ON THE STRUCTURE JACOBI OPERATOR AND RICCI TENSOR OF REAL HYPERSURFACES IN NONFLAT COMPLEX SPACE FORMS

Soo Jin Kim

Abstract. It is known that there are no real hypersurfaces with parallel structure Jacobi operator R_ξ (cf.[16], [17]). In this paper we investigate real hypersurfaces in a nonflat complex space form using some conditions of the structure Jacobi operator R_ξ which are weaker than $\nabla R_\xi = 0$. Under further condition $S\phi = \phi S$ for the Ricci tensor S we characterize Hopf hypersurfaces in a complex space form.

I. Introduction

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature $4c$ is called a complex space form, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$ or a complex hyperbolic space $H_n\mathbb{C}$, according as $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_n(c), c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kähler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is well known that α is a locally constant ([13]) and that M is called a *Hopf hypersurface*.

Typical examples of Hopf hypersurfaces in $P_n\mathbb{C}$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary group $PU(n+1)$. Takagi [18], [19] completely classified such hypersurfaces as six model spaces which are said to be of *type A_1, A_2, B, C, D and E.* Also Berndt ([1]) showed that all Hopf real

hypersurfaces with constant principal curvature in a complex hyperbolic space $H_n\mathbb{C}$ are realized as the tubes of constant radius over certain submanifolds. Nowadays in $H_n\mathbb{C}$ they are said to be of *type A_0, A_1, A_2 and B*. In 2007, Berndt and Tamaru [2] classified all homogeneous real hypersurfaces in $H_n\mathbb{C}$. Real hypersurfaces of each type in $M_n(c)$, $c \neq 0$ have been described in detail by Niebergall and Ryan ([15]).

On the other hand, the structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ for the Riemannian curvature operator R has a fundamental role in contact geometry. Cho and Ki started the study on real hypersurfaces in a complex space form by using the operator R_ξ in [3], [4] and [5]. It is well known that there are no real hypersurfaces with parallel structure Jacobi operator $\nabla R_\xi = 0$ in $M_n(c)$, $c \neq 0$ ([16], [17]).

Motivated by results mentioned above, it is natural to investigate real hypersurfaces in a complex space form by using some conditions (on the derivative of R_ξ) which are weaker than $\nabla R_\xi = 0$. From this point of views, some works have studied real hypersurfaces with $\nabla_\xi R_\xi = 0$ or $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$, and given some results on the classification of real hypersurfaces in a complex space form ([6]~[12], [14]). Two of them we introduce the following without proof.

Theorem 1.1 (Ki, Kurihara, Nagai and Takagi [9]). *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ whose Ricci tensor commutes with R_ξ. Then M satisfies $\nabla_\xi R_\xi = 0$ if and if only M is locally congruent to one of the following:*

(1) In case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,
    (A_1) a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,
    (A_2) a tube of radius r over a totally geodesic $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$ and $r \neq \pi/4$.
    (2) In case $M_n(c) = H_n\mathbb{C}$,
    (A_0) a horosphere,
    (A_1) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,
    (A_2) a tube over a totally geodesic $H_n\mathbb{C}$ $(1 \leq n \leq n-2)$.

Theorem 1.2 (Ki and Kurihara [7]). *Let M be a real hypersurface in a nonflat complex space form which satisfies $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$. M holds $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of (1) and (2) stated in Theorem 1.1.*

Let us denote the Ricci tensor of type (1,1) by S. If we pay a particular attention to the fact that for each Hopf hypersurface M in $M_n(c)$, $c \neq 0$, then $S\xi = g(S\xi, \xi)\xi$ is satisfied. If $S\phi = \phi S$ holds on real hypersurfaces in $M_n(c)$, then we have $S\xi = g(S\xi, \xi)\xi$. The classification of such real hypersurfaces is still open problem. Recently we have the following:

Theorem 1.3 (Kim and Ki [14]). *Let M be a real hypersurface in $M_n(c)$, $c \neq 0$, which satisfies $\nabla_\xi R_\xi = 0$ and at the same time $\phi S = S\phi$. Then M is a Hopf hypersurface. Further, M is locally congruent to one of (1) and (2) stated in Theorem 1.1.*

Theorem 1.4 (Ki and Nagai [11]). *Let M be a real hypersurface in a complex projective space $P_n\mathbb{C}$ which satisfies $R_\xi S = SR_\xi$ and $\nabla_{\phi\nabla_\xi \xi}S = 0$. If $g(S\xi, \xi) =$const., then M is a Hopf hypersurface.*

In this paper, we investigate real hypersurface satisfying $\phi S = S\phi$ and at the same time $\nabla_{\phi\nabla_\xi \xi}R_\xi = 0$ in a complex space form. The main purpose of the present paper is to establish Main Theorem stated in section 5.

