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PEBBLING EXPONENTS OF PATHS

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Abstract. A *pebbling move* on a connected graph G is taking two pebbles off of one vertex and placing one of them on an adjacent vertex. For a connected graph G, G^p $(p > 1)$ is the graph obtained from G by adding the edges (u, v) to G whenever $2 \le dist(u, v) \le p$ in G. And the *pebbling exponent* of a graph G to be the least power of p such that the pebbling number of G^p is equal to the number of vertices of G. We compute the pebbling number of fourth power of paths so that the pebbling exponents of some paths are calculated.

I. Introduction

Pebbling in graphs was first considered by Chung[1]. Consider a connected graph with a fixed number of pebbles distributed on its vertices. A *pebbling move* consists of removing two pebbles from one vertex u and then placing one pebble at an adjacent vertex v. We say that we can pebble to a vertex v, the target vertex, if we can apply pebbling moves repeatedly so that it is possible to reach a configuration with at least one pebble at v. The *pebbling number of a vertex* v for a graph G, denoted by $f(G, v)$, is the smallest integer m which guarantees that any starting pebble configuration with m pebbles allows pebbling to v. And the *pebbling number* of G, denoted by $f(G)$, as the maximum of $f(G, v)$, over all vertices v.

A graph G is called *demonic* if $f(G)$ is equal to the number of its vertices. If one pebble is placed on each vertex other than the vertex v, then no pebble can be moved to v. Also, if w is at distance d from v, and $2^d - 1$ pebbles are placed on w, then no pebble can be moved to v. So it is clear [1] that $f(G) \ge max\{|V(G)|, 2^D\}$, where $|V(G)|$ is the number of the vertices of G and D is the diameter of G. Let $G = (V(G), E(G))$ be a connected graph. Then G^p $(p > 1)$ (the pth power of G) is the graph obtained from G by adding the edges (u, v) to G whenever $2 \le dist(u, v) \le p$ in G. Hence $G^p = (V(G), E(G) \cup F(G))$

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where $F(G) = \{(u,v) : 2 \leq dist(u,v) \leq p \text{ in } G\}$. If $p = 1$, $G^1 = G$. In [5], they proved $f({P_{2k+r}}^2) = 2k + r$ where $o \leq r \leq 1$. Let $|V(G)| = n$. We know that if p is large enough (i.e., $p \geq n-1$) then $G^p = K_n$.

In [6], the *pebbling exponent* of a graph G (denoted by $e(G)$) is defined to be the least power of p such that $f(G^p) = |V(G)|$.

In section 2, we calculate the pebbling number of the fourth power of paths. Using the results in section 2, we give the pebbling exponent of some paths in section 3.

2. The pebbling number of fourth power of P_n

In [6], $f({P_n}^2)$ was calculated but the proof is too long. So that result was reproved by more simple method in [7]. It is known that the pebbling number $f(P_n)$ of the path P_n with n vertices is 2^{n-1} [1]. $p(v)$ will denote the number of pebbles at vertex v.

In [7], the following two lemmas were used to calculate $f({P_n}^2)$ and $f({P_n}^3)$.

Lemma 1. **[7]** Let ${P_n}^2 = x_1x_2\cdots x_{n-1}x_n$ with $n \geq 7$. If $p(x)$ is even for each vertex x of the graph ${P_n}^2$, then $2^{\lceil \frac{n-1}{2} \rceil}$ pebbles are sufficient to pebble x_1 or x_n.

Lemma 2. **[7]** Let ${P_n}^3 = x_1\cdots x_n$ with $n \geq 8$. If $p(x)$ is even for each vertex x of ${P_n}^3$, then $2^{\lceil \frac{n-1}{3} \rceil}$ pebbles are sufficient to pebble x_1 or x_n.

Similarly we can get the following lemma 3.

Lemma 3. Let ${P_n}^k = x_1\cdots x_n$ with $n \geq 2k \geq 8$. If $p(x_i)$ is even for each vertex x_i of ${P_n}^k$, then $2^{\lceil \frac{n-1}{k} \rceil}$ pebbles are sufficient to pebble x_1 or x_n.

