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SURFACES WITH PLANAR LINES OF CURVATURE

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Abstract. We study surfaces in the 3-dimensional Euclidean space with two family of planar lines of curvature. As a result, we establish some characterization theorems for such surfaces.

1. Introduction

Consider a smooth surface M in the Euclidean space $\mathbb{E}^3$ with a unit normal vector field U. Then on each tangent plane T_pM the shape operator S is defined as follows:

$$S(v) = -\nabla_v U,$$

where $\nabla_v U$ denotes the covariant derivative of U in the v direction.

For a unit vector u tangent to M at a point p, the number $k(u) = \langle S(u), u\rangle$ is called the normal curvature of M in the u direction. The maximum and minimum values of the normal curvature $k(u)$ of M at p are called the *principal curvatures* of M at p, and are denoted by k_1 and k_2. The directions in which these extreme values occur are called *principal directions* of M at p.

A regular curve X in M is called a *line of curvature* provided that the velocity X' of X always points in a principal direction. Through each non-umbilic point of M, there are exactly two lines of curvature, which necessarily cut orthogonally across each other.

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The next theorem is useful to find lines of curvature on some classes of surfaces:

Theorem of Joachimstahl. Suppose that M_1 and M_2 intersect along a regular curve X and make an angle $\theta(p)$, $p \in X$. Assume that X is a line of curvature of M_1. Then X is a line of curvature of M_2 if and only if $\theta(p)$ is constant.

Proof. See the proof of Theorem 9 in ([10], p. 296). □

Note that every regular curve on a plane is a line of curvature. Using above theorem, it is easy to show the following: The meridians and parallels on a surface of revolution are its lines of curvature.

For a plane curve X in a plane P, the cylinder M over X is a ruled surface generated by a one-parameter family of straight lines through each point $X(s)$ which are orthogonal to the plane P. Theorem of Joachimstahl also shows that the straight lines, and the intersection of M and each plane parallel to the plane P are lines of curvature of M.

Hence we see that cylinders and surfaces of revolution satisfy the following condition:

(C) Around each point $p \in M$, there exists a local orthonormal frame $\{E_1, E_2\}$ whose integral curves are planar lines of curvature.

In this paper, we study smooth surfaces M in the Euclidean space $\mathbb{E}^3$ which satisfy the condition (C). As a result, we establish some characterization theorems for such surfaces. Furthermore, we give a condition for such a surface to be a surface of revolution.

2. Slant cylinders and generalized slant cylinders

For a fixed unit speed plane curve $X(s) = (x(s), y(s), 0)$, let $T(s) = X'(s)$ and $N(s) = (-y'(s), x'(s), 0)$ denote the unit tangent and principal normal vector, respectively. The curvature $\kappa(s)$ of $X(s)$ is defined by $T'(s) = \kappa(s)N(s)$ and we have $T(s) \times N(s) = V$, where V denotes the unit vector $(0, 0, 1)$. For a constant θ, we let $Y(s) = \cos\theta N(s) + \sin\theta V$. Then the ruled surface M defined by

$$F(s,t) = X(s) + tY(s) \tag{2.1}$$

is regular at (s, t) where $1 - \cos\theta\kappa(s)t$ does not vanish. This ruled surface M is called a *slant cylinder* over $X(s)$. For the unit normal vector $U = -\sin\theta N(s) + \cos\theta V$, M satisfies

$$\langle F_s, F_t\rangle = 0, \langle F_{st}, U\rangle = 0.$$

This shows that the coordinates lines of F are lines of curvature of M with corresponding principal curvatures

$$k_1(s,t) = \frac{-\kappa(s)\sin\theta}{1-\kappa(s)t\cos\theta}, k_2(s,t) = 0, \tag{2.2}$$

respectively. Hence $F(s,t)$ is a principal curvature coordinate system of the flat slant cylinder M([6], p. 53). Since the coordinate lines of F are planar, it follows that the slant cylinder M satisfies the condition (C). The slant cylinder with $\sin\theta = 0$ or $\cos\theta = 0$ is nothing but a parametrization of either a plane or a usual cylinder.

