Honam Mathematical J. **32** (2010), No. 4, pp. 711–738

INTERVAL-VALUED SMOOTH TOPOLOGICAL SPACES

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Abstract. We list two kinds of gradation of openness and we study in the sense of the followings:

(i) We give the definition of IVGO of fuzzy sets and obtain some basic results.

(ii) We give the definition of interval-valued gradation of clopeness and obtain some properties.

(iii) We give the definition of a subspace of an interval-valued smooth topological space and obtain some properties.

(iv) We investigate some properties of gradation preserving (in short, IVGP) mappings.

1. Introduction

In 1965, Zadeh [19] introduced the concept of fuzzy sets as a generalization of (ordinary) subsets. Soon after, Chang [6] was the first to introduce the notion of a fuzzy topology T on a set X by axiomatizing a collection T of fuzzy sets in X as follows:

(i) $\emptyset, X \in T$,

(ii) $A, B \in T \Rightarrow A \cap B \in T$,

(iii) $\{A_\alpha\}_{\alpha\in\Gamma} \subset T \Rightarrow \bigcup_{\alpha\in\Gamma} A_\alpha \in T$,

where he reffered to each member of T as an open set.

Some authors[7,9,10,18] noted that fuzziness in it was absent, and Šostak[18] began the study of fuzzy structures of the topological type and called a function $\tau : I^X \to I$, satisfying the following conditions:

(i) $\tau(\emptyset) = \tau(X) = 1$,

(ii) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B),\ \forall A, B \in I^X$,

Received June 08, 2010. Accepted November 22, 2010.

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2000 Mathematics Subject Classification : 54A40.

Keywords and phrases. t-norm : interval-valued gradation of openness (resp. closedness and clopenness), interval-valued preserving mapping.

[1] This paper was supported by Wonkwang University in 2009.

(iii) $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha),\ \forall\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$,

as a *fuzzy topology on* X. In this case, the pair (X, τ) was called a *fuzzy topological space* (in short, FTS) and $\tau(A)$ was called the *degree of openness of* the fuzzy set A.

On the other hand, various generalizations of the notion of fuzzy set have been done by many authors. Zadeh[20] introduced the idea of interval-valued fuzzy sets. Later, Atanassov[1] introduced the concept of intuitionistic fuzzy set. Moreover, Atanassov and Gargov[2] introduced the notion of interval-valued intuitionistic fuzzy sets as the generalization of both interval-valued fuzzy sets and intuitionistic fuzzy sets. Some researchers [1,2,3,4,5] have worked mainly on operators and relations on intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. Çoker[8] introduced the idea of the topology of intuitionistic fuzzy sets, and Hur et.al[11,12] investigated some properties of intuitionistic fuzzy topological groups and intuitionistic fuzzy topological spaces. Samanta and Mondal[16,17] introduced the definitions of the topology of interval-valued fuzzy sets and the topology of interval-valued intuitionistic fuzzy sets, respectively. In particular, recently, Mondal and Samanta[14,15] introduced the notion of intuitionistic gradation of openness.

In this pater, we list two kinds of gradation of openness and we the sense of the followings:

(i) We give the definition of IVGO of fuzzy sets and obtain some basic results.

(ii) We give the definition of interval-valued gradation of clopeness and obtain some properties.

(iii) We give the definition of a subspace of an interval-valued smooth topological space and obtain some properties.

(iv) We investigate some properties of gradation preserving (In short, IVGP) mappings.

2. Preliminaries

Throughout this paper, X will denote a nonempty set; $I = [0, 1]$, the closed unit interval of the real line; $I_0 = (0, 1]$; $I_1 = [0, 1)$; $I^X =$ the set of all fuzzy sets in X. In particular, $\emptyset$ and X denote the *empty fuzzy set* and the *whole fuzzy set* in X defined by $\emptyset(x) = 0$ and $X(x) = 1$, $\forall x \in X$, respectively. All other notations are standard notations of fuzzy set theory. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ satisfying the condition $\mu_A(x) + \nu_A(x) \leq 1,\ \forall x \in X$, is called an *intuitionistic fuzzy set* in X, and $0_\sim$ and $1_\sim$ denote the *empty intuitionistic fuzzy set* and the *whole intuitionistic fuzzy set* in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0),\ \forall x \in X$, respectively. We will denote the set of all intuitionistic fuzzy sets in X as IFS(X). Also all the notations are standard notations of intuitionistic fuzzy set theory.

Let $D(I)$ be the set of all closed subintervals of the unit interval I. The elements of $D(I)$ are generally denoted by capital letters $M, N, ...$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and upper points respectively. Especially, we denote $\boldsymbol{a} = [a,a]$ for every $a \in (0,1)$. We also note that

(i) $(\forall\ M, N \in D(I))(M = N \Leftrightarrow M^L = N^L,\ M^U = N^U)$.

(ii) $(\forall\ M, N \in D(I))(M \leq N \Leftrightarrow M^L \leq N^L,\ M^U \leq N^U)$.

For every $M \in D(I)$, the complement of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U,\ 1 - M^L]$.

Definition 2.1[20]. Let X be a given nonempty set. A mapping $A = [A^L,\ A^U] : X \to D(I)$ is called an *interval valued fuzzy set* (briefly, *IVFS*) in X, where A^L and A^U are fuzzy sets in X satisfying $A^L(x) \leq A^U(x)$ and $A(x) = [A^L(x),\ A^U(x)]$ for each $x \in X$, and $A^L(x)$ and $A^U(x)$ are called the *lower* and *upper end points* of $A(x)$, respectively

It is clear that every fuzzy set A in X is an IVFS of the form $A = [A, A]$. For any $[a,b] \in D(I)$, the IVFS whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$, *i.e.*, $\widetilde{[a,b]}(x) = [a,b]$ for each $x \in X$. For any $a \in I$, the IVFS whose value is $\boldsymbol{a}$ for all $x \in X$ is denoted by simply $\widetilde{a}$, *i.e.*, $\widetilde{a}(x) = \boldsymbol{a}$ for each $x \in X$. $\widetilde{0}$ and $\widetilde{1}$ denote the *empty interral-valued fuzzy set* and the *whole interval-valued fuzzy set* in X, respectively. For a point $p \in X$ and for $[a,b] \in D(I)$ with $b > 0$, the IVFS which takes the value $[a,b] \in D(I)$ at p and $\mathbf{0}$ elsewhere in X is called an *interval-valued fuzzy point*(briefly, an IVFP) and is denoted by $p_{[a,b]}$. In particular, if $b = a$, it is also denoted by $p_{\boldsymbol{a}}$. We will denote by $D(I)^X$ and $\text{IVF}_P(X)$ the set of all IVFS_S and the set of all IVF points in X by $D(I)^X$ and $\text{IVF}_P(X)$, respectively.

Notation. Let $X = \{x_1, x_2, ..., x_n\}$. Then $A = ([a_1, b_1], [a_2, b_2], ..., [a_n, b_n])$ denotes an IVFS in X such that $A^L(x_i) = a_i$ and $A^U(x_i) = b_i$, for all $i = 1, 2, ..., n$.

Definition 2.2[16]. Let $A, B \in D(I)^X$. Then:

(a) $A \subset B$ iff $A^L(x) \leq B^L(x)$ and $A^U(x) \leq B^U(x)$ for all $x \in X$.

(b) $A = B$ iff $A \subset B$ and $B \subset A$.

(c) The *complement* A^c of A is defined by $A^c = [1 - A^U(x),\ 1 - A^L(x)]$ for all $x \in X$.

(d) If $\{A_\alpha : \alpha \in \Gamma\}$ is an arbitrary subset of $D(I)^X$, then

$$\bigcap A_\alpha(x) = [\ \bigwedge_{\alpha\in\Gamma} A^L_\alpha(x),\ \bigwedge_{\alpha\in\Gamma} A^U_\alpha(x)\ ],$$

$$\bigcup A_\alpha(x) = [\ \bigvee_{\alpha\in\Gamma} A^L_\alpha(x),\ \bigvee_{\alpha\in\Gamma} A^U_\alpha(x)\ ].$$

Definition 2.3[16]. Let $T \subset D(I)^X$. Then T is called an *interval-valued fuzzy topology*(in short, $IVFT$) on X if it satisfies the following conditions:

(i) $\widetilde{0}, \widetilde{1} \in T$,

(ii) $A, B \in T \Rightarrow A \cap B \in T$,

(iii) $\{A_\alpha\}_{\alpha\in\Gamma} \subset T \Rightarrow \bigcup_{\alpha\in\Gamma} A_\alpha \in T$.

In this case, each member of T is called an *IVF open set* and the pair (X, T) is called an *interval-valued fuzzy topological space*(in short, $IVFTS$). $A \in D(I)^X$ is called *closed* in (X, T) if $A^c \in T$.

As in ordinary topologies, the indiscrete topology of IVF sets contains only $\widetilde{1}$ and $\widetilde{0}$, while the discrete topology of IVF sets contains all IVF sets. These two topologies are denoted by T^0 and T^1, respectively.

3. Interval-valued gradation of openness

Definition 3.1[7,18]. A mapping $\tau : I^X \to I$ is called a *gradation of openness* (in short, *GO*) or a *smooth topology* on X if it satisfies the following conditions:

(GO1) $\tau(\emptyset) = \tau(X) = 1$,

(GO2) $\tau(A) \geq r$ and $\tau(B) \geq r \Rightarrow \tau(A \cap B) \geq r$, for any $A, B \in I^X$,

(GO3) $\tau(A_\alpha) \geq r,\ \forall \alpha \in \Gamma \Rightarrow \tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq r$, for any $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$,

where $r \in I_0$; or equivalently:

(GO1)$'$ $\tau(\emptyset) = \tau(X) = 1$,

(GO2)$'$ $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$, for any $A, B \in I^X$, (GO3)$'$ $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$, for any $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.

The pair (X,τ) is called a *smooth topological space*(in short, STS).

