Honam Mathematical J. **32** (2010), No. 4, pp. 763–767

A H^∞-OPTIMIZATION PROBLEM

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Abstract. In this note, we give a solution of a H^∞-optimization problem.

Let $L^2(\mathbb{T})$ be the set of all square-integrable measurable functions on the unit circle $\mathbb{T} \equiv \partial\mathbb{D}$ in the complex plane and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Write $H^\infty(\mathbb{T}) := L^\infty(\mathbb{T}) \cap H^2(\mathbb{T})$. The H^∞-optimization problems naturally arise in several fields of mathematics - for instance, the robust control theory (cf. [FF]). In particular, the hyponormality of Toeplitz operators with bounded type symbols (i.e., quotients of two bounded analytic functions) has a deep connection with the following H^∞-optimization problem :

H^∞-optimization problem. Let $k_0 \in L^\infty(\mathbb{T})$ and θ a fixed inner function in $H^\infty(\mathbb{T})$. Find μ where

$$\mu = \mathrm{dist}\,(k_0,\, \theta H^\infty) \equiv \inf_{h \in H^\infty} ||k_0 - \theta h||_\infty .$$

If P denotes the orthogonal projection from L^2 to H^2, then for every bounded measurable function $\phi \in L^\infty$, the Toeplitz operator T_ϕ and the Hankel operator H_ϕ on H^2 are defined by

$$T_\phi f := P(\phi f) \quad \text{and} \quad H_\phi(f) = J(I-P)(\phi f) \ \ \text{for all } f \in H^2 ,$$

where $J : (H^2)^\perp \to H^2$ is given by $Jz^{-n} = z^{n-1}$ for $n \geq 1$. If $k_0 \in H^\infty$ and θ is an inner function then by Nehari's Theorem [Ne], we have

$$\mathrm{dist}\,(k_0,\ \theta H^\infty) = \inf_{f \in H^\infty} ||\bar{\theta} k_0 + f||_\infty = ||H_{\bar{\theta} k_0}||.$$

2000 *Mathematics Subject Classification.* Primary 47B20, 47B35

Received October 07, 2010. Accepted December 01, 2010.

Key words and phrases: H^∞-optimization problem, Blaschke product, Toeplitz operators, Hankel operators, hyponormal.

This research is financially supported by Changwon National University in 2010 .

It was also known that (see [GS, Theorem 8])

$$T_\phi \text{ is hyponormal } \iff ||H_{\bar{\theta}k_0}|| \le 1.$$

Recently, W.Y. Lee [Le, Lemma 4] has given a solution of a H^∞ -optimization problem: If b and q are finite Blaschke products of the form

$$e^{i\theta}\prod_{j=1}^{n}\frac{z-\alpha_j}{1-\overline{\alpha_j}z} \quad (\alpha_j \in \mathbb{D}),$$

then

$$\deg(b) \ge \deg(q) \iff \operatorname{dist}(b, qH^\infty) < 1.$$

Its proof relies heavily upon the spectral theory of Toeplitz operators. The purpose of this note is to give a direct proof of a slightly extended version without using the spectral theory of Toeplitz operators.

Our main theorem now follows:

Theorem 1. If b is a (possibly infinite) Blaschke product and q is a finite Blaschke product, then

$$\deg(b) \ge \deg(q) \iff \operatorname{dist}(b, qH^\infty) < 1.$$

Proof. We first claim that

$$\deg(b) \ge \deg(q) \implies \operatorname{dist}(b, qH^\infty) < 1. \tag{1}$$

Towards (1), we suppose $\deg(b) =: m \ge n := \deg(q)$ ($m \in \mathbb{N}\cup\{\infty\}$ and $n \in \mathbb{N}$). So we can write

$$b(z) = e^{i\theta}\prod_{j=1}^{m}\frac{z-\alpha_j}{1-\overline{\alpha_j}z} \quad \text{and} \quad q(z) = e^{i\omega}\prod_{j=1}^{n}\frac{z-\beta_j}{1-\overline{\beta_j}z}.$$

If $m > n$, define

$$h(z) := \prod_{j=n+1}^{m}\frac{z-\alpha_j}{1-\overline{\alpha_j}z}\prod_{j=1}^{n}\left(\frac{1-\overline{\beta_j}z}{1-\overline{\alpha_j}z}\right)^2$$

and if instead $m = n < \infty$, define

$$h(z) := \prod_{j=1}^{n}\left(\frac{1-\overline{\beta_j}z}{1-\overline{\alpha_j}z}\right)^2.$$