Proof. Let ${P_n}^k = x_1x_2\cdots x_n$ with $8 \leq 2k \leq n$, $p(x_i)$ be even for x_i. Place $2^{\lceil \frac{n-1}{k} \rceil}$ pebbles on ${P_n}^k$. By symmetry, we assume that $v = x_1$. Let $p(x_1) = 0$. If $\lceil \frac{n-1}{k} \rceil = t$, we can write ${P_n}^k$ as ${P_n}^k = x_{0k+1}x_{0k+2}\cdots x_{1k}x_{1k+1}\cdots x_{1k+k}x_{2k+1}\cdots x_{(t-1)k}x_{(t-1)k+1}\cdots x_n$ where $(t-1)k+1 < n \leq tk+1$. By moving as many pebbles as possible from each x_{jk+r} $(2 \leq r \leq k+1)$ to x_{jk+1} for $j = 0, 1, \cdots, (t-1)$, there can be at least 2^{t-1} pebbles at the vertices $x_{0k+1}x_{1k}\cdots x_{(t-1)k+1}$. Since $x_{0k+1}x_{1k+1}\cdots x_{(t-1)k+1}$ is isomorphic to P_t and $f(P_t) = 2^{t-1}$, we can put a pebble at x_1.

Theorem 4. **[6]** $f({P_{2k+r}}^2) = 2^k + r$ when $0 \leq r \leq 1$.

Theorem 5. [**7**] $f({P_n}^3) = n$ if $1 \le n \le 7$. For $n \ge 8$,

$$f({P_n}^3) = \begin{cases} 2^{\lfloor \frac{n}{3} \rfloor} + 1 & \text{if} \quad n \equiv 0 \pmod 3 \\ 2^{\lfloor \frac{n}{3} \rfloor} + 2 & \text{if} \quad n \equiv 1 \pmod 3 \\ 2^{\lfloor \frac{n}{3} \rfloor + 1} & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}$$

Above two theorems can be restated as followings.

Theorem 4′. $f({P_n}^2) = 2^{\lceil \frac{n-1}{2} \rceil} + r \quad \text{when} \quad 0 \le r \le 1$

Theorem 5′. $f({P_n}^3) = 2^{\lceil \frac{n-1}{3} \rceil} + r \quad \text{when} \quad 0 \le r \le 2,$

$$n \ge 8 \quad \text{and} \quad n - 2 \equiv r \pmod 3$$

First, we note that $f({P_n}^4) = n$ for $1 \le n \le 13$. The reason for that is as follows. We proceed by induction on n. Clearly, the result is correct if $1 \le n \le 5$. Suppose that for all n' with $5 \le n' < n \le 13$ we have $f({P_{n'}}^4) = n'$. We will show that $f({P_n}^4) = n$. Place n pebbles at the vertices of ${P_n}^4 = x_1 x_2 \cdots x_n$ (the edges between x_i and x_{i+4} are implied for $(1 \le i \le 9)$ and assume first that $v \ne x_1$ or x_n. Let $v = x_i$ with $2 \le i \le n-1$. We see that if the subgraph $x_1 \cdots x_i \equiv P_i^4$ contains at least i pebbles and so we are done by induction. Otherwise the subgraph $x_i \cdots x_n \equiv {P_{n-i-1}}^4$ contains at least $(n-i-1)$ pebbles and so we are done by induction. Therefore, we may assume that $v = x_1$ or x_n. By symmetry, we may assume that $v = x_1$. Suppose that $\Sigma_{i=2}^{5} p(x_i) \ge 1$. Let $p(x_i) \ge 1$ for some $2 \le i \le 5$. Then we can put one more pebble at x_i by using the remaining $(n-1)$ pebbles on the subgraph $x_2 \cdots x_n \equiv {P_{n-1}}^4$ by induction. Since $dist(x_1, x_i) = 1$, we are done by moving a pebble at x_1 from x_i. Otherwise $\Sigma_{i=2}^{5} p(x_i) = 0$. Then $\Sigma_{i=6}^{n} p(x_i) = n$. For $6 \le n \le 9$ there are at least two pairs on the subgraph $x_6 \cdots x_n$. Using these two pairs on that subgraph we can put two pebbles at x_5 and so we are done because $dist(x_1, x_5) = 1$. For $10 \le n \le 13$ the subgraph $x_6 \cdots x_n$ contains at least 4 pairs which are used to put two pebbles at x_5. And we are done.