All manifolds in the present paper are assumed to be connected and of C^∞ and real hypersurfaces supposed to be orientable.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$ with parallel almost complex structure J, and constant holomorphic sectional curvature $4c$ and N be a unit normal vector field on M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-study metric $\tilde{g}$ of $M_n(c)$. Then the Gauss and Weingraten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields X and Y on M, where ∇ and g denote the Riemannian connection and the Riemannian metric tensor induced from $\tilde{g}$ respectively, and A denotes the shape operator in the direction of N.

For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

We call ξ the *structure vector field* and its flow also denoted by the same ξ. Then we may see that the aggregate (ϕ, ξ, η, g) is an almost

contact metric structure on M. From the fact $\tilde{\nabla} J = 0$, and by using the Gauss and Weingarten formulas, we have

$$(2.1) \qquad\qquad \nabla_X \xi = \phi A X,$$

$$(2.2) \qquad\qquad (\nabla_X \phi) Y = \eta(Y) A X - g(A X, Y) \xi.$$

Since we consider that the ambient manifold is of constant holomorphic sectional curvature $4c$, we have the following Gauss and Codazzi equations respectively.

$$(2.3) \qquad R(X,Y) Z = c\{g(Y,Z) X - g(X,Z) Y + g(\phi Y, Z)\phi X$$
$$- g(\phi X, Z) \phi Y - 2g(\phi X, Y) \phi Z\}$$
$$+ g(AY, Z) A X - g(A X, Z) A Y,$$

$$(2.4)\ \ (\nabla_X A) Y - (\nabla_Y A) X = c\{\eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi\}$$

for any tangent vector fields X, Y and Z on M, where R denote the Riemannian curvature tensor of M.

Now let us denote by $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$ and $h = \mathrm{Tr} A$, and ∇f the gradient vector field of a function f defined on M.

If we put $U = \nabla_\xi \xi$, then U is orthogonal to ξ. Thus, it is, using (2.1), seen that

$$(2.5) \qquad\qquad \phi U = -A\xi + \alpha\xi,$$

which leads to $g(U, U) = \beta - \alpha^2$. We easily see that ξ is a principal curvature vector field, that is $A\xi = \alpha\xi$, if and only if $\beta - \alpha^2 = 0$. In this case, M is called a *Hopf hypersurface* in $M_n(c)$. If we put

$$(2.6) \qquad\qquad A\xi = \alpha\xi + \mu W,$$

where W is a unit vector field orthogonal to ξ. Then we get $U = \mu\phi W$, which tells us that W is also orthogonal to U. Further, we have

$$(2.7) \qquad\qquad \mu^2 = \beta - \alpha^2.$$

Using (2.1) and (2.6), we see that

$$(2.8) \qquad\qquad \mu g(\nabla_X W, \xi) = g(AU, X),$$

$$(2.9) \qquad\qquad g(\nabla_X \xi, U) = \mu g(AW, X)$$

for any vector field X on M.

Differentiating (2.5) covariantly along M and making use of (2.1) and (2.2), we find

$$(2.10) \quad (\nabla_X A)\,\xi = -\phi \nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX,$$

which enables us to obtain

$$(2.11) \qquad\qquad (\nabla_\xi A)\,\xi = 2AU + \nabla\alpha,$$

where we have used (2.4). From (2.1) and (2.10), we verify that

$$(2.12) \qquad\qquad \nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha.$$

From the Gauss equation (2.3) the structure Jacobi operator R_ξ is given by

$$(2.13) \qquad R_\xi X = R\,(X,\xi)\,\xi = c\,\{X - \eta\,(X)\,\xi\} + \alpha AX - \eta\,(AX)\,A\xi$$

for any vector field X on M.