In general, we consider another unit speed plane curve $W(t) = (z(t), w(t))$. If we let $Y_s(t) = z(t)N(s)+w(t)V$, then the parametrized surface defined by

$$H(s,t) = X(s) + Y_s(t) \tag{2.3}$$

is regular at (s,t) where $1-\kappa(s)z(t)$ does not vanish. This parametrized surface M is called a *generalized slant cylinder* over $X(s)$. For the unit normal vector $U(s,t) = -w'(t)N(s) + z'(t)V$, M satisfies

$$\langle H_s, H_t\rangle = 0, \langle H_{st}, U\rangle = 0.$$

This shows that $H(s,t)$ is a principal curvature coordinate system of M with corresponding principal curvatures

$$k_1(s,t) = \frac{-\kappa(s)w'(t)}{1-\kappa(s)z(t)}, k_2(s,t) = \kappa(t), \tag{2.4}$$

respectively, where $\kappa(t) = z'(t)w''(t) - z''(t)w'(t)$ denotes the curvature of $W(t)$. It is obvious that the coordinate lines of H are planar. Hence we see that the generalized slant cylinder also satisfies the condition (C).

If $W(t)$ is a straight line, then the generalized slant cylinder $H(s,t)$ is nothing but a slant cylinder. Furthermore, we prove the following.

Proposition 1. If a plane curve $X(s)$ is a circle, then the generalized slant cylinder M over $X(s)$ is a surface of revolution.

Proof. Suppose that $X(s)$ is a circle of radius r. Then it is straightforward to show that for each fixed t, s curve of the generalized slant cylinder H defined in (2.4) is a circle of radius $r-z(t)$ with principal normal vector $N(s)$. Hence the s curve through $H(0,t)$ is a circle centered at

$$C(t) = H(0,t) + \{r - z(t)\}N(0) = X(0) + rN(0) + w(t)V,$$

which parametrizes a fixed straight line l in the direction of V. Thus M is a surface of revolution with axis l. □

Therefore the class of generalized slant cylinders contains both the class of slant cylinders and the class of surfaces of revolution.

3. Some characterizations

Suppose that a smooth surface M in the Euclidean space $\mathbb{E}^3$ satisfies the condition (C). If we let $E_3 = E_1 \times E_2$, then $\{E_1, E_2, E_3\}$ is a principal frame field on M([9], p. 261). For the dual 1-forms θ_1, θ_2 of E_1, E_2 the connection forms are given by

$$\omega_{12} = g_1\theta_1 + g_2\theta_2, \omega_{13} = k_1\theta_1, \omega_{23} = k_2\theta_2, \tag{3.1}$$

where g_1, g_2 are some functions and k_1, k_2 denote the principal curvatures in the direction of E_1, E_2, respectively. Hence the covariant derivatives of $E_i (i = 1, 2, 3)$ with respect to $E_j (j = 1, 2)$ are given by

$$\nabla_{E_1}E_1 = g_1E_2 + k_1E_3, \nabla_{E_1}E_2 = -g_1E_1, \nabla_{E_1}E_3 = -k_1E_1, \tag{3.2}$$

$$\nabla_{E_2}E_1 = g_2E_2, \nabla_{E_2}E_2 = -g_2E_1 + k_2E_3, \nabla_{E_2}E_3 = -k_2E_2, \tag{3.3}$$

respectively.

From the Codazzi equations we have([9], p. 262)

$$E_1(k_2) = (k_1 - k_2)g_2, \tag{3.4}$$

$$E_2(k_1) = (k_1 - k_2)g_1. \tag{3.5}$$

For the Gaussian curvature K of M the second structural equation gives([9], p. 263)

$$K = k_1k_2 = E_2(g_1) - E_1(g_2) - g_1^2 - g_2^2. \tag{3.6}$$

It follows from (3.2) that the integral curves of E_1 are planar if and only if

$$g_1E_1(k_1) - k_1E_1(g_1) = 0. \tag{3.7}$$

Similarly, we see that the integral curves of E_2 are planar if and only if

$$g_2 E_2(k_2) - k_2 E_2(g_2) = 0. \tag{3.8}$$

Furthermore, for each $i = 1, 2$, the integral curves of E_i lie on a plane $V_i^{\perp}$ normal to V_i, which is given by

$$V_1 = \frac{-k_1 E_2 + g_1 E_3}{\sqrt{k_1^2 + g_1^2}}, V_2 = \frac{k_2 E_1 + g_2 E_3}{\sqrt{k_2^2 + g_2^2}}, \tag{3.9}$$

unless the denominators vanish. It is obvious from the condition (C) that

$$\nabla_{E_i} V_i = 0, i = 1, 2. \tag{3.10}$$

First of all we prove the following:

Theorem 2. A flat surface M in the Euclidean space $\mathbb{E}^3$ satisfies the condition (C) if and only if it is locally a slant cylinder over a plane curve.