Definition 3.2[14]. A complex mapping $\tau = (\mu_\tau, \nu_\tau) : I^X \to I \times I$ is called *an intuitionistic gradation of openness*(in short, IGO) an *intuitionistic smooth topology* on X if it satisfies the following conditions:

(IGO1) $\mu_\tau(A) + \nu_\tau(A) \leq 1$, for each $A \in I^X$,

(IGO2) $\tau(\emptyset) = \tau(X) = (1,0)$,

(IGO3) $\mu_\tau(A \cap B) \geq \mu_\tau(A) \wedge \mu_\tau(B)$ and $\nu_\tau(A \cap B) \leq \nu_\tau(A) \vee \nu_\tau(B)$, for any $A, B \in I^X$,

(IGO4) $\mu_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mu_\tau(A_\alpha)$ and $\nu_\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \leq \bigvee_{\alpha\in\Gamma} \nu_\tau(A_\alpha)$, for any $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.

The triple (X, μ_τ, ν_τ) is called an *intuitionistic smooth topological space*(in short, $ISTS$), and μ_τ and ν_τ may be interpreted as gradation of openness and nonopenness, respectively.

Definition 3.3. A mapping $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ is called an *interval-valued gradation of openess*(in short, $IVGO$) or an *interval-valued smooth topology* on X if it satisfies the following conditions:

(IVGO1) $\tau^L(A) \leq \tau^U(A)$, for each $A \in I^X$,

(IVGO2) $\tau(\emptyset) = \tau(X) = \mathbf{1}$,

(IVGO3) $\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B)$ and $\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B)$, for any $A, B \in I^X$,

(IVGO4) $\tau^L(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau^L(A_\alpha)$ and $\tau^U(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau^U(A_\alpha)$, for any $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.

The pair (X,τ) is called an *interval-valued smooth topological space*(in short, $IVSTS$).

We will denote the set of all GOs[resp. IGOs and IVGOs] on X as GO(X)[resp, IGO(X) and IVGO(X)].

Example 3.3. (a) Let T be the topology on $\mathbb{R}$ generated by $\mathcal{B} = \{(a,b] : a, b \in \mathbb{R}$ and $a < b\}$ as a subbase, and let T_o be the family of all open sets in $\mathbb{R}$ with respect to (in short, w.r.t.) the usual topology on $\mathbb{R}$, where $\mathbb{R}$ denotes the set of all real numbers. We define the mapping $\tau = [\tau^L, \tau^U] : I^{\mathbb{R}} \to D(I)$ as follows: For each $A \in I^X$,

$$\tau(A)=\begin{cases}\mathbf{1} & \text{if } A\in T_o,\\ [0.5,0.7] & \text{if } A\in T\backslash T_o,\\ \mathbf{0} & \text{otherwise.}\end{cases}$$

Then it can easily seen that $\tau\in$IVGO(X).

(b) Let $a<b$ in $\mathbb{R}$ and let $\lambda\in I_o$. We define the mapping $A:\mathbb{R}\to I$ as follows: For each $x\in\mathbb{R}$,

$$A(x)=\begin{cases}1 & \text{if } x\in(a,b),\\ \lambda & \text{if } x=b,\\ 0 & \text{othewise.}\end{cases}$$

Then clearly $A\in I^X$ and we write $A=(a,b)_\lambda$. Let $\mathcal{B}=\{(a,b)_\lambda : a,b\in\mathbb{R},\ a<b \text{ and } \lambda\in I_o\}$, let T be the chang's fuzzy topology generated by $\mathcal{B}$ as a subbase and let $T_o=\{\chi_o : O \text{ is an open set in } \mathbb{R}\}$. Any $A\in T\backslash T_o$ can be expressed as

$$A=\bigcup_{\alpha\in\Gamma}A_\alpha \quad (3.1)$$

where $A_\alpha=(a_\alpha,b_\alpha)_\lambda$ and Γ is countable. We define the mapping $\tau=[\tau^L,\tau^U]: I^{\mathbb{R}}\to D(I)$ as follows: For each $A\in I^X$,

$$\tau(A)=\begin{cases}\mathbf{1} \text{ if } A\in T_o,\\ [1\text{-}0.5\lambda,0.7\lambda] \text{ if } A=(a,b)_\lambda,\\ [\bigwedge_{\alpha\in\Gamma}\tau^L(A_\alpha),\bigwedge_{\alpha\in\Gamma}\tau^U(A_\alpha)] \text{ if } A \text{ is expressed in the form (3.1)},\\ \mathbf{0} \text{ otherwise.}\end{cases}$$

Then we can easily see that $\tau\in$ IVGO(X). □

The following is the immediate result of Definitions 3.1, 3.2 and 3.3.

Proposition 3.4. (a) If $\tau\in$GO(X), then $(\tau,\tau^c)\in$ IGO(X) and $\tau=[\tau,\tau]\in$ IVGO(X), where $\tau^c(A)=1-\tau(A),\ \forall\ A\in I^X$.

(b) If $\tau\in$ IGO(X) [resp. IVGO(X)], then $\mu_\tau,\nu_\tau^c\in$GO(X) [resp. $\tau^L,\tau^U\in$ GO(X)].

Proposition 3.5. We define two mappings f :IVGO$(X)\to$ IGO(X) and g :IGO$(X)\to$ IVGO(X) as follows, respectively:

$$f(\tau)=f([\tau^L,\tau^U])=(\tau^L,(\tau^U)^c),\ \forall\ \tau\in \text{IVGO}(X)$$

and

$$g(\tau)=g((\mu_\tau,\nu_\tau))=[\mu_\tau,\nu_\tau^c],\ \forall\ \tau\in \text{IGO}(X).$$

Then $g \circ f = 1_{\mathrm{IVGO(X)}}$ and $f \circ g = 1_{\mathrm{IGO(X)}}$.

Proof. It can be easily seen that f and g are functions. Let $\tau \in \mathrm{IVGO}(X)$. Then

$g \circ f(\tau) = g((\tau^L, (\tau^U)^c))$
$= [\tau^L, ((\tau^U)^c)^c]$
$= [\tau^L, \tau^U] = \tau = 1_{\mathrm{IVGO(X)}}$

Now let $\tau \in \mathrm{IGO}(X)$. Then

$f \circ g(\tau) = f([\mu_\tau, (\nu_\tau)^c])$
$= (\mu_\tau, ((\nu_\tau)^c)^c)$
$= (\mu_\tau, \nu_\tau) = \tau = 1_{\mathrm{IGO(X)}}$.

This completes the proof. □

Remark 3.5. Proposition 3.5 shows the concepts of IVGO and IGO to be equipollent generalizations of one of GO.

Definition 3.6[7]. A mapping F : $I^X \to I$ is called a *gradation of closedness*(in short, *GC*) or a *smooth cotopology* on X if it satisfies the following conditions:

(GC1) $\mathcal{F}(\emptyset) = \mathcal{F}(X) = 1$,
(GC2) $\mathcal{F}(A) \geq r$ and $\mathcal{F}(B) \geq r \Rightarrow \mathcal{F}(A \cup B) \geq r$, for any $A, B \in I^X$,
(GC3) $\mathcal{F}(A_\alpha) \geq r,\ \forall\ \alpha \in \Gamma \Rightarrow \mathcal{F}(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq r$, for any $\{A_\alpha\} \subset I^X$,

where $r \in I_o$; or equivalently:

$(\mathrm{GC1})'$ $\mathcal{F}(\emptyset) = \mathcal{F}(X) = 1$,
$(\mathrm{GC2})'$ $\mathcal{F}(A \cup B) \geq \mathcal{F}(A) \cap \mathcal{F}(B)$, for any $A, B \in I^X$,
$(\mathrm{GC3})'$ $\mathcal{F}(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \mathcal{F}(A_\alpha)$, for any $\{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$.

Definition 3.7[14]. A complex mapping $\mathcal{F} = (\mu_{\mathcal{F}}, \nu_{\mathcal{F}}) : I^X \to I \times I$ is called an *intuitionistic gradation of closedness*(in short, *IGC*) an *intuitionistic smooth cotopology* on X if it satisfies the following conditions:

(IGC1) $\mu_{\mathcal{F}}(A) + \nu_{\mathcal{F}}(A) \leq 1$, for each $A \in I^X$,
(IGC2) $\mathcal{F}(\emptyset) = \mathcal{F}(X) = (1, 0)$,
(IGC3) $\mu_{\mathcal{F}}(A \cup B) \geq \mu_{\mathcal{F}}(A) \wedge \mu_{\mathcal{F}}(B)$ and $\nu_{\mathcal{F}}(A \cup B) \leq \nu_{\mathcal{F}}(A) \vee \nu_{\mathcal{F}}(B)$, for any $A, B \in I^X$.
(IGC4) $\mu_{\mathcal{F}}(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \mu_{\mathcal{F}}(A_\alpha)$ and $\nu_{\mathcal{F}}(\bigcap_{\alpha \in \Gamma} A_\alpha) \leq \bigvee_{\alpha \in \Gamma} \nu_{\mathcal{F}}(A_\alpha)$, for any $\{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$.

Definition 3.8. A mapping $\mathcal{F} = [\mathcal{F}^L, \mathcal{F}^U] : I^X \to D(I)$ is called an *interval-valued gradation of closedness*(in short, *IVGC*) an *interval-valued smooth cotopology* on X if it satisfies the following conditions:
(IVGC1) $\mathcal{F}^L(A) \leq \mathcal{F}^U(A)$, for each $A \in I^X$,
(IVGC2) $\mathcal{F}(\emptyset) = \mathcal{F}(X) = 1$,
(IVGC3) $\mathcal{F}^L(A \cup B) \geq \mathcal{F}^L(A) \wedge \mathcal{F}^L(B)$ and $\mathcal{F}^U(A \cup B) \geq \mathcal{F}^U(A) \wedge \mathcal{F}^U(B)$, for any $A, B \in I^X$,
(IVGC4) $\mathcal{F}^L(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mathcal{F}^L(A_\alpha)$ and $\mathcal{F}^U(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mathcal{F}^U(A_\alpha)$, for any $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.

We will denote the set of all GCs[resp. IGCs and IVGCs] an X as GC(X)[resp. IGC(X) and IVGC(X)].

The following is the generalization of Propositions 2.3, 2.4 and Corollary 2.5 in[7], as well as the analogue to Theorem 2.6 in[14].