We set $\Omega = \{p \in M | \mu\,(p) \neq 0\}$, and suppose that $\Omega \neq \varnothing$, that is ξ is not principal curvature vector on M. Hereafter, unless otherwise stated, we continue our discussions on the open set Ω of M.

Differentiating (2.13) covariantly along Ω, we find

$$(2.14) \quad g\,((\nabla_X R_\xi)\,Y, Z)$$

$$= g(\nabla_X (R_\xi Y) - R_\xi(\nabla_X Y), Z)$$
$$= -c\,\{\eta\,(Z)\,g(\nabla_X \xi, Y) + \eta\,(Y)\,g(\nabla_X \xi, Z)\} + (X\alpha)g(AY, Z)$$
$$+\alpha g\,((\nabla_X A)\,Y, Z) - \eta\,(AZ)\,\{g\,((\nabla_X A)\xi, Y) + g(A\phi AX, Y)\}$$
$$-\eta\,(AY)\,\{g\,((\nabla_X A)\xi, Z) + g(A\phi AX, Z)\}\,.$$

If we put $X = W$ and $Y = \xi$ in (2.14), then we get

$$g((\nabla_W R_\xi)\xi, Z)$$
$$= -cg(\phi AW, Z) + (W\alpha)g(A\xi, Z) - \eta(AZ)g((\nabla_W A)\xi, \xi)$$
$$-\eta(AZ)g(A\phi AW, Z) - \alpha g(A\phi AW, Z),$$

or using (2.11)

$$(2.15)$$
$$g((\nabla_W R_\xi)\xi, Z) = -\alpha g(A\phi AW, Z) - cg(\phi AW, Z) - g(AU, W)\eta(AZ).$$

3. Real hypersurfaces satisfying $S\phi = \phi S$

In the following we shall denote the Ricci tensor of type $(1,1)$ by S. Then it follows from (2.3) that

$$(3.1) \qquad\qquad S = c\{(2n + 1)I - 3\eta \otimes \xi\} + hA - A^2,$$

where I is an identity map, which implies

(3.2) $S\xi = 2c(n-1)\xi + hA\xi - A^2\xi.$

Now, suppose that $S\phi = \phi S$ is satisfied on a real hypersurface M of $M_n(c)$, $c \neq 0$. Then we have from (3.1) $A^2\phi - \phi A^2 = h(A\phi - \phi A)$, which enables us to obtain $\phi(A^2\xi - hA\xi) = 0$. Because of properties of the almost contact metric structure (ϕ, g, ξ, η), it follows, using this, that

(3.3) $A^2\xi = hA\xi + (\beta - h\alpha)\xi.$

Combining (2.6) to this, we verify that

(3.4) $AW = \mu\xi + (h-\alpha)W$

and hence

(3.5) $A^2W = hAW + (\beta - h\alpha)W$

because of $\mu \neq 0$. Differentiating (3.4) covariantly along Ω, we find

(3.6) $(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(h-\alpha)W + (h-\alpha)\nabla_X W.$

If we take an inner product with W in the last equation, then we find

(3.7) $g((\nabla_X A)W, W) = -2g(AU, X) + Xh - X\alpha$

since W is a unit vector field orthogonal to ξ. We also obtain by applying ξ to (3.6),

$\mu g((\nabla_X A)W, \xi) = (h-2\alpha)g(AU,X) + \mu(X\mu),$

where we have used (2.8), which together with the Codazzi equation (2.4) implies that

(3.8) $\mu(\nabla_\xi A)W = (h-2\alpha)AU - cU + \mu\nabla\mu.$

Replacing X by $\mu\xi$ in (3.6) and taking account of (3.8), we get

(3.9) $(h-2\alpha)AU - cU + \mu\nabla\mu + \mu\{A\nabla_\xi W - (h-\alpha)\nabla_\xi W\}$

$$= \mu(\xi\mu)\xi + \mu^2 U + \mu(\xi h - \xi\alpha)W.$$

By the way, from $\phi U = -\mu W$ we see, using (2.2,) that

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W.$$

Putting $X = \xi$ in this and making use of (2.5) and (2.12), we obtain

(3.10) $\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W.$

Substituting this into (3.9), we have

(3.11) $3A^2 U - 2hAU + A\nabla\alpha + \dfrac{1}{2}\nabla\beta - h\nabla\alpha + (\alpha h - \beta - c)U$

$$= 2\mu(W\alpha)\xi + \mu(\xi h)W - (h-2\alpha)(\xi\alpha)\xi,$$

where we have used (2.7). Taking an inner product ξ to this and using (2.6) and (2.7), we find