Proof. Suppose that a flat surface M satisfies the condition (C). We denote by P the set of planar points and by $W = M - P$ the set of parabolic points. Then P is closed and W is open in M. On a connected component W_1 of W, we may assume that k_1 does not vanish. Hence k_2 vanishes identically on W_1. By reversing the direction of E_1 if necessary, we may assume that $k_1 > 0$. Hence (3.4) shows that $g_2 = 0$. Thus it follows from (3.3) that the E_2 curve through a point $p \in W_1$ is an open segment of a straight line, which parametrizes a unique asymptotic line segment through p. Using (3.7), we see that $g_1 = h_1 k_1$ for a function h_1 satisfying $E_1(h_1) = 0$. Therefore we get from (3.5) and (3.6) that

$$g_1^2 = E_2(g_1) = g_1^2 + E_2(h_1)k_1,$$

which shows that h_1 is a constant c, that is, $g_1 = ck_1$. Thus we obtain from (3.9) that

$$V_1 = \frac{-E_2 + cE_3}{\sqrt{1 + c^2}}. \tag{3.11}$$

Since $g_2 = k_2 = 0$, (3.3) and (3.10) show that V_1 is a constant vector. Hence every E_1 curve lies in a plane $V_1^{\perp}$.

We now prove Theorem 2 in the following procedures.

Step 1. Let $\ell(p)$ be the maximal asymptotic line segment through a point $p \in W$. Then we have $\ell(p) \subset W$.

Proof. We parametrize $\ell(p)$ by $p + tE_2(p)$. Since $g_1 = ck_1$, it follows from (3.5) that $\frac{dk_1}{dt} = ck_1^2$. Hence we have $k_1(t) = \frac{1}{c-dt}$, which cannot vanish along $\ell(p)$. This completes the proof. □

For a point p in the boundary $bd(W)$ of the set W, we prove the following.

Step 2. Let $p \in bd(W) \subset M$. Then through p there passes a unique open segment of straight line $\ell(p) \subset M$. Furthermore, $\ell(p) \subset bd(W)$, that is, $bd(W)$ consists of open segments of asymptotic lines.

Proof. Let $p \in bd(W)$. On a neighborhood O around p, let $\{E_1, E_2\}$ be a principal orthonormal frame on O with principal curvatures k_1, k_2, respectively, which appears in the condition (C). On $O \cap W$ the Gaussian curvature k_1k_2 vanishes everywhere, but k_1 and k_2 does not vanish simultaneously. Since p is a limit point of W, it is possible to choose a sequence $\{p_n\}$ in $O \cap W$ which converges to p as $n \to \infty$.

Without loss of generality, we may assume that there exists such a sequence $\{p_n\}$ as above with $k_1(p_n) \neq 0, n = 1, 2, \cdots$. Then in a neighborhood of p_n, k_2 vanishes identically. Put $\phi : (-\delta_1, \delta_1) \times U \to O$ be the unique trajectory of E_2 with $\phi(0, q) = q$ in a neighborhood U of p. Then $\phi(t, p_n)$ is nothing but a parametrization of the asymptotic line segment $\ell(p_n)$ through p_n. This shows that $\nabla_{E_2}E_2(\phi(t, p_n)) = 0$ for each $n = 1, 2, \cdots$ and $|t| < \delta_1$. By letting $n \to \infty$, we see that $\nabla_{E_2}E_2(\phi(t, p)) = 0$ for all t with $|t| < \delta_1$. Thus $\phi(t, p)$ is an asymptotic line segment through p in the direction of E_2.

Suppose that there exists another sequence $\{q_n\}$ in $O \cap W$ with $k_2(q_n) \neq 0, n = 1, 2, \cdots$, which converges to p as $n \to \infty$. Then, as before, we see that the unique trajectory $\psi(t, q_n)$ of E_1, $|t| < \delta_2$, converges to a line segment $\psi(t, p)$ through p. For sufficiently large n, the line segment $\phi(t, p_n)$ through p_n should meet the line segment $\psi(t, p)$ at a point q in O. This is a contradiction, because Step 1 shows that $\phi(t, p_n)$ and $\psi(t, p)$ belong to the sets W and P, respectively. This contradiction shows that for a sufficiently small neighborhood O of p, k_1 does not vanish on $O \cap W$ and the integral curve $\phi(t, p)$ of E_2 is the unique asymptotic line segment through p, which we will denote by $\ell(p)$.