Proposition 3.9. (a) For each $\tau \in$ IVGO(X), we define the mapping $\mathcal{F}_\tau : I^X \to D(I)$ as follows: For each $A \in I^X$,
$\mathcal{F}_\tau(A) = \tau(A^c)$.
Then $\mathcal{F}_\tau \in$ IVGC(X).
(b) For each $\mathcal{F} \in$ IVGC(X), we define the mapping $\tau_{\mathcal{F}} : I^X \to D(I)$ as follows: For each $A \in I^X$,
$\tau_{\mathcal{F}}(A) = \mathcal{F}(A^c)$.
Then $\tau_{\mathcal{F}} \in$ IVGO(X).
(c) $\tau_{\mathcal{F}_\tau} = \tau$ and $\mathcal{F}_{\tau_{\mathcal{F}}} = \mathcal{F}$.

Proof. (a) It is clear that $\mathcal{F}_\tau$ satisfies the conditions (IVGC1) and (IVGC2). Let $A, B \in I^X$. Then
$\mathcal{F}^L_\tau(A \cup B) = \tau^L((A \cup B)^c) = \tau^L(A^c \cap B^c)$
$\geq \tau^L(A^c) \wedge \tau^L(B^c)$ [By the condition (IVGO3)]
$= \mathcal{F}^L_\tau(A) \wedge \mathcal{F}^L_\tau(B)$. [By the definition of $\mathcal{F}_\tau$]
Similarly, we have $\mathcal{F}^U_\tau(A)(A \cup B) \geq \mathcal{F}^U_\tau(A) \wedge \mathcal{F}^U_\tau(B)$. Thus $\mathcal{F}_\tau$ satisfies the condition (IVGC3). Now let $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$. Then
$\mathcal{F}^L_\tau(\bigcap_{\alpha\in\Gamma} A_\alpha) = \tau^L((\bigcap_{\alpha\in\Gamma} A_\alpha)^c) = \tau^L(\bigcup_{\alpha\in\Gamma} A^c_\alpha)$
$\geq \bigwedge_{\alpha\in\Gamma} \tau^L(A^c_\alpha)$ [By the condition (IVGO4)]
$= \bigwedge_{\alpha\in\Gamma} \mathcal{F}^L_\tau(A^c_\alpha)$, [By the definition of $\mathcal{F}_\tau$]

Similarly, we have $\mathcal{F}_\tau^L(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \mathcal{F}_\tau^L(A_\alpha)$. So $\mathcal{F}_\tau$ satisfies the condition (IVGC4). Hence $\mathcal{F}_\tau \in \mathrm{IVGC}(X)$.

The proof of (b) is similar to one of (a) and (c) are the immediate results of the definitions of $\mathcal{F}_\tau$ and $\tau_{\mathcal{F}}$. □

Definition 3.10. Let $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset \mathrm{IVGO}(X)$. Then the *intersection* of $\{\tau_\alpha\}_{\alpha\in\Gamma}$, denoted by $\bigcap_{\alpha\in\Gamma} \tau_\alpha$, is defined as follows: For each $A \in I^X$,

$$(\bigcap_{\alpha\in\Gamma} \tau_\alpha)(A) = [\bigwedge_{\alpha\in\Gamma} \tau_\alpha^L(A), \bigwedge_{\alpha\in\Gamma} \tau_\alpha^U(A)].$$

The following is the immediate result of Definitions 3.3 and 3.10.

Proposition 3.11. Let $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset \mathrm{IVGO}(X)$. Then $\bigcap_{\alpha\in\Gamma} \tau_\alpha \in \mathrm{IVGO}(X)$.

Definition 3.12. We define a relation " $\leq$ " on $\mathrm{IVGO}(X)$ as follows:
$\tau \leq \eta \Leftrightarrow \tau^L \leq \eta^L$ and $\tau^U \leq \eta^U$, for any $\tau, \eta \in \mathrm{IVGO}(X)$.

It can be easily seen that $(\mathrm{IVGO}(X), \leq)$ is a partially ordered set.

Remark 3.13. We define two mappings $\tau_{\mathbf{0}}, \tau_{\mathbf{1}} : I^X \to D(I)$ as follows: For each $A \in I^X$,

$$\tau_{\mathbf{0}}(A) = \begin{cases} \mathbf{1} \text{ if } A = \varnothing \textit{ or } A = X, \\ \mathbf{0} \text{ if } A \in I^X \backslash \{\varnothing, X\} \end{cases}$$

and

$$\tau_{\mathbf{1}}(A) = \mathbf{1}.$$

Then we can easily see that $\tau_{\mathbf{0}}, \tau_{\mathbf{1}} \in \mathrm{IVGO}(X)$ and $\tau_{\mathbf{0}} \leq \tau \leq \tau_{\mathbf{1}},\ \forall\ \tau \in \mathrm{IVGO}(X)$.

The followings is the immediate result of Proposition 3.11 and Remark 3.13.

Proposition 3.14. $(\mathrm{IVCO}(X), \leq)$ is a complete lattice with the smallest element $\tau_{\mathbf{0}}$ and the largest element $\tau_{\mathbf{1}}$.

Proposition 3.15. Let (X, τ) be an IVFTS, where $\tau \in \mathrm{IVGO}(X)$ and let $[\lambda, \mu] \in D(I)$. Then

$$\tau_{[\lambda,\mu]} = \{A \in I^X : \tau(A) \geq [\lambda, \mu],\ \textit{i.e.},\ \tau^L(A) \geq \lambda \text{ and } \tau^U(A) \geq \mu\}$$

is a Chang's fuzzy topology on X. In this case, $\tau_{[\lambda,\mu]}$ [resp. $\tau_{\boldsymbol{\lambda}}$] is called the $[\lambda,\mu]$-*level* [resp. $\boldsymbol{\lambda}$-*level*] *Chang's fuzzy topology on X w.r.t.* τ.

Proof. Since $\tau \in \text{IVGO}(X)$, $\tau(\emptyset) = \tau(X) = \mathbf{1}$. Then

$$\tau^L(\emptyset) = 1 \geq \lambda,\ \tau^U(\emptyset) = 1 \geq \mu$$

and

$$\tau^L(X) = 1 \geq \lambda,\ \tau^U(X) = 1 \geq \mu.$$

Thus $\varnothing, X \in \tau_{[\lambda,\mu]}$. Let $A, B \in \tau_{[\lambda,\mu]}$. Then

$$\tau^L(A) \geq \lambda,\ \tau^U(A) \geq \mu$$

and

$$\tau^L(B) \geq \lambda,\ \tau^U(B) \geq \mu.$$

Since $\tau \in \text{IVGO}(X)$,

$$\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B) \geq \lambda$$

and

$$\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B) \geq \mu.$$

Thus $A \cap B \in \tau_{[\lambda,\mu]}$. Now let $\{A_\alpha\}_{\alpha\in\Gamma} \subset \tau_{[\lambda,\mu]}$. Then

$$\tau^L(A_\alpha) \geq \lambda \text{ and } \tau^U(A_\alpha) \geq \mu,\ \forall\ \alpha \in \Gamma.$$

Since $\tau \in \text{IVGO}(X)$,

$$\tau^L(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau^L(A_\alpha) \geq \lambda$$

and

$$\tau^U(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau^U(A_\alpha) \geq \mu.$$

Thus $\bigcup_{\alpha\in\Gamma} A_\alpha \in \tau_{[\lambda,\mu]}$. So $\tau_{[\lambda,\mu]}$ is a Chang's fuzzy topology on X. By the process of the proof of $\tau_{[\lambda,\mu]}$, it is clear that $\tau_{\boldsymbol{\lambda}}$ is a Chang's fuzzy topology on X. □

Proposition 3.16. Let (X,τ) be an IVFTS and let $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu]\in D(I)}$ be the family of all $[\lambda,\mu]$-level Chang's fuzzy topologies w.r.t. τ. Then $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu]\in D(I)}$ is dscending and for each $[\lambda,\mu] \in D(I_o)$, $\tau_{[\lambda,\mu]} = \bigcap_{[a,b]<[\lambda,\mu]} \tau_{[a,b]}$. In this case, $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu]\in D(I_o)}$ is called the *family of Chang's fuzzy topologies associated with the gradation of* τ.

Proof. Suppose $[a,b] \leq [\lambda,\mu]$. Then clearly $\tau_{[\lambda,\mu]} \subset \tau_{[a,b]}$. Thus $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu]\in D(I)}$ is a descending family of Chang's fuzzy topologies. So

$$\tau_{[\lambda,\mu]} \subset \bigcap_{[a,b]<[\lambda,\mu]} \tau_{[a,b]}, \text{ for each } [\lambda,\mu] \in D(I_o).$$

Assume that $A \notin \tau_{[\lambda,\mu]}$. Then $\tau^L(A) < \lambda$ or $\tau^U(A) < \mu$. Thus $\exists\ [a,b] \in D(I_o)$ such that $\tau^L(A) < a < \lambda$ or $\tau^U(A) < b < \mu$. So $A \notin \bigcap_{[a,b]<[\lambda,\mu]} \tau_{[a,b]}$. Hence $\bigcap_{[a,b]<[\lambda,\mu]} \tau_{[a,b]} \subset \tau_{[\lambda,\mu]}$. Therefore $\tau_{[\lambda,\mu]} = \bigcap_{[a,b]<[\lambda,\mu]} \tau_{[a,b]}$. □

The following is the immediate result of Proposition 3.16.

Corollary 3.16. Let (X, τ) be an IVFTS and let $\{\tau_{\mathbf{r}}\}_{\mathbf{r}\in D(I)}$ be the family of all **r**-level Chang's fuzzy topologies w.r.t. τ. Then $\{\tau_{\mathbf{r}}\}_{\mathbf{r}\in D(I)}$ is descending and for each $\mathbf{r} \in D(I_o)$, $\tau_{\mathbf{r}} = \bigcap_{\mathbf{s}<\mathbf{r}} \tau_{\mathbf{s}}$.

Proposition 3.17. Let $\{T_{[\lambda,\mu]}\}_{[\lambda,\mu]\in D(I_o)}$ be a nonempty descending family of Chang's fuzzy topologies on X. We define the mapping $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ as follows:

$$\tau(A) = \bigvee\{[\lambda, \mu] \in D(I_o) : A \in T_{[\lambda,\mu]}\},\ \forall\ A \in I^X.$$

Then $\tau \in \text{IVGO}(X)$. If, for each $[a, b] \in D(I_o)$,

$$T_{[\lambda,\mu]} = \bigcap_{[a,b]<[\lambda,\mu]} T_{[a,b]}, \quad (3.2)$$

then $\tau_{[\lambda,\mu]} = T_{[\lambda,\mu]}$ for each $[\lambda, \mu] \in D(I_o)$.