(3.12) $$\xi\mu = W\alpha.$$

From (3.7) and (3.8), it is seen that

(3.13) $$W\mu = \xi h - \xi\alpha.$$

Differentiating (3.3) covariantly and making use of (2.1), we find

$$(\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX - hA\phi AX$$
$$= (Xh)A\xi + h(\nabla_X A)\xi + X(\beta - h\alpha)\xi + (\beta - h\alpha)\phi AX,$$

which together with the Codazzi equation (2.4) implies that

$$c\{u(Y)\eta(X) - u(X)\eta(Y)\} + 2c(h - \alpha)g(\phi X, Y) - g(A^2\phi AX, Y)$$
$$+ g(A^2\phi AY, X) + 2hg(\phi AX, Y) - (\beta - h\alpha)\{g(\phi AY, X) - g(\phi AX, Y)\}$$
$$= g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + (Yh)g(A\xi, X) - (Xh)g(A\xi, Y)$$
$$+ Y(\beta - h\alpha)\eta(X) - X(\beta - h\alpha)\eta(Y),$$

where we have defined a 1-form u by $u(X) = g(U, X)$ for any vector field X. If we replace X by μW to the both sides of the last equation and make use of (2.4), (2.11), (3.4), (3.5) and (3.8), then we get

(3.14) $$(3\alpha - 2h)A^2U + 2(h^2 + \beta - 2h\alpha + c)AU + (h - \alpha)(\beta - h\alpha - 2c)U$$
$$= \mu A\nabla\mu + (\alpha h - \beta)\nabla\alpha - \frac{1}{2}(h - \alpha)\nabla\beta + \mu^2\nabla h - \mu(Wh)A\xi$$
$$- \mu W(\beta - h\alpha)\xi.$$

Applying by U or W in (3.1), we have respectively

$$SU = (2n + 1)cU + hAU - A^2U,$$

$$SW = (2n + 1)cW + (h\alpha - \beta)W$$

by virtue of (3.5). If we take an inner product U to $S\phi - \phi S = 0$ and make use of the last two equations, then we obtain

$$-\mu SW + (2n + 1)c\mu W - h\phi AU + \phi A^2U = 0,$$

or equivalently $(\beta - h\alpha)\mu W - h\phi AU + \phi A^2U = 0$. Thus, it follows that

(3.15) $$A^2U = hAU + (\beta - h\alpha)U.$$

4. Real hypersurfaces satisfying $S\phi = \phi S$ and $\nabla_{\phi U} R_\xi = 0$

We will continue our arguments on Ω under the assumptions $S\phi = \phi S$ and at the same time $\nabla_{\phi \nabla_\xi \xi} R_\xi = 0$. Then the second hypothesis implies that $\nabla_W R_\xi = 0$ because of (2.5) and (2.6). So see from (2.15) that

$$(4.1) \qquad \alpha A\phi AW + c\phi AW + g(AU, W)\xi = 0.$$

Since the first assumption implies (3.4). So we obtain $g(AW, U) = 0$. Therefore (4.1) reformed as

$$(4.2) \qquad \alpha A\phi AW + c\phi AW = 0.$$

Combining this to (3.4), we see that

$$(4.3) \qquad \lambda(\alpha AU + cU) = 0.$$

because of (2.5), where we have put $\lambda = h - \alpha$.

We notice here that the following :

Remark 1. $\alpha \neq 0$ on Ω.