Next, we show that $f({P_{14}}^4) = 2^4$, $f({P_{15}}^4) = 2^4 + 1$, $f({P_{16}}^4) = 2^4 + 2$ and $f({P_{17}}^4) = 2^4 + 3$.

(a) We show that $f({P_{14}}^4) = 2^4$.

First, we will show that $f({P_{14}}^4) \ge 2^4$. Let ${P_{14}}^4 = x_1 x_2 \cdots x_{14}$ (the edges between x_i and x_{i+4} are implied for $1 \le i \le 10$) and place $(2^4 - 1)$ pebbles at x_{14}. Then no pebble can be moved to x_1 because $dist(x_1, x_{14}) = 4$, therefore $f({P_{14}}^4) \ge 2^4$. Place 2^4 pebbles at the vertices of ${P_{14}}^4 = x_1x_2\cdots x_{14}$. By symmetry, we may assume that our target vertex is $v = x_1, x_2, \cdots, x_8$, or x_9. If $\Sigma_{i=1}^{9}p(x_i) \geq 9$, then we are done because the subgraph $x_1\cdots x_9$ is isomorphic to ${P_9}^4$ and $f({P_9}^4) = 9$. If $\Sigma_{i=1}^{9}p(x_i) = 8$, then $\Sigma_{j=10}^{14}p(x_j) = 8$. By moving as many pebbles as possible from $x_{10}, x_{11}, \cdots, x_{13}$ or x_{14} to x_9, we see that the subgraph $x_1\cdots x_9 \equiv {P_9}^4$ contains at least 9 pebbles. So we are done. If $\Sigma_{i=1}^{9}p(x_i) \leq 7$, then $\Sigma_{j=10}^{14}p(x_j) \geq 9$ and there are at least 4 pairs on the subgraph $x_9\cdots x_{14} \equiv {P_6}^4$. By Lemma 3, We can put 4 pebbles at x_9 by moving 4 pairs on the subgraph $x_9\cdots x_{14} \equiv {P_6}^4$ to x_9. So we are done because $dist(x_k, x_9) \leq 2$ for $1 \leq k \leq 9$.

(b) We show that $f({P_{15}}^4) = 2^4 + 1$.

First, we will show that $f({P_{15}}^4) \geq 2^4+1$. Let ${P_{15}}^4 = x_1x_2\cdots x_{15}$ (the edges between x_i and x_{i+4} are implied for $1 \leq i \leq 11$) and place $(2^4 - 1)$ pebbles at x_{15} and one pebble at x_{14}. Then no pebble can be moved to x_1. Place $(2^4 + 1)$ pebbles at the vertices of ${P_{15}}^4 = x_1x_2\cdots x_{15}$. By symmetry, we may assume that our target vertex is $v = x_1, x_2, \cdots, x_8$, or x_9. If $\Sigma_{i=1}^{9}p(x_i) \geq 9$, then we are done because the subgraph $x_1\cdots x_9$ is isomorphic to ${P_9}^4$ and $f({P_9}^4) = 9$. If $\Sigma_{i=1}^{9}p(x_i) = 8$, then $\Sigma_{j=1}^{15}p(x_j) = 9$. By moving as many pebbles as possible from $x_{10}, \cdots, x_{14}$ or x_{15} to x_9, we see that the subgraph $x_1\cdots x_9 \equiv {P_9}^4$ contains at least 9 pebbles and we are done. If $\Sigma_{i=1}^{9}p(x_i) \leq 7$, then $\Sigma_{j=10}^{15}p(x_j) \geq 10$ and there are at least 4 pairs on the subgraph $x_9\cdots x_{15} \equiv {P_7}^4$. By Lemma 3, we can put 4 pebbles at x_9 by moving 4 pairs on the subgraph $x_9\cdots x_{15} \equiv {P_7}^4$. So we are done because $dist(x_k, x_9) \leq 2$ for $1 \leq k \leq 9$.