Next, we assert that every point of $\ell(p)$ on M is a boundary point of W. In fact, if $q \in \ell(p)$, there exists a sequence $q_n = \phi(t, p_n)$ in W with $p_n \to p$, and hence $q_n \to q$ as $n \to \infty$. Thus q belongs to the closure of W. Assume that q does not belong to $bd(W)$. Then $q \in W$. Since $\ell(p)$ is the unique asymptotic line segment through $q \in W$, we get $p \in W$, which is a contradiction. □

Note that each connected component of $int(P)$ is an open part of a plane.

Now we give a proof of Theorem 2. It suffices to show that the theorem holds in a neighborhood of a point $p \in bd(W)$. Let p be a point in the boundary of W, and $\{E_1, E_2\}$ an orthonormal frame in a neighborhood of p as in the proof of Step 2. Without loss of generality, we may assume that the line segment $\ell(p)$ is in the direction of E_2. Then the proof of Step 2 shows that there exists a neighborhood O of p such that $\nabla_{E_2} E_2 = 0$ and k_1 does not vanish on $O \cap W$. It follows from the condition (C) that for the constant vector V_1 in (3.11), every E_1 curve on $O \cap int(P)$ parametrizes an open segment of the straight line $V_1^{\perp} \cap int(P)$ which is orthogonal to $\ell(p)$. Every E_2 curve on $O \cap int(P)$ is also an open segment of a straight line which is parallel to $\ell(p)$.

Let $X(s)$ denote an E_1 curve through p which lies in the plane $V_1^{\perp}$ and $N(s) = V_1 \times E_1(s)$ the principal normal. It follows from Theorem of Joachimstahl that $\langle E_3, V_1 \rangle$ is constant along $X(s)$, hence we have for a constant θ, $E_2(s) = \cos\theta N(s) + \sin\theta V_1$. Hence O is an open part of the following slant cylinder:

$$F(s,t) = X(s) + tE_2(s).$$

This completes the proof of Theorem 2. □

Example 1 in ([4], p.409) describes a flat surface which satisfies the condition (C). It is locally(but not globally) an open part of a slant cylinder.

Now, suppose that a non-flat surface M satisfies the condition (C). Then by reversing the unit vector E_1(hence $E_3 = E_1 \times E_2$ is also reversed) if necessary, we may assume that $k_1 > 0, k_2 \neq 0$. It follows from (3.7) and (3.8) that

$$g_i = h_i k_i, E_i(h_i) = 0, i = 1, 2. \tag{3.12}$$

We prove the following :

Theorem 3 Suppose that a non-flat surface M satisfies the condition (C). Then every E_2 curve is a geodesic(that is, $g_2 = 0$) if and only if it is a generalized slant cylinder over an E_1 curve. In either case, we have

$$E_2(h_1) = (1 + h_1^2)k_2. \tag{3.13}$$

Proof. Suppose that g_2 vanishes identically on M. Then from (3.3) we get

$$\nabla_{E_2} E_1 = 0, \nabla_{E_2} E_2 = k_2 E_3, \nabla_{E_2} E_3 = -k_2 E_2, \tag{3.14}$$

Furthermore, (3.13) follows from (3.5), (3.6) and (3.12). Since M is non-flat, it follows from (3.9) that

$$V_1 = \frac{-E_2 + h_1 E_3}{\sqrt{1+h_1^2}}, V_2 = E_1, \tag{3.15}$$

which shows that V_1, V_2 are orthogonal to each other. By differentiating V_1 in (3.15) with respect to E_2, (3.14) shows that

$$\begin{aligned}(1+h_1)^{3/2}\nabla_{E_2}V_1 =& g_2(1+h_1^2)E_1 + h_1\{E_2(h_1) - (1+h_1^2)k_2\}E_2 \\ &+ \{E_2(h_1) - (1+h_1^2)k_2\}E_3.\end{aligned} \tag{3.16}$$

Together with (3.10), (3.13) and (3.16) show that V_1 is a constant vector.