In fact, if not, then we get $\alpha = 0$. So we verify, using (4.3), that $\lambda = 0$ and hence $h = 0$ on this subset. Consequently (3.11), (3.14) and (3.15) are reduced respectively to

$$3A^2 U + \frac{1}{2}\nabla\beta - (\beta + c)U = 0,$$

$$2(\beta + c)AU = \frac{1}{2}A\nabla\beta - \mu(W\beta)\xi,$$

$$A^2 U = \beta U$$

on the subset, where we have used (2.7). Combining these we obtain $4\beta AU + \mu(W\beta)\xi = 0$, which together with (2.6) yields $W\beta = 0$ and thus $AU = 0$ on the set because $\beta \neq 0$ on Ω. From the equation $A^2 U = \beta U$ we have $U = 0$, a contradiction. Therefore $\alpha \neq 0$ is proved on Ω.

Putting $X = W$ in (2.14) and using (4.2) and Remark 1, we find

$$\alpha(\nabla_W A)X = -(W\alpha)AX + g(A\xi, X)(\nabla_W A)\xi + g((\nabla_W A)\xi, X)A\xi$$

$$-\frac{c}{\alpha}\{w(X)\phi AW + g(\phi AW, X)W\},$$

where we have defined a 1-form w by $w(X) = g(W, X)$ for any vector field X, which connected to (3.4) and (3.8) gives

$$(4.4) \qquad \alpha(\nabla_W A)X = -(W\alpha)AX - \frac{c}{\alpha}\lambda(w(X)U + u(X)W)$$

$$+ \frac{1}{\mu}\{(\lambda - \alpha)AU - 2cU + \mu\nabla\mu\}g(A\xi, X)$$

$$+ \frac{1}{\mu}g((\lambda - \alpha)AU - 2cU + \mu\nabla\mu, X)A\xi.$$

If we put $X = W$ in (4.4) and take account of (2.7), (3.7) and (3.13), then we obtain

$$(4.5) \quad \frac{1}{2}\alpha\nabla\beta - \alpha^2\nabla h = c(h + \alpha)U - \alpha hAU + \alpha(W\alpha)AW - \alpha(\xi h - \xi\alpha)A\xi.$$

First of all, we shall prove the following lemma, which will be used later.

Lemma 1. $\alpha AU + cU = 0$ *on* Ω.

Proof. We suppose that $\alpha AU + cU \neq 0$ on Ω, and then we restrict the arguments on such a place. From (4.3) we then have $\lambda = 0$, that is, $h - \alpha = 0$. Accordingly (3.4) is reduced to $AW = \mu\xi$. So (3.14) turns out to be

$$(4.6) \quad \mu(W\alpha)A\xi = \mu A\nabla\mu - \alpha A^2U - 2(\mu^2 + c)AU,$$

where we have used (2.7) and (3.13). In the same way we see from (4.5) that

$$\mu\nabla\mu = 2cU - \alpha AU + \mu(W\alpha)\xi$$

because of (2.7) and Remark 1, which implies

$$\mu A\nabla\mu = 2cAU - \alpha A^2U + \mu(W\alpha)A\xi.$$

Thus, (4.6) together with this yields

$$(4.7) \quad \alpha A^2U + \mu^2 AU = 0,$$

or using (2.7) and (3.15) $\beta AU + \alpha\mu^2 U = 0$. From this we have $\beta A^2U + \alpha\mu^2 AU = 0$. Hence we have $A^2U = 0$ by virtue of (4.7). So we obtain $AU = 0$ and therefore $U = 0$, a contradiction. This completes the proof. ∎

Using (3.15) and Lemma 1, we have

$$(4.8) \quad \alpha^2(\beta - h\alpha) = c(h\alpha + c).$$

By Lemma 1, (4.5) turns out to be

$$(4.9) \quad \frac{1}{2}\alpha^2\nabla\beta = \alpha^2\nabla h + c(\alpha + 2h)U + \alpha(W\alpha)AW - \alpha(\xi h - \xi\alpha)A\xi.$$

Applying (4.9) by W and making use of (2.7) and (3.13,) we find

$$(4.10) \quad W\beta = \alpha(Wh) + h(W\alpha).$$

Because of Lemma 1 and Remark 1, we can write (4.4) as

(4.11) $\alpha(\nabla_W A)X = -(W\alpha)AX - \dfrac{c\lambda}{\alpha}\{u(X)W + w(X)U\}$