(c) The proof of $f({P_{16}}^4) = 2^4 + 2$ or $f({P_{17}}^4) = 2^4 + 3$ are similar to those of $f({P_{14}}^4) = 2^4$ or $f({P_{15}}^4) = 2^4 + 1$.

Theorem 6. For $n \geq 14$, $f({P_n}^4) = 2^{\lceil \frac{n-1}{4} \rceil} + r$ where $0 \leq r \leq 3$ and $n - 2 \equiv r \pmod 4$.

Proof. First, we will show that $f({P_n}^4) \geq 2^{\lceil \frac{n-1}{4} \rceil} + 3$ for $n - 2 \equiv 3 \pmod 4$. Let ${P_n}^4 = x_1x_2\cdots x_n$ (the edges between x_i and x_{i+4} are implied for $1 \leq i \leq n-4$) and place $(2^{\lceil \frac{n-1}{4} \rceil} - 1)$ pebbles at x_n and one pebble at each x_j with $j = n-3, n-2$, and $n-1$. It is easy to see that a pebble can not be moved to x_1, therefore $f({P_n}^4) \geq 2^{\lceil \frac{n-1}{4} \rceil} + 3$.

Since the diameter of ${P_n}^4$ is $\lceil \frac{n-1}{4} \rceil$ for $n - 2 \equiv 0 \pmod 4$, we have $f({P_n}^4) \geq 2^{\lceil \frac{n-1}{4} \rceil}$.

Similarly, $f({P_n}^4) \geq 2^{\lceil \frac{n-1}{4} \rceil} + r$ for $n - 2 \equiv r \pmod 4$, $r = 1$ or 2.

We proceed by induction on n. We have already showed that our theorem is correct if n=14, 15, 16, or 17.

Case (a) $r = 0$.

Suppose that for all n' with $n' < n$ and $n' - 2 \equiv 0 \pmod 4$ we have $f({P_{n'}}^4) = 2^{\lceil \frac{n'-1}{4} \rceil}$. We will show that $f({P_n}^4) = 2^{\lceil \frac{n-1}{4} \rceil}$. Let $n = 4l + 2$ for some $l \geq 4$. Then $\lceil \frac{n-1}{4} \rceil = l + 1$. Place $2^{\lceil \frac{n-1}{4} \rceil} (= 2^{l+1})$ pebbles at the vertices of ${P_n}^4 = x_1 x_2 \cdots x_n$. Let v be the target vertex. Then there are the following three possible cases (a.1), (a.2) and (a.3).

(a.1) $v \neq x_1, \cdots, x_4, x_{n-3}, \cdots, x_{n-1}$ or x_n.

Let P_A be the subgraph $x_5 \cdots x_n$ and P_B be the subgraph $x_1 \cdots x_{n-4}$. Then both P_A and P_B are isomorphic to ${P_{n-4}}^4$. It is easy to see that P_A or P_B contains at least $2^{\lceil \frac{(n-4)-1}{4} \rceil}$ pebbles. By induction we are done.

(a.2) $v = x_1$ or x_n.