We denote by X(s) an E_1 curve. Then $X(s)$ lies on a plane $V_1^\perp$ perpendicular to V_1 and $N(s) = V_1 \times E_1(s)$ is the principal normal to $X(s)$. Note that for each s, the E_2 curve through $X(s)$ lies in the plane $V_2^\perp$. Since $V_2^\perp$ is orthogonal to $V_2(s) = E_1(s)$, it is spanned by $\{N(s), V_1\}$. Thus we see that

$$H(s,t) = X(s) + z(s,t)N(s) + w(s,t)V_1 \tag{3.17}$$

is a parametrization of the surface M, where $z(s,t)$ and $w(s,t)$ are some functions which satisfy

$$z(s,0) = w(s,0) = 0, z_t^2 + w_t^2 = 1. \tag{3.18}$$

Now we show that $z(s,t), w(s,t)$ can be chosen so that they depend only on t. For this purpose, first of all we assert that for any (s_0,t_0), $w_t(s_0,t_0) \neq 0$. Otherwise, differentiating the last equation in (3.18) with respect to t, we have $z_{tt}(s_0,t_0) = 0$. Hence we get at (s_0,t_0)

$$k_2E_3 = \nabla_{E_2}E_2 = H_{tt} = w_{tt}V_1, \tag{3.19}$$

where the first equality follows from (3.14). Since M is non-flat, $k_2(s_0,t_0)$ $\neq 0$. Thus (3.19) shows that

$$V_1 = \pm E_3(s_0,t_0),$$

which contradicts to (3.15). This contradiction implies that $w_t(s_0,t_0) \neq$ 0.

Note that the E_1 curve through $H(s_0,t_0)$ is contained in the plane $V_1^\perp$ through $H(s_0,t_0)$. Hence it follows from (3.17) that the E_1 curve is contained in the set $\{H(s,t)|w(s,t) = w(s_0,t_0)\}$. Since $w_t(s_0,t_0) \neq 0$, we see that

$$X_{t_0}(s) = H(s, f(s)), \tag{3.20}$$

is a reparametrization of the E_1 curve through $H(s_0, t_0)$, where $f(s)$ satisfies

$$f(s_0) = t_0, w(s, f(s)) = w(s_0, t_0). \tag{3.21}$$

By differentiating (3.20) with respect to s, (3.17) and (3.21) show that

$$X'_{t_0}(s) = \{1 - \kappa(s)z(s, f(s))\}E_1(s) + \{\frac{d}{ds}z(s, f(s))\}N(s). \tag{3.22}$$

On the other hand, it follows from (3.20) that $X'_{t_0}(s)$ is proportional to $E_1(s, f(s))$. Furthermore, the first equation in (3.14) shows that E_1 is parallel along $t-$curve of H so that we have $E_1(s, f(s)) = E_1(s, 0) = E_1(s)$. Hence it follows from (3.21) and (3.22) that

$$z(s, f(s)) = z(s_0, t_0). \tag{3.23}$$

Thus we have

$$\begin{aligned} X_{t_0}(s) &= X(s) + z(s, f(s))N(s) + w(s, f(s))V_1 \\ &= X(s) + z(s_0, t_0)N(s) + w(s_0, t_0)V_1, \end{aligned} \tag{3.24}$$

where the second equality follows from (3.21) and (3.23). Since t_0 is arbitrary, if we let $z(t) = z(s_0, t), w(t) = w(s_0, t)$, then (3.24) implies that

$$H(s, t) = X(s) + z(t)N(s) + w(t)V_1$$

is a reparametrization of M. This shows that M is a generalized slant cylinder over an E_1 curve $X(s)$.

Finally, suppose that M is a generalized slant cylinder over an E_1 curve $X(s)$ of which parametrization $H(s, t)$ is given in (2.3). Then every E_2 curve is a $t-$curve of H. Since H_{tt} is orthogonal to H_t and H_s, every t curve of H is a geodesic of M, that is, g_2 vanishes identically. Together with (3.16), constancy of $V = V_1$ shows that (3.13) holds. This completes the proof. □

There exist surfaces in the Euclidean space $\mathbb{E}^3$ which satisfy the condition (C), but not an open part of a generalized slant cylinder. For example, the Enneper's minimal surface and the family of associated Bonnet surfaces are cases of these kinds([1], [3], [8]).