$+g(\nabla\mu - \dfrac{ch}{\alpha\mu}U, X)A\xi + g(A\xi, X)(\nabla\mu - \dfrac{ch}{\alpha\mu}U).$

By putting $X = U$ in this, and making use of Lemma 1, we obtain

(4.12) $\alpha(\nabla_W A)U = (U\mu)A\xi + \dfrac{c}{\alpha}((W\alpha)U - \lambda\mu^2 W - h\mu A\xi).$

On the other hand, if we take an inner product U to (3.6) and make use of (2.4) and Lemma 1, then we obtain

$(\alpha\lambda + c)g(\nabla_X W, U) = \alpha g((\nabla_W A)X, U) + c\alpha\mu\eta(X) - \alpha\mu^2 g(AW, X),$

which together with (4.12) implies that

(4.13) $(\alpha\lambda + c)g(\nabla_X W, U) = \dfrac{c}{\alpha}g((W\alpha)U - \lambda\mu^2 W - \mu h A\xi, X)$

$+(U\mu)g(A\xi, X) + \alpha\mu\eta(X) - \alpha\mu^2 g(AW, X).$

Putting $X = U$ in this and using (3.4), we have

(4.14) $(\alpha\lambda + c)g(\nabla_U W, U) = \dfrac{c}{\alpha}\mu^2(W\alpha).$

Now, if we differentiate $\alpha AU + cU = 0$ covariantly, and use itself again, then we get

(4.15) $-\dfrac{c}{\alpha}(X\alpha)U + \alpha(\nabla_X A)U + \alpha A\nabla_X U + c\nabla_X U = 0,$

which together with the Codazzi equation (2.4) and (2.5) yields

$\dfrac{c}{\alpha}\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + c\alpha\mu\{\eta(X)w(Y) - \eta(Y)w(X)\}$

$+\alpha\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + cdu(X, Y) = 0,$

where du is the exterior derivative of a 1-form u given by

$du(X, Y) = X(u(Y)) - Y(u(X)) - u([X, Y]).$

Replacing X by ξ in the last equation and taking account of (2.6) and (2.9), we obtain

$\alpha\mu g(W, \nabla_X U) = -\dfrac{c}{\alpha}(\xi\alpha)u(X) + c\alpha\mu w(X) + \mu(\alpha^2 + c)g(AW, X)$

$+g(\alpha A\nabla_\xi U + c\nabla_\xi U, X).$

If we put $X = U$ in this and make use of $\alpha AU + cU = 0$, then we have $\alpha^2 g(\nabla_U W, U) = c\mu(\xi\alpha)$, which together with (4.14) implies that

(4.16) $\alpha\mu(W\alpha) = (\alpha\lambda + c)\xi\alpha.$

Differentiating (4.8) along Ω and using itself again, we find

$$(4.17) \qquad \alpha^2 \nabla \beta = -\frac{2c}{\alpha}(h\alpha + c)\nabla\alpha + (\alpha^2 + c)(\alpha\nabla h + h\nabla\alpha).$$

If we take an inner product W to this and make use of (4.10), then we obtain

$$(4.18) \qquad \alpha(W\beta) = 2(h\alpha + c)W\alpha.$$

Finally we prove

Lemma 2. $\xi\alpha = 0$, $\xi h = 0$, $\xi\beta = 0$ and $W\alpha = 0$ on Ω.

Proof. It is verify, using (2.7), that we have from (3.13) $\frac{1}{2}W\beta = \mu(\xi\lambda) + \alpha(W\alpha)$, which together with (4.18) implies that $(\alpha\lambda + c)W\alpha = \mu\alpha(\xi\lambda)$, where we have used $\lambda = h - \alpha$. Thus, it follows that

$$(4.19) \qquad (\alpha\lambda + c)W\alpha = \mu\alpha(\xi h) - \mu\alpha(\xi\alpha).$$

On the other hand, applying (4.17) by ξ we obtain,

$$(4.20) \qquad \alpha^2(\xi\beta) = -\frac{2c}{\alpha}(h\alpha + c)\xi\alpha + (\alpha^2 + c)(\alpha\xi h + h\xi\alpha).$$