Proof. Since $T_{[\lambda,\mu]}$ is a Chang's fuzzy topology on X, $\emptyset,\ X \in T_{[\lambda,\mu]}$. Then, by the definition of τ,

$$\tau(\emptyset) = \tau(X) = \mathbf{1}.$$

Furthermore, $\tau^L(A) \le \tau^U(A)$, for each $A \in I^X$. Thus τ satisfies the conditions (IVGO1) and (IVGO2).

For any $A_i \in I^X$, let $\tau(A_i) = [a_i, b_i]$ for $i = 1, 2$. Suppose $\tau(A_i) = \mathbf{0}$ for some i. Then clearly

$$\tau(A_1 \cap A_2) \ge \tau(A_1) \wedge \tau(A_2).$$

Thus, without loss of generality, suppose $[a_i, b_i] > \mathbf{0}$ for $i = 1, 2$. Let $[s, t] \le \tau(A_i)$ for $i = 1, 2$ and let $\varepsilon > 0$. Then, by the definition of τ,

$\exists[\lambda_1, \mu_1], [\lambda_2, \mu_2] \in D(I_o)$ such that

$a_i - \varepsilon < \lambda_i \le a_i,\ b_i - \varepsilon < \mu_i \le b_i$ and $A_i \in T_{[\lambda_i,\mu_i]}$ for $i = 1, 2$.

Let $[\lambda, \mu] = [\lambda_1, \mu_1] \wedge [\lambda_2, \mu_2]$ and let $[a, b] = [a_1, b_1] \wedge [a_2, b_2]$. Then clearly $A_1, A_2 \in T_{[\lambda,\mu]}$. Thus $A_1 \cap A_2 \in T_{[\lambda,\mu]}$. So

$$\tau^L(A_1 \cap A_2) \ge \lambda > a - \varepsilon > s - \varepsilon$$

and

$$\tau^U(A_1 \cap A_2) \ge \mu > b - \varepsilon > t - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\tau^L(A_1 \cap A_2) \geq s \text{ and } \tau^U(A_1 \cap A_2) \geq t.$$

Hence $\tau(A_1, A_2) \geq [s, t]$, i.e., $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$. Therefore τ satisfies the condition (IVGO3).

Now suppose $\tau(A_\alpha) = [l_\alpha, m_\alpha]$ for each $\alpha \in \Gamma$ and let $[l, m] = \bigwedge_{\alpha\in\Gamma} [l_\alpha, m_\alpha]$. Suppose $[l, m] = \mathbf{0}$. Then it is obvious that

$$\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha).$$

Suppose $[l, m] > 0$ and let $[l, m] > \boldsymbol{\varepsilon} > \mathbf{0}$. Then $0 < l - \varepsilon < l_\alpha$ and $0 < m - \varepsilon < m_\alpha$ for each $\alpha \in \Gamma$. Thus $A_\alpha \in T_{[l-\varepsilon, m-\varepsilon]}$, $\forall\ \alpha \in \Gamma$. Since $T_{[l-\varepsilon,m-\varepsilon]}$ is a Chang's fuzzy topology, $\bigcup_{\alpha\in\Gamma} A_\alpha \in T_{[l-\varepsilon,m-\varepsilon]}$. So

$$\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq [l - \varepsilon, m - \varepsilon].$$

Since $\varepsilon > 0$ is arbitrary,

$$\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq [l, m] = \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha).$$

Hence τ satisfies the condition (IVGO.4). Therefore $\tau \in \text{IVGO}(X)$.

Finally, suppose $\{T_{[\lambda,\mu]}\}_{[\lambda,\mu]\in D(I_o)}$ satisfies the condition (3.2) and let $A \in T_{[\lambda,\nu]}$. Then clearly $\tau(A) \geq [\lambda, \mu]$. Thus $A \in \tau_{[\lambda,\mu]}$. So $T_{[\lambda,\mu]} \subset \tau_{[\lambda,\mu]}$. Now let $A \in \tau_{[\lambda,\mu]}$. Then $\tau(A) \geq [\lambda, \mu]$. Thus, by the definition of τ,

$$\bigvee\{[a, b] \in D(I_o) : A \in T_{[a,b]}\} = [s, t] \geq [\lambda, \mu].$$

Let $\varepsilon > 0$. Then $\exists [a, b] \in D(I_o)$ such that

$$s - \varepsilon < a,\ t - \varepsilon < b \text{ and } A \in T_{[a,b]}.$$

Thus

$$\lambda - \varepsilon \leq s - \varepsilon < a,\ \mu - \varepsilon \leq t - \varepsilon < b \text{ and } A \in T_{[a,b]}.$$

So $A \in T_{[\lambda-\varepsilon,\mu-\varepsilon]}$. Since $\varepsilon > 0$ is arbitrary, by the condition (3.2), $A \in T_{[\lambda,\mu]}$. Hence $\tau_{[\lambda,\mu]} \subset T_{[\lambda,\mu]}$. Therefore $\tau_{[\lambda,\mu]} = T_{[\lambda,\mu]}$. This completes the proof. □

The followings are the immediate results of Corollary 3.16 and Proposition 3.17.

Corollary 3.17-1. Let $\tau, \eta \in \text{IVGO}(X)$. Then $\tau = \eta$ if and only if $\tau_{[\lambda,\mu]} = \eta_{[\lambda,\mu]}$, $\forall [\lambda, \mu] \in D(I_o)$.

Corollary 3.17-2. Let $\{T_{\mathbf{r}}\}_{\mathbf{r}\in D(I_o)}$ be a nonempty dscending family of Chang's fuzzy topologices on X and let $\tau : I^X \to D(I)$ be a mapping defined as follows: For each $A \in I^X$,

$$\tau(A) = \bigvee\{\mathbf{r} \in D(I_o) : A \in T_{\mathbf{r}}\}.$$

Then $\tau \in \mathrm{IVGO}(X)$. If, for each $\mathbf{r} \in D(I_o)$,

$$T_{\mathbf{r}} = \bigcap_{\mathbf{s}<\mathbf{r}} T_{\mathbf{s}}$$

then $\tau_{\mathbf{r}} = T_{\mathbf{r}}$ for all $\mathbf{r} \in D(I_o)$.

Proposition 3.18. Let (X, T) be a Chang's fuzzy topological space. For each $[\lambda, \mu] \in D(I_o)$, we define a mapping $T^{[\lambda,\mu]} : I^X \to D(I)$ as follows: For each $A \in I^X$,

$$T^{[\lambda,\mu]}(A) = \begin{cases} \mathbf{1} \text{ if } A = \emptyset \textit{ or } A = X, \\ [\lambda, \mu] \text{ if } A \in T\backslash\{\emptyset, X\}, \\ \mathbf{0} \text{ otherwise.} \end{cases}$$

Then $T^{[\lambda,\mu]} \in \mathrm{IVGO}(X)$ such that $(T^{[\lambda,\mu]})_{[\lambda,\mu]} = T$.

In this case, $T^{[\lambda,\mu]}$ [resp. $T^{\boldsymbol{\lambda}}$] is called a $[\lambda, \mu]$-*th* [resp. $\boldsymbol{\lambda}$-*th*] *interval-valued gradation* [in short, *IVG*] an X, and $(X, T^{[\lambda,\mu]})$[resp. $(X, T^{\boldsymbol{\lambda}})$] is called a $[\lambda, \mu]$-*th* [resp. $\boldsymbol{\lambda}$-*th*] *interval-valued graded fuzzy topological space.*

Proof. By the definition of $T^{[\lambda,\mu]}$, $T^{[\lambda,\mu]^L}(A) \le T^{[\lambda,\mu]^U}(A),\ \forall\ A \in I^X$. Then (IVGO1) holds. Also, it is clear that (IVGO2) holds.

Let $A_i \in I^X,\ i = 1, 2$. Suppose $A_i = \varnothing$ for some i. Then $A_1 \cap A_2 = \emptyset$. Thus

$$T^{[\lambda,\mu]}(A_1 \cap A_2) = \mathbf{1} \ge T^{[\lambda,\mu]}(A_1) \wedge T^{[\lambda,\mu]}(A_2).$$

Suppose $A_i = X$, for some i(say A_1). Then $A_1 \cap A_2 = A_2$. Thus

$$T^{[x,\mu]}(A_1 \cap A_2) = T^{[\lambda,\mu]}(A_2) \ge T^{[\lambda,\mu]}(A_1) \wedge T^{\lambda,\mu}(A_2).$$

Suppose $A_1, A_2 \in T \setminus \{\emptyset, X\}$. Then $A_1 \cap A_2 \in T$. Thus

$$T^{[x,\mu]}(A_1 \cap A_2) \ge [\lambda, \mu] = T^{[\lambda,\mu]}(A_1) \wedge T^{[\lambda,\mu]}(A_2).$$

Suppose $A_i \in I^X - T$ for some i (say A_1) Then $T^{[\lambda,\mu]}(A_1) = \mathbf{0}$. Thus

$$T^{[\lambda,\mu]}(A_1 \cap A_2) \ge \mathbf{0} = T^{[\lambda,\mu]}(A_1) \wedge T^{[\lambda,\mu]}(A_2).$$

In all cases, $T^{[\lambda,\mu]}$ satisfies the condition (IVGO3).