4. Linear Weingarten surfaces with planar lines of curvature

Suppose that a non-flat and non-minimal linear Weingarten surface M in the Euclidean space $\mathbb{E}^3$ satisfies the condition (C). Hence we have $k_2 = ak_1+b, k_1 \neq 0, k_2 \neq 0$, where a, b are constant with $(a+1)^2+b^2 \neq 0$ and $a^2 + b^2 \neq 0$. Furthermore we assume that M has no umbilic points, that is, $k_1 \neq k_2$. By reversing the unit vector E_1(hence $E_3 = E_1 \times E_2$ is also reversed) if necessary, we assume that $k_1 > 0$.

From (3.4), (3.5) and (3.6) we obtain

$$aE_1(k_1) = \{(1-a)k_1 - b\}g_2, \tag{4.1}$$

$$E_2(k_1) = \{(1-a)k_1 - b\}g_1, \tag{4.2}$$

$$k_1(ak_1 + b) = E_2(g_1) - E_1(g_2) - g_1^2 - g_2^2. \tag{4.3}$$

By differentiating (4.1) and (4.2) with respect to E_2, E_1, respectively, we obtain

$$aE_2E_1(k_1) = (1-a)\{(1-a)k_1 - b\}g_1g_2 + \{(1-a)k_1 - b\}E_2(g_2), \tag{4.4}$$

$$aE_1E_2(k_1) = (1-a)\{(1-a)k_1 - b\}g_1g_2 + a\{(1-a)k_1 - b\}E_1(g_1). \tag{4.5}$$

On the other hand, from (3.2), (3.3), (4.1) and (4.2) we have

$$\begin{aligned} a\{E_2E_1(k_1) - E_1E_2(k_1)\} =& a\{\nabla_{E_2}E_1(k_1) - \nabla_{E_1}E_2(k_1)\} \\ =& a\{g_2E_2(k_1) + g_1E_1(k_1)\} \\ =& (a+1)\{(1-a)k_1 - b\}g_1g_2. \end{aligned}$$

Hence (4.4) and (4.5) show that

$$E_2(g_2) - aE_1(g_1) = (a+1)g_1g_2. \tag{4.6}$$

1) First, we consider the case $a \neq 0$. It follows from (3.12) that $g_1 = h_1k_1, g_2 = h_2(ak_1+b)$ for some functions satisfying $E_1(h_1) = E_2(h_2) = 0$. Substituting these into (4.6), we get

$$h_1h_2\{a(a+1)k_1^2 + 2bk_1 - b^2\} = 0. \tag{4.7}$$

Suppose that $h_1h_2 \neq 0$ on an open set W. Then (4.7) shows that k_1 is a root of a nontrivial polynomial of degree 1 or 2. Hence k_1(and hence k_2)is constant. This shows that W is an open part of either a circular cylinder(flat) or a sphere(umbilic)([C]), which contradicts to the hypotheses. Thus h_1h_2(hence g_1g_2) vanishes identically on M.

Since M is non-flat, (4.3) shows that g_1, g_2 cannot vanish simultaneously on an open set. Hence we may assume that $W_1 = \{p \in M | g_1(p) \neq 0\}$ is nonempty. Since g_2 vanishes identically on W_1, Theorem 3 shows that W_1 is a generalized slant cylinder over an E_1 curve $X(s)$. It follows from (4.1) and (4.6) that

$$E_1(k_1) = E_1(g_1) = 0. \tag{4.8}$$

Since $X''(s) = \nabla_{E_1} E_1 = g_1 E_2 + k_1 E_3$, (4.8) shows that the plane curve $X(s)$ has nonzero constant curvature $\sqrt{k_1^2 + g_1^2}$. Hence $X(s)$ is a circle. It follows from Proposition 1 that W_1 is a surface of revolution and each parallel(that is, E_1 curve) on W_1 lies on a plane $V_1^{\perp}$, where V_1 is given by

$$V_1 = \frac{-E_2 + h_1 E_3}{\sqrt{1 + h_1^2}}. \tag{4.9}$$

It follows from (4.8) that g_1 is constant on each parallel. Hence the closure $\overline{W}_1$ of $W_1 \subset M$ has boundary $bd(\overline{W}_1)$(if any) consisting of open segments of parallels which lie on some planes $V_1^{\perp}$.