By the way, (2.7) and (3.12) implies that $\alpha(\xi\beta) = 2\mu\alpha(W\alpha) + 2\alpha^2(\xi\alpha)$, which connected to (4.16) gives $\alpha(\xi\beta) = 2(\alpha h + c)\xi\alpha$. Substituting the last equation into (4.20), we find

$$(4.21) \qquad (\alpha^2 + c)\{\alpha^2(\xi h) - (h\alpha + 2c)\xi\alpha\} = 0.$$

Let Ω_0 be the set of points such that $(\xi\alpha)_p \neq 0$, $p \in \Omega$ and suppose that Ω_0 be nonempty. Then we have on Ω_0 $\alpha^2(\xi h) - (h\alpha + 2c)\xi\alpha = 0$.

This, together with (4.19) implies that $\alpha(\alpha\lambda + c)W\alpha = \mu(\alpha\lambda + 2c)\xi\alpha$. Accordingly we have from this and (4.16) on Ω_0.

$$(\alpha\lambda + c)^2 = \mu^2(\alpha\lambda + 2c).$$

Since we have $\alpha^2\mu^2 = \alpha^3\lambda + c(\alpha\lambda + \alpha^2 + c)$ by virtue of (4.8), the last relationship reformed as $(\alpha\lambda + c)(\alpha\lambda + \alpha^2 + 2c) = 0$ on Ω_0. If $\alpha\lambda + c = 0$, then $\alpha h + c = \alpha^2$. Thus, (4.8) implies $\beta - h\alpha = c$ and hence $\beta = \alpha^2$ i.e $\mu = 0$, it cannot occur on Ω, a contradiction. Therefore we have $\alpha\lambda + \alpha^2 + 2c = 0$ on Ω_0. So we have $\alpha h + 2c = 0$ and therefore $\alpha(Wh) + h(W\alpha) = 0$ on Ω_0, which together with (4.10) yields $W\beta = 0$. Thus, (4.18) gives $W\alpha = 0$ and hence $\xi\alpha = 0$ on Ω_0 by virtue of (4.16). Thus $\xi\alpha = 0$ every where on Ω is proved. Therefore we have $W\alpha = 0$, $\xi h = 0$ and $\xi\beta = 0$ because of (4.16), (4.19) and (4.20) respectively. This completes the proof of Lemma2. ■

5. Main theorem

We will continue our arguments on Ω under the hypotheses as those stated in section 4. We prove

Theorem. *Let M be a real hypersurface with almost contact metric structure (ϕ, g, ξ, η) in a complex space form which satisfies $\phi S = S\phi$ and at the same time $\nabla_{\phi \nabla_\xi \xi} R_\xi = 0$. Then M is a Hopf hypersurface, where S denotes the Ricci tensor of M.*

Proof. Because of Lemma 2. we can write (4.9) as

$$(5.1) \qquad \frac{1}{2}\alpha \nabla \beta = \alpha^2 \nabla h + c(\alpha + 2h)U,$$

which together with (4.17) implies that

$$(5.2) \qquad (ch\alpha - h\alpha^3 + 2c^2)\nabla\alpha + \alpha^2(\alpha^2 - c)\nabla h = -2c\alpha^2(2h + \alpha)U.$$

This is rewritten as

$$(ch\alpha - h\alpha^3 + 2c^2)(Y\alpha) + \alpha^2(\alpha^2 - c)(Yh) + 2c\alpha^2(2h + \alpha)u(Y) = 0$$

for any vector field Y. Differentiating this with respect to a vector field X, and taking the skew-symmetric parts for X and Y, we eventually have

$$3(c-\alpha^2)\{(Xh)(Y\alpha)-(Yh)(X\alpha)\}+2c(4h+3\alpha)\{(X\alpha)u(Y)-(Y\alpha)u(X)\}$$
$$+4c\alpha\{(Xh)u(Y)-(Yh)u(X)\}+2c\alpha(2h+\alpha)du(X,Y) = 0.$$