Let $\{A_\alpha\}_{\alpha\in P} \subset I^X$. Suppose $A_{\alpha_0} = \emptyset$ for some $\alpha_0 \in \Gamma$. Then

$$\bigcup_{\alpha\in\Gamma} A_\alpha = \bigcup_{\alpha\in\Gamma(\alpha\neq\alpha_0)} A_\alpha.$$

Thus

$$\bigwedge_{\alpha\in\Gamma} T^{[\lambda,\mu]}(A_\alpha) = \bigwedge_{\alpha\in\Gamma(\alpha\neq\alpha_0)} T^{[\lambda,\mu]}(A_\alpha).\ [\text{Since } T^{\lambda,\mu}(A_{\alpha_0}) = \mathbf{1}]$$

So, without loss of generality, assume that $A_\alpha \neq \emptyset\ \forall\alpha \in \Gamma$.
Suppose $A_{\alpha_0} = X$ for some $\alpha_0 = \Gamma$. Then

$$T^{[\lambda,\mu]}(\bigcup_{\alpha\in\Gamma} A_\alpha) = T^{[\lambda,\mu]}(X) = \mathbf{1} \geq \bigwedge_{\alpha\in\Gamma} T^{[\lambda,\mu]}(A_\alpha).$$

Suppose $A_\alpha \in T\setminus\{\emptyset, X\}\forall\alpha \in \Gamma$. Then clearly $(\bigcup_{\alpha\in\Gamma} A_\alpha \in T$. Thus

$$T^{[\lambda,\mu]}(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq [\lambda,\mu] = (\bigwedge_{\alpha\in\Gamma} T^{[\lambda,\mu]}(A_\alpha).$$

Suppose $A_{\alpha_0} \in I^X - T$ for some $\alpha_0 \in \Gamma$. Then

$$T^{[\lambda,\mu]}(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \mathbf{0} = T^{[\lambda,\mu]}(A_{\alpha_0}) = \bigwedge_{\alpha\in\Gamma} T^{[\lambda,\mu]}(A_\alpha).$$

In all cases, $T^{[\lambda,\mu]}$ satisfies the condition (IVGO4). Hence $T^{[\lambda,\mu]} \in$ IVGO(X).

By the above result and Proposition 3.15,

$(T^{[\lambda,\mu]})_{[\lambda,\mu]} = \{A \in I^X : T^{[\lambda,\mu]}(A) \geq [\lambda,\mu]\}$=$T$.

From the process of the above proof, it can be easily seen that the remainder holds. □

4. Interval-valued gradation of clopenness

Definition 4.1. A mapping $\tau : I^X \to D(I)$ is called an *interval-valued gradation of clopenness* (in short, IVGCO) on X if $\tau \in$ IVGO$(X)\cap$ IVGC(X). We will denote the set of all IVGCOs on X as IVGCO(X). It is clear that $\tau_{\mathbf{0}}, \tau_{\mathbf{1}} \in$ IVGCO(X).

Example 4.1. Let $[\lambda,\mu] \in D(I)$ be fixed. We define the mapping $\tau : I^X \to D(I)$ as follows : For each $A \in I^X$,

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A = \emptyset \text{ or } A = X, \\ [\lambda,\mu] & \text{if } A \neq \emptyset \text{ and } A \neq X. \end{cases}$$

Then it is obvious that $\tau \in$ IVGCO(X). In this case, τ is called an *interval-valued constant gradation* and we will denote it by $[\lambda,\mu]$. □

The following is the characterization of IVGCO.

Theorem 4.2. $\tau \in$ IVGCO(X) if and only if

(i) $\tau^L(A) \leq \tau^U(A), \forall A \in I^X$,
(ii) $\tau(\emptyset) = \tau(X) = \mathbf{1}$,
(iii) $\tau(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$, $\forall \{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$,
(iv) $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$, $\forall \{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.

Proof. From Definitions 3.3, 3.8 and 4.1, it is obvious. □

Definition 4.3. In Proposition 3.9, for each $\tau \in$IVGO(X), $\mathcal{F}_\tau$ is called *an inter- val-valued conjugate gradation* of τ. By Proposition 3.9(c), τ is the interval-valued conjugate gradation of $\mathcal{F}_\tau$.

It is clear that if $\tau \in$IVGCO(X), $\mathcal{F}_\tau = \tau$.

The following gives a nice IVGCO.

Proposition 4.4. We define the mapping $\sigma : I^X \to D(I)$ as follows:

$$\sigma(A) = \begin{cases} \mathbf{1} & \text{if } A = \emptyset, \\ [\bigwedge_{x\in \mathrm{supp(A)}} A(x), \bigwedge_{x\in \mathrm{supp(A)}} A(x)] & \text{if } A \neq \emptyset, \end{cases}$$

for each $A \in I^X$, where supp(A)=$\{x \in X : A(x) > 0\}$. Then $\sigma \in$IVGCO(X). In this case, σ is called the *interval-valued support gradation.*

Proof. It is obvious that $\sigma(\emptyset) = \sigma(X) = \mathbf{1}$ and $\sigma^L(A) \leq \sigma^U(A)$ for each $A \in I^X$.

Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$, let $\lambda = \sigma(\bigcup_{\alpha\in\Gamma} A_\alpha)$ and let $\lambda_\alpha = \sigma(A_\alpha)$ $\forall \alpha \in \Gamma$.

Suppose $\bigwedge_{\alpha\in\Gamma} \lambda_\alpha = \mu > \lambda$ and let $x \in \mathrm{supp}(\bigcup_{\alpha\in\Gamma} \mathrm{A}_\alpha)$. Since $\mathrm{supp}(\bigcup_{\alpha\in\Gamma} \mathrm{A}_\alpha) = \bigcup_{\alpha\in\Gamma} \mathrm{supp}(\mathrm{A}_\alpha)$, $\exists \alpha_0 \in \Gamma$ such that $x \in$ supp (A_{α_0}). Thus

$A_{\alpha_0}(x) \geq \bigwedge\{A_{\alpha_0}(y) : y \in \mathrm{supp}(\mathrm{A}_{\alpha_0})\} = \lambda_{\alpha_0} \geq \mu$.

So $(\bigcup_{\alpha\in\Gamma} A_\alpha)(x) \geq \mu$ and hence $\sigma(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \mu$. This is a contradiction from the fact that $\sigma(\bigcup_{\alpha\in\Gamma} A_\alpha) = \lambda < \mu$. Therefore $\sigma(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \sigma(A_\alpha)$

Now let $\lambda = \bigwedge\{(\bigcap_{\alpha\in\Gamma} A_\alpha)(x) : x \in \mathrm{supp}(\bigcap_{\alpha\in\Gamma} \mathrm{A}_\alpha)\}$. Then

$$\lambda = \bigwedge\{\bigwedge_{\alpha\in\Gamma} A_\alpha(x) : x \in \mathrm{supp}(\bigcap_{\alpha\in\Gamma} \mathrm{A}_\alpha)\}$$
$$= \bigwedge_{\alpha\in\Gamma} (\bigwedge\{A_\alpha(x) : x \in \mathrm{supp}(\bigcap_{\alpha\in\Gamma} \mathrm{A}_\alpha)\})$$
$$\geq \bigwedge_{\alpha\in\Gamma} (\bigwedge\{A_\alpha(x) : x \in \mathrm{supp}(\mathrm{A}_\alpha)\}).$$

Thus, by the definition of σ,

$$\sigma(\bigcap_{\alpha\in\Gamma} A_\alpha) = \lambda$$
$$\geq \bigwedge_{\alpha\in\Gamma} [\bigwedge\{A_\alpha(x) : x \in \mathrm{supp}(\mathrm{A}_\alpha)\}, \bigwedge\{\mathrm{A}_\alpha(\mathrm{x}) : \mathrm{x} \in \mathrm{supp}(\mathrm{A}_\alpha)\}]$$
$$= \bigwedge_{\alpha\in\Gamma} \sigma(A_\alpha).$$

Hence $\sigma \in$IVGCO(X). □

Remark 4.4. Let σ be the IVGCO on X given by Proposition 4.4. Then its conjugate gradation $\mathcal{F}_\sigma$ is given by, for each $A \in I^X$,

$$\mathcal{F}_a(A) = \sigma(A^c)$$
$$= [\bigwedge\{A^c(x) : x \in \mathrm{supp}(A^c)\},\ \bigwedge\{A^c(x) : x \in \mathrm{supp}(A^c)\}]$$
$$= [\bigwedge\{1 - A(x) : A(x) \neq 0\},\ \bigwedge\{1 - A(x) \neq 0\}]$$
$$= [1 - \bigvee\{A(x) : A(x) \neq 0\},\ 1 - \bigvee\{A(x) : A(x) \neq 0\}].$$

Example 4.4. Let X be a set with two points at least. We define the mapping $\delta : I^X \to D(I)$ as follows : For each $A \in I^X$,

$$\sigma(A) = \begin{cases} \mathbf{1} & \text{if } A = \emptyset \text{ or } A = X \text{ or } \mathrm{supp}(A) = X, \\ 0 & \text{if } \mathrm{supp}(A) \neq X. \end{cases}$$

Then it can be easily seen that $\delta \in$IVGO(X). For a fixed point $p \in X$ and for n=1,2,$\cdots$, we define the mapping $G_n : X \to I$ as follows : For each $x \in X$,

$$G_n(x) = \begin{cases} \frac{1}{n} \text{ if } x \neq p, \\ 1 \text{ if } x = p. \end{cases}$$

Then clearly $\{G_n\}_{n\in\mathbb{N}} \subset I^X$ and $\delta(G_n) = \mathbf{1}\ \forall n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. But, $\delta(\bigcap_{n\in\mathbb{N}} G_n) = \mathbf{0}$. Thus $\delta(\bigcap_{n\in\mathbb{N}} G_n) <$

$\mathbf{1} = \bigwedge_{n\in\mathbb{N}} \delta(G_n)$. So $\delta \notin$IVGC(X). Hence $\delta \notin$IVGCO(X). □

The following gives a sufficient condition to be an IVGCO.

Proposition 4.5. Let $\delta : I^X \to D(I)$ be a mapping. Consider the following conditions :
(a) $\sigma^L(A) \leq \sigma^U(A), \forall A \in I^X$,
(b) $\sigma(\emptyset) = \mathbf{1}$,
(c) $\sigma(A) = \sigma(A^c), \forall A \in I^X$,
(d) $\sigma(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \sigma(A_\alpha)$, $\forall\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$,
(e) $\sigma(\bigcap_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \sigma(A_\alpha)$, $\forall\{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.
If σ satisfies the condition (a)∼(d) or (a)∼(c) and (e), then $\sigma \in$ IVGCO(X).

Proof. The condition (e) is deduced from the condition (b) and (c). Also the condition (d) is deduced from the condition (b) and (e). Hence, by Theorem 4.2, $\sigma \in$ IVGCO(X). □

The following is the immediate result of Theorem 4.2 and Proposition 4.5.

Corollary 4.5. If $\sigma \in$ IVGO(X) or $\sigma \in$IVGC(X), and $\sigma(A) = \sigma(A^c)$ for each $A \in I^X$, then $\sigma \in$ IVGCO(X).