By symmetry we assume that $v = x_1$. Suppose that $\Sigma_{i=2}^{5} p(x_i) = 0$. Then there are 2^{l+1} pebbles on the subgraph $x_5 \cdots x_n$ which is isomorphic to ${P_{n-4}}^4$. By induction we can put two pebbles at x_5 using $2 \cdot 2^l$ pebbles on the subgraph $x_5 \cdots x_n$. Because $dist(x_1, x_4) = 1$, we are done. Otherwise $\Sigma_{i=2}^{4} p(x_i) \geq 1$. Let $j = \min_{2 \leq i \leq 5} \{i \mid p(x_i) \geq 1\}$. For $p(x_j) \geq 2$, we are done. Let $p(x_j) = 1$. Then there are 2^{l-1} pairs on the subgraph $x_j \cdots x_n$ because $\lceil \frac{(2^{l+1} - (n-j+1)}{2} \rceil \geq 2^l = 2 \cdot 2^{l-1}$. By using these 2^{l-1} pairs on that subgraph we can put one more pebble at x_j and we are done.

(a.3) $v = x_2, x_3, x_4, x_{n-3}, x_{n-2}$, or x_{n-1}.

By symmetry we assume that $v = x_k$ for $2 \leq k \leq 4$. If $p(x_1) \geq 2$, then we can put a pebble at x_k because $dist(x_1, x_k) = 1$. Otherwise $p(x_1) \leq 1$ and then $\Sigma_{j=2}^{n} p(x_j) \geq 2^{l+1} - 1$. Since $(2^{l+1} - 1) - (n-1) = (2^{l+1} - 1) - (4l + 1) = 2^{l+1} - 4l - 2 \geq 2^{l-1}$ for $l \geq 4$, there are 2^{l-2} pairs on the subgraph $x_2 \cdots x_n \equiv {P_{n-1}}^4$. By Lemma 3, we can put a pebble at x_k by using 2^{l-2} pairs on the subgraph $x_2 \cdots x_n$ with $2^{\lceil \frac{(n-1)-1}{4} \rceil} = l$.

Case (b) r=1.

Suppose that for all n' with $n' < n$ and $n' - 2 \equiv 1 \pmod 4$ we have $f({P_{n'}}^4) = 2^{\lceil \frac{n'-1}{4} \rceil} + 1$. We will show that $f({P_n}^4) = 2^{\lceil \frac{n-1}{4} \rceil} + 1$. Let $n = 4l{+}3$ for some $l \geq 4$. Then $\lceil \frac{n-1}{4} \rceil = l{+}1$. Place $2^{\lceil \frac{n-1}{4} \rceil}{+}1 (= 2^{l+1}{+}1)$ pebbles at the vertices of ${P_n}^4 = x_1 x_2 \cdots x_n$. Let v be the target vertex. Then there are the following three possible cases (b.1), (b.2) and (b.3).

(b.1) $v \neq x_1, \cdots, x_4, x_{n-3}, \cdots, x_{n-1}$ or x_n.

This case can be proved by the same way to (a.1).

(b.2) $v = x_1$ or x_n.

By symmetry we assume that $v = x_1$. Suppose that $\Sigma_{i=2}^4 p(x_i) = 0$. Then there are $(2^{l+1}+1)$ pebbles on the subgraph $x_5 \cdots x_n$ which is isomorphic to ${P_{n-4}}^4$. Since $(2^{l+1}+1)-(n-4) = (2^{l+1}+1)-(4l-1) \geq 2^l$ for $l \geq 4$, there are 2^{l-1} pairs on the subgraph $x_5 \cdots x_n$ and so we can put a pebble at x_5 using these 2^{l-1} pairs on that subgraph with $\lceil \frac{(n-4)-1}{4} \rceil = l$ by Lemma 3. And we put one more pebble at x_5 using the remaining (2^l+1) pebbles on the subgraph $x_5 \cdots x_n \equiv {P_{n-4}}^4$ by induction. Since $dist(x_1, x_5) = 1$, we can put a pebble at x_1.

(b.3) $v = x_2, x_3, x_4, x_{n-2}, x_{n-1}$, or x_n.

By symmetry we assume that $v = x_k$ for $2 \leq k \leq 4$. If $p(x_1) \geq 2$, then we can put a pebble at x_k because of $dist(x_1, x_k) = 1$. Otherwise $p(x_1) \leq 1$. Then there are at least 2^{l+1} pebbles on the subgraph $x_2 \cdots x_n$ which is isomorphic to ${P_{n-1}}^4$. Then by Case (a) we can put a pebble at x_k.

