

Conversely, if there exists an element $b \in \mathbb{Z}[G]$ with $\chi_1(b) = \beta_1$ and $\chi_2(b) = \beta_2$, then the element

$$(0, \beta_1\lambda_1^n, \beta_2\lambda_2^n) \in \prod_{i=0}^{2} \mathbb{Z}[\zeta_i]$$

actually belongs to $\chi(I_G^n)$.

To determine whether there exists an element $b \in \mathbb{Z}[G]$ with $\chi_1(b) = \beta_1$ and $\chi_2(b) = \beta_2$ for given $\beta_1 \in \mathbb{Z}[\zeta_1]$ and $\beta_2 \in \mathbb{Z}[\zeta_2]$, we have the following proposition which is a generalization of the Chinese remainder theorem.

Proposition 2. *Let R be a commutative ring with 1, and I, J be ideals of R. There exists a short exact sequence of R-modules*

$$0 \to R/(I \cap J) \to R/I \times R/J \to R/(I + J) \to 0,$$

where the first map sends r to (r, r) and the second sends (r_1, r_2) to $r_1 - r_2$.

Proof. We show that if

$$r_1 \equiv r_2 \pmod{I + J}$$

then there exists an element $r \in R$ such that

$$\begin{aligned} r &\equiv r_1 \pmod{I}, \\ r &\equiv r_2 \pmod{J}. \end{aligned}$$

Write

$$r_1 - r_2 = i + j$$

for some $i \in I, j \in J$. Then the element

$$r = r_1 - i = r_2 + j$$

satisfies the requirement. □

We apply Proposition 2 to the case when $R = \mathbb{Z}[x], I = (f_1(x))$ and $J = (f_2(x))$. In this case, $R/I = \mathbb{Z}[\zeta_1]$, $R/J = \mathbb{Z}[\zeta_2]$. As $f_2(\zeta_1) = l$ and $f_1(\zeta_2) = \eta$, we have

$$R/(I + J) = \mathbb{Z}[\zeta_1]/(l) = \mathbb{Z}[\zeta_2]/(\eta).$$

We note that for $p(x) \in \mathbb{Z}[x]$, $p(\zeta_1) \in \mathbb{Z}[\zeta_1]/(l)$ is identified with $p(\zeta_2) \in \mathbb{Z}[\zeta_2]/(\eta)$.

Proposition 2 states that for $\beta_1 \in \mathbb{Z}[\zeta_1]$ and $\beta_2 \in \mathbb{Z}[\zeta_2]$, there exists an element $b \in \mathbb{Z}[G]$ such that $\chi_1(b) = \beta_1$ and $\chi_2(b) = \beta_2$ if and only if

$$\beta_1 \pmod{l} = \beta_2 \pmod{\eta}$$

holds. Hence the following theorem is proved.

Theorem 3. *Suppose $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ is an element of $\prod_{i=0}^{2} \mathbb{Z}[\zeta_i]$, and n is a positive integer. Then $\alpha \in \chi(I_G^n)$ if and only if the following conditions hold:*

1. $\alpha_0 = 0$,
2. $\lambda_i^n \mid \alpha_i$ *for* $i = 1, 2$,
3. $\alpha_1/\lambda_1^n \pmod{l} = \alpha_2/\lambda_2^n \pmod{\eta}$.

Applying Theorem 3 to $\chi(\theta_G - m \cdot R_G)$, we obtain the following result:

Theorem 4. *Suppose K/k is a cyclic extension of degree l^2. We have*

1. $\zeta_{S,T}(0) = 0$,
2. $\lambda_1^{n+1} \mid L_{S,T}(\chi_1, 0) - m \cdot \chi_1(R_G)$,
3. $\lambda_2^{n+1} \mid L_{S,T}(\chi_2, 0) - m \cdot \chi_2(R_G)$.

Furthermore, write

$$\begin{aligned} L_{S,T}(\chi_1, 0) - m \cdot \chi_1(R_G) &= \beta_1 \cdot \lambda_1^{n+1}, \\ L_{S,T}(\chi_2, 0) - m \cdot \chi_2(R_G) &= \beta_2 \cdot \lambda_2^{n+1}. \end{aligned}$$

Then

$$\beta_1 \pmod{l} = \beta_2 \pmod{\eta}.$$

Corollary 5. *Under the same hypothesis as Theorem 4, we have*

1. $\lambda_1^n \mid L_{S,T}(\chi_1, 0)$,
2. $\lambda_2^n \mid L_{S,T}(\chi_2, 0)$,
3. $L_{S,T}(\chi_1, 0)/\lambda_1^n \pmod{l} = L_{S,T}(\chi_2, 0)/\lambda_2^n \pmod{\eta}$

Proof. This comes from the weaker version of the conjecture, namely $\theta_G \in I_G^n$. □

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