The following is the immediate result of Definition 4.3 and Proposition 3.11.

Proposition 4.6. Let $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset$IVGC(X) [resp. IVGCO(X)]. Then $\bigcap_{\alpha\in\Gamma} \tau_\alpha \in$ IVGC(X) [resp. IVGCO(X)].

Definition 4.7. Let $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset$IVGO(X). Then the union of $\{\tau_\alpha\}_{\alpha\in\gamma}$, denoted by $\bigcup_{\alpha\in\Gamma} \tau_\alpha$, is defined as follows : For each $A \in I^X$,

$$(\bigcup_{\alpha\in\Gamma} \tau_\alpha)(A) = [\bigvee_{\alpha\in\Gamma} {\tau_\alpha}^L(A), \bigvee_{\alpha\in\Gamma} {\tau_\alpha}^U(A)].$$

The following example shows that the union of two IVGCOs is not, in general, an IVGO(IVGC) even they are conjugate.

Example 4.7. Let X be a set with two points at least. Let $\{M, N\}$ be a partition of X, let $\frac{1}{2} < \lambda < 1$ and let $\mu = 1 - \lambda$. Consider two fuzzy sets A and B in X defined as follows : For each $x \in X$,

$$A(x) = \begin{cases} \mathbf{0} & \text{if } x \in M \\ \lambda & \text{if } x \in N \end{cases}$$

and

$$B(x) = \begin{cases} \mu \text{ if } x \in M \\ 0 \text{ if } x \in N. \end{cases}$$

Then $A \cup B$ is the fuzzy set in X given by, for each $x \in X$,

$$(A \cup B)(x) = \begin{cases} \mu \text{ if } x \in M \\ \lambda \text{ if } x \in N. \end{cases}$$

Let σ be the interval-valued support gradation and let δ be its conjugate gradation. Then

$(\sigma \cup \delta)(A \cup B) = [\mu, \mu]$,

and

$(\sigma \cup \delta)(A) = [\lambda, \lambda]$, $(\sigma \cup \delta)(B) = [1 - \mu, 1 - \mu] = [\lambda, \lambda]$.

Since $\frac{1}{2} < \lambda < 1$ and $\mu = 1 - \lambda$, $\mu < \lambda$. Thus

$(\sigma \cup \delta)(A \cup B) = [\mu, \mu] < [\lambda, \lambda] = (\sigma \cup \delta)(A) \wedge (\sigma \cup \delta)(B)$

So $\sigma \cup \delta \notin$IVGCO(X). □

Definition 4.8[9]. Let (X, T) be a Chang's fuzzy topological space. Then the fuzzy space X (the fuzzy topology T) is said to be *interpreservative*[resp. *super 0-dimensional*] if the intersection of each family of open sets is open [resp. each open set is closed or equivalently if the family of closed sets in X agrees with T.

It is clear that if X is super 0-dimensional, then X is interpreservative.

Definition 4.9. Let $\sigma \in$IVGO(X) and let T be a Chang's fuzzy topology on X. We define the mapping $\sigma^* : I^X \to B(I)$ as follows : For each $A \in I^X$,

$$\sigma^*(A) = \begin{cases} \sigma(A) \text{ if } A \in T, \\ \mathbf{0} \text{ otherwise.} \end{cases}$$

Then σ^* is called the *deduced gradation from σ and T*.

It is clear that $\sigma^* \in$IVGO(X) and its $[\lambda,\mu]$-level $\sigma^*_{[\lambda,\mu]}$ is $\sigma^*_{[\lambda,\mu]} = \sigma_{[\lambda,\mu]} = \sigma_{[\lambda,\mu]} \cap T$ for each $[\lambda,\mu] \in D(I)$.

The following is the immediate result of Definitions 4.8 and 4.9.

Proposition 4.10. Let $\sigma \in$IVGO(X) and let T be a Chang′s fuzzy topology. Then

(a) If σ^* is deduced gradation from σ and T, then $\sigma^* \in$IVGCO(X).

(b) If δ is the conjugate gradation of σ and T is super 0-dimensional, then δ^* is the conjugate gradation of σ^* and hence $\delta^* \in$IVGCO(X).

Example 4.10. Let σ be the interval-valued support gradation on $\mathbb{R}$, let δ be its conjugate and let T be the laminated indiscrete topology on $\mathbb{R}$[13], i.e., T is constituted by the constant mappings on $\mathbb{R}$. Then clearly σ and T satisfies (b) of Proposiotion 4.10. Let $f_\alpha \in T$ be the constant mapping given by $f_\alpha(x) = \alpha$ for each $x \in X$. Then, the deduced gradation gradation σ^* from σ and T is given by : For each $A \in I^X$,

$$\sigma(A) = \begin{cases} \mathbf{1} \text{ if } A = \emptyset \text{ or } A = X, \\ [\alpha,\alpha] \text{ if } A = f_\alpha \in T \text{ and } \alpha \neq 0, \\ \mathbf{0} \text{ otherwise.} \end{cases}$$

Then

$\delta^*(\emptyset) = \delta^*(X) = \mathbf{1}$,

$\delta^*(f_\alpha) = \delta(f_\alpha) = \sigma(1 - f_\alpha) = [1-\alpha, 1-\alpha]$

$= \sigma^*(1 - f_\alpha) = \sigma^*({f_\alpha}^c)$, if $\alpha \neq 1$.

By the definition of T, it is clear that $A \in T$ if and only if $A^c(= 1 - A) \in T$. Thus, for $A \notin T$, $\delta^*(A) = \sigma^*(A^c) = \mathbf{0}$. So σ^* and δ^* are conjugate. □

Definition 4.11. Let $\tau, \eta \in$ IVGO(X). Then we say that τ *is equivalent to* η, $\mathcal{F}$ denoted by $\tau \approx \eta$, if their families $[\lambda,\mu]$-levels agree, i.e., $\{\tau_{[\lambda,\mu]}\}_{[\lambda,\mu]\in D(I)} = \{\eta_{[a,b]}\}_{[a,b]\in D(I)}$.

Proposition 4.12. Let $\sigma \in$IVGO(X) [resp. IVGC(X)] and let $\varphi : I \to I$ be an increasing continuous mapping with $\varphi(1) = 1$. Then $\varphi \circ \sigma = [\varphi \circ \sigma^L, \varphi \circ \sigma^U] \in$IVGO(X) [resp. IVGC(X)]. Moreover, if φ is strictly increasing, then $\sigma \approx \varphi \circ \sigma$.

Proof. Suppose $\sigma \in$IVGO(X). Then it is clear that the condition (IVGO1) holds. On the other hand,

$(\varphi \circ \sigma)(\emptyset) = [(\varphi \circ \sigma^L)(\emptyset), (\varphi \circ \sigma^U)(\emptyset)]$
$= [(\varphi(\sigma^L)(\emptyset)), (\varphi(\sigma^U)(\emptyset))]$
$= [(\varphi(1), (\varphi(1)]$
$= [1, 1] = \mathbf{1}$.

Similarly, $(\varphi \circ \sigma)(X) = \mathbf{1}$. Thus the condition(IVGO2) holds.

Let $A, B \in I^X$. Then

$(\varphi \circ \sigma)(A \cap B) = [(\varphi \circ \sigma^L)(A \cap B), (\varphi \circ \sigma^U)(A \cap B)]$
$\geq [(\varphi(\sigma^L(A \cap B)), (\varphi(\sigma^U(A \cap B))]$
$\geq [(\varphi(\sigma^L(A) \wedge \sigma^L(B)), (\varphi(\sigma^U(A) \wedge \sigma^U(B))]$. (4.1)
[Since $\sigma \in$ IVGO(X)]

Suppose $\sigma^L(A) \leq \sigma^L(B)$. Since φ is increasing continuous,

$\varphi(\sigma^L(A)) \leq \varphi(\sigma^L(B))$.

Thus

$\varphi(\sigma^L(A)) \leq \varphi(\sigma^L(B)) = \varphi(\sigma^L(A)) = \varphi(\sigma^L(A)) \wedge \varphi(\sigma^L(B))$
$= (\varphi \circ \sigma^L)(A) \wedge (\varphi \circ \sigma^L)(B)$. (4.2)

Similarly, we have

$\varphi(\sigma^U)(A) \wedge \sigma^U(B) = (\varphi \circ \sigma^U)(A) \wedge (\varphi \circ \sigma^U)(B)$. (4.3)

So, by (4.1),(4,2) and (4,3),

$(\varphi \circ \sigma)(A \cap B) \geq (\varphi \circ \sigma)(A) \wedge (\varphi \circ \sigma)(B)$.

Hence the condition (IVGO3) holds.

Now let $\{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$ and let $[\lambda, \mu] = \bigwedge_{\alpha \in \Gamma} \sigma(A_\alpha)$.

Suppose $\exists \alpha_0 \in \Gamma$ such that $[\lambda, \mu] = \sigma(A_{\alpha_0})$. Then

$(\varphi \circ \sigma)(\bigcup_{\alpha \in \Gamma} A_\alpha) = [\varphi(\sigma^L(\bigcup_{\alpha \in \Gamma} A_\alpha)), \varphi(\sigma^U(\bigcup_{\alpha \in \Gamma} A_\alpha))]$
$\geq [\varphi(\bigwedge_{\alpha \in \Gamma} \sigma^L(A_\alpha)), \varphi(\bigwedge_{\alpha \in \Gamma} \sigma^U(A_\alpha))]$ [Since $\sigma \in$IVGO(X)]
$= [\varphi(\sigma^L(A_{\alpha_0})), \varphi(\sigma^U(A_{\alpha_0}))]$ [By the hypothesis]
$= [(\varphi \circ \sigma)(A_{\alpha_0})]$
$\geq \bigwedge_{\alpha \in \Gamma} (\varphi \circ \sigma)(A_\alpha)$.