Now suppose that $W_2 = \{p \in M | g_2(p) \neq 0\}$ is nonempty. Then, as before, it follows from Proposition 1 and Theorem 3 that W_2 is a surface of revolution and each parallel(that is, E_2 curve) on W_2 lies on a plane $V_2^{\perp}$, where V_2 is given by

$$V_2 = \frac{-E_1 + h_2 E_3}{\sqrt{1 + h_2^2}}. \tag{4.10}$$

For a point $p \in bd(\overline{W}_2)$, the parallel $C(p)$ through p on $\overline{W}_2$ is also a parallel on $\overline{W}_1$. This implies that $C(p)$ lies on both $V_1^{\perp}$ and $V_2^{\perp}$, which shows that V_1 is parallel to V_2. But from (4.9) and (4.10) we see that V_1 cannot be parallel to V_2. This contradiction shows that W_2 is empty, and hence M is a surface of revolution.

2) Finally, we consider the case $a = 0$. Then we have $k_2 = b(\neq 0)$. Hence, (3.4) shows that g_2 vanishes identically. It follows from (3.3) that every E_2 curve $Y(t)$ is a circle of radius $1/|b|$. Thus Theorem 3 shows that M is a tube along an E_1 curve $X(s)$.

5. Weingarten surfaces with planar lines of curvature

Suppose that a non-flat surface M satisfying the condition (C) also satisfies the Weingarten condition:

$$\text{(W)} \qquad k_2 = f(k_1),$$

for some polynomial function $f(x)$ of degree $n(\geq 2)$ in x. Furthermore we assume that M has no umbilic points. As in Section 4, we may assume that $k = k_1 > 0$. From (3.4), (3.5) and (3.6) we obtain

$$f'(k)E_1(k) = \{k - f(k)\}g_2, \tag{5.1}$$

$$E_2(k) = \{k - f(k)\}g_1, \tag{5.2}$$

$$kf(k) = E_2(g_1) - E_1(g_2) - g_1^2 - g_2^2. \tag{5.3}$$

By differentiating (5.1) and (5.2) with respect to E_2, E_1, respectively, we obtain

$$\begin{aligned} &f'(k)\{E_2E_1(k) - E_1E_2(k)\} \\ =&\{k - f(k)\}\{-f''(k)E_1(k)g_1 + E_2(g_2) - f'(k)E_1(g_1)\}. \end{aligned} \tag{5.4}$$

On the other hand, from (3.2), (3.3), (5.1) and (5.2) we have

$$\begin{aligned} f'(k)\{E_2E_1(k) - E_1E_2(k)\} =& f'(k)\{\nabla_{E_2}E_1(k) - \nabla_{E_1}E_2(k)\} \\ =& f'(k)\{g_2E_2(k) + g_1E_1(k)\} \\ =& \{f'(k) + 1\}\{k - f(k)\}g_1g_2. \end{aligned} \tag{5.5}$$

Hence (5.4) and (5.5) show that

$$E_2(g_2) - f'(k)E_1(g_1) = f''(k)E_1(k)g_1 + \{f'(k) + 1\}g_1g_2. \tag{5.6}$$

It follows from (3.12) that $g_1 = h_1k, g_2 = h_2f(k)$ for some functions h_1 and h_2 satisfying $E_1(h_1) = E_2(h_2) = 0$. Substituting these into (5.6), we get

$$h_1h_2[\{k - f(k)\}\{f''(k)kf(k) - f'(k)^2k + f(k)\} + \{f'(k) + 1\}kf(k)] = 0. \tag{5.7}$$

Suppose that $h_1h_2 \neq 0$ on an open set W. Then (5.7) shows that $k = k_1$ is a root of some nontrivial polynomial of degree $3n - 1$. Hence $k = k_1$(and hence k_2)is constant there. Thus W is an open part of either a circular cylinder(flat) or a sphere(umbilic)([2]). This contradiction shows that h_1h_2(and hence g_1g_2) vanishes identically on M. Hence we can proceed as in Section 4 to conclude that M is a surface of revolution.

Summarizing the results in Section 4 and 5, we establish the following.

Theorem 4 Let M be a non-flat and non-minimal surface without umbilic points which satisfies the condition (C). Suppose that M is a Weingarten surface with

$$k_2 = f(k_1), \tag{W}$$

where f is a polynomial of degree $n(\geq 1)$. Then M is a surface of revolution.

It is well-known that every surface of revolution is a Weingarten surface ([7], pp. 91-92). According to H. Hopf([5]), surfaces of revolution satisfying $k_2 = ak_1 (a \in R)$ are classified in ([7], pp. 92-93).

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