Case (c) $r = 2$.

Suppose that for all n' with $n' < n$ and $n' - 2 \equiv 2 \pmod 4$ we have $f({P_{n'}}^4) = 2^{\lceil \frac{n'-1}{4} \rceil} + 2$. We will show that $f({P_n}^4) = 2^{\lceil \frac{n-1}{4} \rceil} + 2$. Let $n = 4l+4$ for some $l \geq 4$. Then $\lceil \frac{n-1}{4} \rceil = l+1$. Place $2^{\lceil \frac{n-1}{4} \rceil}+2 (= 2^{l+1}+2)$ pebbles at the vertices of ${P_n}^4 = x_1 x_2 \cdots x_n$. Let v be the target vertex. Then there are the following three possible cases (c.1), (c.2) and (c.3).

(c.1) $v \neq x_1, \cdots, x_4, x_{n-3}, \cdots, x_{n-1}$ or x_n.

This case can be proved by the same way to (a.1)

(c.2) $v = x_1$ or x_n.

By symmetry we assume that $v = x_1$. If $\Sigma_{i=2}^4 p(x_i) = 0$, then there are $(2^{l+1}+2)$ pebbles on the subgraph $x_5 \cdots x_n$ which is isomorphic to ${P_{n-4}}^4$. Since $(2^{l+1}+2) - (n-4) \geq 2^l$ for $l \geq 4$, there are 2^{l-1} pairs on the subgraph $x_5 \cdots x_n$ and so we can put a pebble at x_5 using these 2^{l-1} pairs on that subgraph with $\lceil \frac{(n-4)-1}{4} \rceil = l$ by Lemma 3. And we put one more pebble at x_5 using the remaining (2^l+2) pebbles on the subgraph $x_5 \cdots x_n \equiv {P_{n-4}}^4$ by induction. Since $dist(x_1, x_5) = 1$, we can put a pebble at x_1.

(c.3) $v = x_2, x_3, x_4, x_{n-3}, x_{n-2}$, or x_{n-1}.

By symmetry we assume that $v = x_k$ for $2 \leq k \leq 4$. If $\Sigma_{j=1}^3 p(x_j) \geq 4$, then we are done. Otherwise $\Sigma_{j=1}^3 p(x_j) \leq 3$ and $\Sigma_{j=5}^n p(x_j) \geq (2^{l+1} + 2) - 3 (= 2^{l+1} - 1)$. Since $\lceil \frac{(2^{l+1}-1)-(n-4)}{2} \rceil \geq 2^l$ for $l \geq 4$, there are 2^{l-1} pairs on the subgraph $x_5 \cdots x_n \equiv {P_{n-4}}^4$ with $\lceil \frac{(n-4)-1}{4} \rceil = l$ and so we can put a pebble at x_5 using these 2^{l-1} pairs on that subgraph by Lemma 3. And we can put one more pebble at x_5 using the remaining

$(2^l + 2)$ pebbles on the subgraph $x_5 \cdots x_n \equiv P_{n-4}{}^4$ by induction. Since $dist(x_k, x_5) = 1$ for $2 \leq k \leq 4$, we are done.

Case (d) $r = 3$.

The proof is similar to Case(c).

3. The exponents of Paths

Using Theorem 6 we can get the following Theorem 7.

Theorem 7.

$$e(P_n) = \begin{cases} 2 & \text{if} \quad n \leq 6 \\ 3 & \text{if} \quad 7 \leq n \leq 10 \\ 4 & \text{if} \quad 11 \leq n \leq 13 \\ 5 & \text{if} \quad 14 \leq n \leq 21 \end{cases}$$

For $2 \leq k \leq n$, consider the set

$$E = \{k \mid 2^{\lceil \frac{n-1}{k} \rceil} + r \leq n \quad \text{where} \quad 0 \leq r \leq k-1$$

$$\text{and} \quad n - 2 \equiv r \pmod{k}\}$$

Conjecture. $e(P_n) = minE$.

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