Suppose $\nexists \alpha_0 \in \Gamma$ such that $[\lambda, \mu] = \sigma(A_{\alpha_0})$. Then $\lambda \in ac\{a^L(A_\alpha) : \alpha \in \Gamma\}$ and $\mu \in ac\{\sigma^U(A_\alpha) : \alpha \in \Gamma\}$. Thus $\exists$ strictly decreasing sequences $\{\sigma^L(A_n)\}_{n=1}^\infty$ and $\{\sigma^U(A_n)\}_{n=1}^\infty$ such that they converge to λ and μ, respectively. So $\{(\varphi \circ \sigma^L)(A_n)\}_{n=1}^\infty$ and $\{(\varphi \circ \sigma^U)(A_n)\}_{n=1}^\infty$

are lower bounded sequences and thus they converge to their infimums, respectively. Hence

$$\lim_{n\to\infty}(\varphi\circ\sigma^L)(A_n)=\bigwedge_n(\varphi\circ\sigma^L)(A_n)$$
$$\geq\bigwedge_{\alpha\in\Gamma}(\varphi\circ\sigma^L)(A_\alpha)\ (4.4)$$

Similarly, we have

$$\lim_{n\to\infty}(\varphi\circ\sigma^U)(A_n)\geq\bigwedge_{\alpha\in\Gamma}(\varphi\circ\sigma^L)(A_\alpha).\ (4.5)$$

On the other hand,

$$(\varphi\circ\sigma)(\bigcup_{\alpha\in\Gamma}A_\alpha)\geq[\varphi(\bigwedge_{\alpha\in\Gamma}\sigma^L(A_\alpha)),\varphi(\bigwedge_{\alpha\in\Gamma}\sigma^U(A_\alpha))]\ [\text{Since } \sigma\in \text{IVGO}(X)]$$
$$=[\varphi(\lambda),\varphi(\mu)]\ [\text{Since } [\lambda,\mu]=\bigwedge_{\alpha\in\Gamma}\sigma(A_\alpha)]$$
$$=[\varphi(\lim_{n\to\infty}\sigma^L(A_n)),\varphi(\lim_{n\to\infty}\sigma^U(A_n))]$$
$$=[\lim_{n\to\infty}(\varphi\circ\sigma^L)(A_n),\lim_{n\to\infty}(\varphi\circ\sigma^U(A_n))].\ (4.6)$$

[Since φ is continuous]

From (4.4), (4.5) and (4.6),

$$(\varphi\circ\sigma)(\bigcup_{\alpha\in\Gamma}A_\alpha)\geq\bigwedge_{\alpha\in\Gamma}(\varphi\cdot\alpha)(A_\alpha).$$

So $\varphi\circ\sigma$ satisfies the condition (IVGO4). Hence $\varphi\circ\sigma\in$ IVGO(X).

Suppose $\sigma\in$ IVGC(X). By the similar way, we can prove that

$$(\varphi\circ\sigma)(\bigcap_{\alpha\in\Gamma}A_\alpha)\geq(\bigwedge_{\alpha\in\Gamma})(\varphi\circ\alpha)(A_\alpha)$$

for each $\{A_\alpha\}_{\alpha\in\Gamma}\subset I^X$. Also we can easily see that the remainders hold. Hence $\varphi\circ\sigma\in$ IVGC(X). □

The following example shows that the continuity condition for the mapping φ in Proposition 4.12 cannot be removed. The following is the modification of Example 2.16 in [9].

Example 4.12. Let δ^* be same as in Example 4.10. Let $\varphi: I\to I$ be the mapping defined as follows : For each $x\in I$,

$$\varphi(x)=\begin{cases}\frac{1}{2}x \text{ if } x<\frac{1}{2},\\ \frac{1}{2} \text{ if } x=\frac{1}{2},\\ \frac{1}{2}x+\frac{1}{2} \text{ if } x>\frac{1}{2}.\end{cases}$$

Then φ is strictly increasing and $\varphi(1)=1$. But it is not continuous at $x=\frac{1}{2}$. We will show that $\varphi\circ\delta^*$ is not an IVGO :

Consider a strictly increasing sequence $\{k_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} k_n = \frac{1}{2}$ and $0 \leq k_n \leq \frac{1}{2}$ $\forall n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we define the contant mapping $K_n : \mathbb{R} \to I$ as follows : For each $x \in \mathbb{R}$, $K_n(x) = k_n$. Then $\delta^*(K_n) = [1-k_n, 1-k_n]$ and $\{1-k_n\}_{n=1}^{\infty}$ is a strictly decreasing sequence contained in I such that $(1-k_n) = \frac{1}{2}$. Thus

$$(\varphi \circ \delta^*)(K_n) = [(\varphi \circ {\delta^*}^L)(K_n), (\varphi \circ {\delta^*}^U)(K_n)]$$
$$= [\varphi(1-k_n), \varphi(1-k_n)]$$
$$= [\tfrac{1-k_n}{2} + \tfrac{1}{2}, \tfrac{1-k_n}{2} + \tfrac{1}{2}].$$

So $\varphi(1-k_n)_{n=1}^{\infty}$ is a strictly decreasing sequence such that $\lim_{n\to\infty} \varphi(1-k_n) = 3/4$. Hence $(\varphi \circ \delta^*)(K_n) \geq \mathbf{3/4}$, for $n = 1, 2, \cdots$.

On the other hand, $\bigcup_{n=1}^{\infty} K_n$ is the constant mapping $f_{\frac{1}{2}} : \mathbb{R} \to$ I given by $f_{\frac{1}{2}}(x) = \frac{1}{2}$ for each $x \in \mathbb{R}$. Then

$$(\varphi \circ \delta^*)(\bigcup_{n=1}^{\infty} K_n) = [1-1/2, 1-1/2] = \mathbf{1/2}.$$

Thus

$$(\varphi \circ \delta^*)(\bigcup_{n=1}^{\infty} K_n) = [\varphi(1/2), \varphi(1/2)] = \mathbf{1/2}.$$

So

$$(\varphi, \delta^*)(\bigcup_{n=1}^{\infty} K_n) < \bigwedge_{n=1}^{\infty} (\varphi \circ \delta^*) K_n.$$

Hence $\varphi \circ \delta^* \notin$ IVGO(X). □

5. Interval-valued fuzzy subspace.

Definition 5.1[10]. Let Y be a subset of X and let $A \in I^X$. Then the *restriction* of A on Y is denoted by $A|_Y$. For each $B \in I^Y$, the *extension* of B, on X, denoted by B_X, is defined by

$$B_X(x) = \begin{cases} B(x) \text{ if } x \in Y, \\ 0 \text{ if } x \in X \backslash Y, \text{ for each } x \in X. \end{cases}$$

Proposition 5.2. Let (X, τ) be an IVFTS and let $Y \subset X$. We define the mapping $\tau_Y : I^Y \to D(I)$ as follows : For each $A \in I^Y$,

$$\tau_Y(A) = \bigvee \{\tau(B) : B \in I^X \text{ and } \mathrm{A} = \mathrm{B}|_{\mathrm{Y}}\}.$$

Then $\tau_Y \in$ IVGO(Y) and $\tau_Y(A) \geq \tau(A_X)$. In this case, the IVFTS (Y, τ_Y) is called a *subspace* of (X, τ) and τ_Y is called the *induced IVGO on Y from (X,τ)*.

Proof. For each $A \in I^Y$, let $B \in I^X$ such that $A = B|_Y$. Then
$\tau^L(B) \leq \tau^U(B)$.
Thus
$\bigvee\{\tau^L(B) : A = B|_Y\} \leq \bigvee\{\tau^U(B) : A = B|_Y\}$
So, by the definition of τ_Y,
${\tau_Y}^L(A) \leq {\tau_Y}^U(A)$.
Hence τ_Y satisfies the condition (IVGO1). It is obvious that (IVGO2) holds.

Let $A_1, A_2 \in I^Y$. Then
$\tau_Y(A_1 \cap A_2) = \bigvee\{\tau(B) : B \in I^X \mathrm{and} \mathrm{A}_1 \cap \mathrm{A}_2 = \mathrm{B}|_\mathrm{Y}\}$.
Suppose $\tau_Y(A_1) \wedge \tau_Y(A_2) = \mathbf{0}$. Then clearly
$\tau_Y(A_1 \cap A_2) \geq \mathbf{0} = \tau_Y(A_1) \wedge \tau_Y(A_2)$.
Suppose $\tau_Y(A_1) \wedge \tau_Y(A_2) > \mathbf{0}$. Let $\mathbf{0} < [\lambda, \mu] < \tau_Y(A_1) \wedge \tau_Y(A_2)$. Then $\exists B_i \in I^X$ such that $A_i = B_i|_Y$ and $\tau(B_i) > [\lambda, \mu]$, $i = 1, 2$. Since $\tau \in$ IVGO(X),
$\tau(B_1 \cap B_2) \geq \tau(B_1) \wedge \tau(B_2) > [\lambda, \mu]$.
On the other hand,
$(B_1 \cap B_2)|_Y = (B_1|_Y) \cap (B_2|_Y) = A_1 \cap A_2$.
Thus
$\tau_Y(A_1 \cap A_2) \geq \tau(B_1 \cap B_2) > [\lambda, \mu]$.
So, by the definition of τ_Y,
$\tau_Y(A_1 \cap A_2) \geq \tau_Y(A_1) \wedge \tau_Y(A_2)$.
In either cases,
$\tau_Y(A_1 \cap A_2) \geq \tau_Y(A_1) \wedge \tau_Y(A_2)$.
Hence the condition (IVGO3) holds.

Now let $\{A_\alpha\}_{\alpha \in \Gamma} \subset I^X$. Then
$\tau_Y(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigvee\{\tau(B) : B \in I^X \text{ and } \bigcup_{\alpha \in \Gamma} \mathrm{A}_\alpha = \mathrm{B}|_\mathrm{Y}\}$.
Suppose $\bigwedge_{\alpha \in \Gamma} \tau_Y(A_\alpha) = \mathbf{0}$. Then clearly
$\tau_Y(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \mathbf{0} = \bigwedge_{\alpha \in \Gamma} \tau_Y(A_\alpha)$.
Suppose $\bigwedge_{\alpha \in \Gamma} \tau_Y(A_\alpha) > \mathbf{0}$ and let $\mathbf{0} < [\lambda, \mu] < \bigwedge_{\alpha \in \Gamma} \tau_Y(A_\alpha)$. Then
$\tau_Y(A_\alpha) > [\lambda, \mu]$, $\forall \alpha \in \Gamma$.
Thus $\exists B_\alpha \in I^X$ such that $A_\alpha = B_\alpha|_Y$ and $\tau(B_\alpha) > [\lambda, \mu]$, $\forall \alpha \in \Gamma$.