Then we have that if $c > 0$ then for all $z \in \mathbb{T}$,

$$\begin{aligned}|b(z) - q(z)ce^{i(\theta-\omega)}h(z)| &= \left|\prod_{j=1}^{n} \frac{z-\alpha_j}{1-\overline{\alpha_j}z} - c\prod_{j=1}^{n}\frac{z-\beta_j}{1-\overline{\beta_j}z}\prod_{j=1}^{n}\left(\frac{1-\overline{\beta_j}z}{1-\overline{\alpha_j}z}\right)^2\right| \\ &= \left|1 - c\prod_{j=1}^{n}\left(\frac{z-\beta_j}{z-\alpha_j}\cdot\frac{1-\overline{\beta_j}z}{1-\overline{\alpha_j}z}\right)\right| \\ &= \left|1 - c\prod_{j=1}^{n}\left|\frac{1-\overline{\beta_j}z}{1-\overline{\alpha_j}z}\right|^2\right|.\end{aligned}$$

But since

$$\frac{\prod_{j=1}^{n}(1-|\beta_j|)^2}{2^{2n}} < \prod_{j=1}^{n}\left|\frac{1-\overline{\beta_j}z}{1-\overline{\alpha_j}z}\right|^2 < \frac{2^{2n}}{\prod_{j=1}^{n}(1-|\alpha_j|)^2} \qquad \text{on } \mathbb{T},$$

it follows that if we choose

$$c = \frac{2^{2n}}{\prod_{j=1}^{n}(1-|\alpha_j|)^2}$$

then $|b(z) - q(z)ce^{i(\theta-\omega)}h(z)| < 1$, which implies that

$$\inf_{f\in H^\infty} ||b - qf||_\infty < 1$$

because $f \equiv ce^{i(\theta-\omega)}h(z) \in H^\infty$. This proves the assertion (1). To complete the proof we suppose $\deg(b) < \deg(q)$. Assume to the contrary that $\operatorname{dist}(b, qH^\infty) < 1$. Then there exists a function $f \in H^\infty$ such that $||b - qf||_\infty < 1$, and hence

$$|b(z) - q(z)f(z)| < 1 = |b(z)| \quad \text{on } \mathbb{T}.$$

By Rouché's theorem,

$$\#(\text{zeros of } b \text{ in } \mathbb{D}) = \#(\text{zeros of } qf \text{ in } \mathbb{D}),$$

which contradicts the assumption that $\deg(b) < \deg(q)$. This completes the proof. □

A bounded linear operator A on a Hilbert space $\mathcal{H}$ is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ is positive semidefinite. The problem of determining which symbols induce hyponormal

Toeplitz operators was completely solved by C. Cowen [Co] in 1988: If $\phi \in L^\infty(\mathbb{T})$ and

$$\mathcal{E}(\phi) := \{k \in H^\infty(\mathbb{T}) : ||k||_\infty \le 1 \text{ and } \phi - k\overline{\phi} \in H^\infty(\mathbb{T})\},$$

then T_ϕ is hyponormal if and only if the set $\mathcal{E}(\phi)$ is nonempty. This is called the *Cowen's theorem*. Recently, W.Y. Lee [Le] give a complete description on the Cowen set $\mathcal{E}(\phi)$ if the selfcommutator $[T_\phi^*, T_\phi]$ is of finite rank, in which a H^∞-optimization problem like Theorem 1 is employed. However, if the selfcommutator $[T_\phi^*, T_\phi]$ is of infinite rank, then a description on the set $\mathcal{E}(\phi)$ is not definite.

Example 2. *If ϕ is of bounded type such that the rank of the selfcommutator $[T_\phi^*, T_\phi]$ is of infinite then $\mathcal{E}(\phi)$ may contain either a unique function or infinitely many functions in H^∞.*

Proof. Suppose

$$\phi(z) = \frac{1}{2}\overline{z} + z\,b \quad (b \text{ is an infinite Blaschke product}).$$

Clearly, ϕ is of bounded type. Since $[T_\phi^*, T_\phi] = H_{\overline{\phi}}^* H_{\overline{\phi}} - H_\phi^* H_\phi$, it follows that

$$[T_\phi^*, T_\phi] = H_{\overline{zb}}^* H_{\overline{zb}} - \frac{1}{4} H_{\overline{z}}^* H_{\overline{z}},$$

and hence rank $[T_\phi^*, T_\phi] = \deg(zb) = \infty$. Observe that if $|c| \le \frac{1}{2}$ then each function $(\frac{1}{2} + c\,z)b$ is contained in $\mathcal{E}(\phi)$, so that $\mathcal{E}(\phi)$ contains infinitely many functions. If instead

$$\psi(z) = \overline{z} + zb \quad (b \text{ is an infinite Blaschke product}),$$

then by the same argument as above, $[T_\psi^*, T_\psi]$ has infinite rank. Evidently, $\mathcal{E}(\psi)$ contains the function b. We now claim that $\mathcal{E}(\psi)$ contains exactly one element. Indeed, if k is in $\mathcal{E}(\psi)$ then $\overline{z} - k\,\overline{zb} \in H^\infty$ and so $k\,\overline{b} \in 1 + zH^\infty$ and $||k\overline{b}||_\infty \le 1$, which forces $k = b$. □

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