Putting $X = \xi$ in this and taking account of Remark 1 and Lemma 2, we find

$$(5.3) \qquad (2h + \alpha)du(\xi, X) = 0.$$

Now, suppose that $\Omega_1 = \{p \in \Omega | du(\xi, X)_p \neq 0\}$ and $\Omega_1 \neq \varnothing$. Then we have $2h+\alpha = 0$ and hence $h\alpha = -\frac{1}{2}\alpha^2$ on Ω_1. Therefore (5.2) implies $\nabla\alpha = 0$ and hence $\nabla h = 0$ on Ω_1. Accordingly $\nabla\beta = 0$ because of (4.17) and Remark 1. Using these facts, (3.11) turns out to be

$$3A^2U - 2hAU + (\alpha h - \beta - c)U = 0,$$

which together with Lemma 1 and (4.8) gives $\alpha^2 = h\alpha - 2c$ and consequently $3\alpha^2 = -4c$ because we have already $h\alpha = -\frac{1}{2}\alpha^2$ on Ω_1. Hence we see that $c < 0$ on Ω_1. By the way, (4.8) implies $2\beta + \alpha^2 = 2c$ on Ω_1. It is contradictory by virtue of $c < 0$. Therefore we have $\Omega_1 = \varnothing$ and thus $du(\xi, X) = 0$ on Ω because of (5.3). From this fact, we have

$$g(\nabla_\xi U, X) + g(\nabla_X \xi, U) = 0.$$

which together with (2.9) and (2.12) implies that

$$3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha + \mu AW = 0$$

or, using (2.6), (3.4) and Lemma 1, $(3c + h\alpha)\mu W + \phi\nabla\alpha = 0$. Thus, it follows that

(5.4) $$\alpha\nabla\alpha = (\alpha h + 3c)U.$$

From this and (5.2) we verify that

(5.5) $$\alpha^3(c - \alpha^2)\nabla h = (2c\alpha^4 - h^2\alpha^4 + ch\alpha^3 + c\alpha^2 h^2 - 5c^2\alpha h + 6c^3)U.$$

On the other hand, we have from (3.11)

$$3A^2 U - 2hAU + A\nabla\alpha + \frac{1}{2}\nabla\beta - h\nabla\alpha + (\alpha h - \beta - c)U = 0$$

by virtue of Lemma 2, which together with (5.1), (5.4) and Lemma 1 implies that

(5.6) $$\alpha^2\nabla h = (\alpha h^2 + \frac{c^2}{\alpha} + ch)U.$$

Combining this to (5.5), we obtain

(5.7) $$\alpha^4 + h\alpha^3 + \frac{c}{2}\alpha^2 + 2ch\alpha + \frac{5}{2}c^2 = 0.$$

Differentiation this gives

$$(4\alpha^3 + 3\alpha^3 h + c\alpha + 2ch)\alpha\nabla\alpha + (\alpha^2 + 2c)\alpha^2\nabla h = 0.$$

which together with (5.4), (5.6) and (5.7) yields

$$12\alpha^4 + 9\alpha^2(h\alpha) - 4(\alpha h)^2 - 2c(h\alpha) + 4c\alpha^2 + 2c^2 = 0.$$

If we combine this to (5.7), then we get

$$4(\alpha^2 + \frac{c}{2}\alpha^2 + \frac{5}{2}c^2)^2 + (9\alpha^2 - 2c)(\alpha^2 + 2c)(\alpha^4 + \frac{c}{2}\alpha^2 + \frac{5}{2}c)$$
$$= (\alpha^2 + 2c)^2(12\alpha^4 + 4c\alpha^2 + 2c^2),$$

or equivalently $2\alpha^8 + 55c\alpha^6 + \cdots + 7c^4 = 0$. Thus, α is a root of the algebraic equation with constant coefficient. So we verify that α is a constant on Ω. Accordingly we have $\alpha h + 3c = 0$ because of (5.4) and hence $h = constant$. with the aid of Remark 1. Consequently we see, using (5.6), that $(\alpha h)^2 + c(\alpha h) + c^2 = 0$ and hence $c = 0$ since $\alpha h + 3c = 0$. It is contradictory. Thus, $\Omega = \varnothing$, namely $A\xi = \alpha\xi$ on M. This completes the proof. ∎

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Soo Jin Kim
Department of Mathematics,
Chosun University,
Kwangju, 501-759, S.Korea.
E-mail: `sj-kim@chosun.kr`