So

$$\tau(\bigcup_{\alpha\in\Gamma} B_\alpha) \geq [\lambda, \mu].$$

On the other hand,

$$(\bigcup_{\alpha\in\Gamma} A_\alpha)|_Y = \bigcup_{\alpha\in\Gamma}(A_\alpha|_Y) = (\bigcup_{\alpha\in\Gamma} B_\alpha).$$

Thus, by the definition of τ_Y,

$$\tau_Y(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau_Y(A_2).$$

In either cases, τ_Y satisfies the condition (IVGO4). Hence $\tau \in \text{IVGO}(Y)$.

It is clearly that $\tau_Y(A) \geq \tau(A_X),\ \forall A \in I^Y$. □

Proposition 5.3. Let (Y, τ_Y) be an interval-valued fuzzy subspace of the IVFTS (X, τ) and let $A \in I^Y$. Then

(a) $\mathcal{F}_{\tau_Y}(A) = \bigvee\{\mathcal{F}_\tau(B) : B \in I^X \text{ and } A = B|_Y\}$.

(b) If $Z \subset Y \subset X$, then $\tau_Z = (\tau_Y)_Z$.

Proof. The proofs are very similar to that of Proposition 3.3 in (7). So they are omitted. □

6. Interval-valued gradation of preserving mappings

Definition 6.1. Let (X, τ) and (Y, η) be two IVSTSs and let $f : X \to Y$ be a mapping. Then f is called an *interval-valued gradation preserving mapping* (in short, an *IVGP-mapping*) or *interval-valued smooth continuous* if for each $B \in I^Y$,

$$\eta(B) \leq \tau(f^{-1}(B)), \text{ i.e., } [\eta^L(B), \eta^U(B)] \leq [\tau^L(f^{-1}(B)), \tau^U(f^{-1}(B))].$$

Definition 6.1′[7]. Let (X, τ) and (Y, η) be two STSs and let $f : X \to Y$ be a mapping. Then f is called a *gradation preserving mapping* (in short, an *GP-mapping*) or *smooth continuous* if for each $B \in I^Y$, $\eta(B) \leq \tau(f^{-1}(B)$.

Remark 6.1. (a) If a mapping $f : (X, \tau) \to (Y, \eta)$ is a GP-mapping, then $f : (X, [\tau, \tau]) \to (Y, [\eta, \eta])$ is an IVGP-mapping.

(b) If a mapping $f : (X, \tau) \to (Y, \eta)$ is an IVGP-mapping, then $f : (X, \tau^L) \to (Y, \eta^L)$ and $f : (X, \tau^U) \to (Y, \eta^U)$ are GP-mappings, respectively.

Theorem 6.2. Let (X, τ) and (Y, η) be two IVSTSs and let $f : X \to Y$ be a mapping. Then $f : (X, \tau) \to (Y, \eta)$ is an IVGP-mapping if and only if $f : (X, \tau_{[\lambda,\mu]}) \to (Y, \eta_{[\lambda,\mu]})$ is continuous w.r.t. Chang, for each $[\lambda, \mu] \in D(I_o)$.

Proof. ($\Rightarrow$): Suppose f is an IVGP-mapping. Let $[\lambda, \mu] \in D(I_0)$ and let $B \in \eta_{[\lambda,\mu]}$. Since $\eta \in \text{IVGO}(Y)$, $\eta(B) \geq [\lambda, \mu]$. Then, by the hypothesis, $\eta(B) \leq \tau(f^{-1}(B))$. Thus

$$\tau(f^{-1}(B)) \geq [\lambda, \mu].$$

So $f^{-1}(B) \in \tau_{\lambda,\mu}$. Hence $f : (X, \tau_{[\lambda,\mu]}) \to (Y, \eta_{[\lambda,\mu]})$ is continuous w.r.t. Chang.

($\Leftarrow$): Suppose $f : (X, \tau_{[\lambda,\mu]}) \to (Y, \eta_{[\lambda,\mu]})$ is continuous for each $[\lambda, \mu] \in D(I_0)$. Let $B \in I^Y$. If $\eta(B) = \mathbf{0}$, then clearly $\eta(B \leq \tau(f^{-1}(B)))$. If $\eta(B) = [\lambda, \mu]$, then $B \in \eta_{[[\lambda,\mu]}$. Thus, by the hypothesis, $f^{-1}(B) \in \tau_{[\lambda,\mu]}$. So $\tau(f^{-1}(B)) \geq [\lambda, \mu] = \eta(B)$. In either cases, $\eta(B) \leq \tau(f^{-1}(B))$. Hence f is an IVGP-mapping. □

Theorem 6.3. Let (X,T) and (Y,T′) be two Chang's fuzzy topological space and let $f : X \to Y$ be a mapping. Then $f : (X, T) \to (Y, T')$ is continuous if and only if $f : (X, T^{[\lambda,\mu]}) \to (Y, (T')^{[\lambda,\mu]})$ is an IVGP-mapping, for each $[\lambda, \mu] \in D(I_0)$.

Proof. ($\Rightarrow$) : Suppose $f : (X, T) \to (Y, T')$ is continuous, let $B \in I^Y$ and let $[\lambda, \mu] \in D(I_0)$. Then we have the following cases:

(i) $B = \phi$ or Y,

(ii) $B \in T'$,

(iii) $B \notin T'$.

Case (i) : $f^{-1}(\phi) = \phi$ or $f^{-1}(Y) = X$. Thus

$$(T')^{[\lambda,\mu]}(B) \leq T^{[\lambda,\mu]}(f^{-1}(B)).$$

Case (ii) : Clearly $(T')^{[\lambda,\mu]}(B) = [\lambda, \mu]$. Since f is continuous, $f^{-1}(B) \in T$. Thus $T^{[\lambda,\mu]}(f^{-1}(B)) = [\lambda, \mu]$. So

$$(T')^{[\lambda,\mu]} \leq T^{\lambda,\mu}(f^{-1}(B)).$$

Case(iii) : It is clear that $(T')^{\lambda,\mu}(B) = \mathbf{0}$. Thus

$$\mathbf{0} = (T')^{[\lambda,\mu]}(B) \leq T^{\lambda,\mu}(f^{-1}(B)).$$

So, in all cases, $f : (X, T^{\lambda,\mu}) \to (Y, (T')^{[\lambda,\mu]})$ is an IVGP-mapping.

($\Leftarrow$) : It follows from Proposition 3.18 and Theorem 6.2. □

The following is the immediate result of Definition 6.1.

Proposition 6.4. Let $(X, \tau), (Y, \eta)$ and (Z, ξ) be IVSTSs.

(a) $1_X : (X, \tau) \to (X, \tau)$ is an IVGP-mapping.

(b) If $f : (X, \tau) \to (Y, \eta)$ and $g : (Y, \eta) \to (Z, \xi)$ is IVGP-mappings, then $g \circ f : (X, \tau) \to (Z, \xi)$ is an IVGP-mapping.

We can easily see that the collection of all IVFTSs and IVGP-mapping between then forms a concrete category and we will denote it by **IVTop**.

Theorem 6.5. Let (X, τ) be an IVFTS and let $f : X \to Y$ be a mapping. Let $\{T_{\lambda,\mu}\}_{\lambda,\mu} \in D(I_0)$ be a descending family of chang's fuzzy topologies on Y. Let η be the IVGO on X generated by this family. For each $[\lambda, \mu] \in D(I_0)$, suppose $\mathfrak{B}_{[\lambda,\mu]}$ or $\mathcal{S}_{[\lambda,\mu]}$ is a base or a subbase for $T_{[\lambda,\mu]}$, respectively. Then the followings are equivalent:

(a) $f : (X, \tau) \to (y, \eta)$ is an IVGP-mapping.

(b) $\tau(f^{-1}(B)) \geq [\lambda, \mu], \forall B \in T_{[\lambda,\mu]},\ \forall[\lambda, \mu] \in D(I_0)$.

(c) $\tau(f^{-1}(B)) \geq [\lambda, \mu], \forall B \in \mathfrak{B}_{[\lambda,\mu]},\ \forall[\lambda, \mu] \in D(I_0)$.

(d) $\tau(f^{-1}(B)) \geq [\lambda, \mu], \forall B \in \mathcal{S}_{[\lambda,\mu]},\ \forall[\lambda, \mu] \in D(I_0)$.

Proof. (a) $(\Rightarrow)$ (b) : Suppose (a) holds. Let $[\lambda, \mu] \in D(I_0)$ and let $B \in T_{[\lambda,\mu]}$.
Then $\tau(f^{-1}(B)) \geq \zeta(B) \geq [\lambda, \mu]$.

It is obvious that (b) $\Rightarrow$ (c) $\Rightarrow$ (d) hold.

(d) $(\Rightarrow)$ (a) : Suppose (d) holds. Let $B \in I^Y$ and, without loss of generality, let $\eta(B) = [\lambda, \mu] > \mathbf{0}$. Then $B \in T_{[\lambda,\mu]}$. Now, B is of the form, $B = \bigcup_{\alpha\in\Gamma} B_\alpha$, where $B_\alpha \in \mathfrak{B}_{[\lambda,\mu]},\ \forall\alpha \in \Gamma$. Also, for each $\alpha \in \Gamma$, B_α is of the form, $B_\alpha = \bigcup_{j=1}^{n_\alpha} S_{\alpha,j}$, where $S_{\alpha,j} \in \mathcal{S}_{[\lambda,\mu]}, \forall\ j = 1, 2, \cdots, n_\alpha$. Thus

$$\tau(f^{-1}(B)) = \tau(f^{-1}(\bigcup_{\alpha\in\Gamma}(\bigcap_{j=1}^{n_\alpha} S_{\alpha,j})))$$

$$= \tau(\bigcup_{\alpha\in\Gamma}(\bigcap_{j=1}^{n_\alpha} f^{-1}(S_{\alpha,j})))$$

$$\geq \bigwedge_{\alpha\in\Gamma}(\bigwedge_{j=1}^{n_\alpha} \tau f^{-1}(S_{\alpha,j})))\ \text{[Since } \tau \in \text{IVGO(X)]}$$

$$\geq [\lambda, \mu].\ \text{[By the condition (d)]}$$

So $\tau(f^{-1}(B)) \geq \eta(B)$. Hence $f : (X, \tau) \to (Y, \eta)$ is on IVGP-mapping. This completes the proof. □

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