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INTERVAL-VALUED FUZZY IRRESOLUTE MAPPINGS ON INTERVAL-VALUED FUZZY TOPOLOGICAL SPACES

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Abstract. We introduce the concepts of IVF irresolute mappings and IVF irresolute open mappings, and investigate characterizations for such mappings on the interval-valued fuzzy topological spaces.

1. Introduction

Zadeh [4] introduced the concept of fuzzy set and investigated basic properties. Gorzalczany [1] introduced the concept of interval-valued fuzzy set which is a generalization of fuzzy sets. In [3], Mondal and Samanta introduced the concepts of interval-valued fuzzy topology, continuity and compactness and studied some topological properties. In [2], Jun et al. introduced the concepts of IVF semiopen sets and IVF semiopen mappings and studied some results about them.

In this paper, we introduce the concepts of IVF irresolute mappings and IVF irresolute open mappings, and investigate characterizations for such mappings.

2. Preliminaries

Let $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$ and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the

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upper end points respectively. Especially, we denote $\mathbf{0} = [0, 0], \mathbf{1} = [1, 1]$, and $\mathbf{a} = [a, a]$ for $a \in (0, 1)$. We also note that

(1) $(\forall M, N \in D[0, 1])(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$.
(2) $(\forall M, N \in D[0, 1])(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D[0, 1]$, the complement of M, denoted by M^c, is defined by $M^c = \mathbf{1} - M = [1 - M^U, 1 - M^L]$. Let X be a nonempty set. A mapping $A : X \to D[0, 1]$ is called an interval-valued fuzzy set (simply, IVF set) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^L$ and $A(x)^U$, respectively. For any $[a, b] \in D[0, 1]$, the IVF set whose value is the interval $[a, b]$ for all $x \in X$ is denoted by $\widetilde{[a, b]}$. In particular, for any $c \in [a, b]$, the IVF set whose value is $\mathbf{c} = [c, c]$ for all x X is denoted by simply $\widetilde{c}$. For a point $p \in X$ and for $[a, b] \in D[0, 1]$ with $b > 0$, the IVF set which takes the value $[a, b]$ at p and $\mathbf{0}$ elsewhere in X is called an interval-valued fuzzy point (simply, IVF point) and is denoted by $[a, b]_p$. In particular, if $b = a$, then it is also denoted by a_p. Denoted by D^X the set of all IVF sets in X.

For every $A, B \in D^X$, we define

$A = B \Leftrightarrow (\forall x \in X)([A(x)]^L = [B(x)]^L$ and $[A(x)]^U = [B(x)]^U)$,

$A \subseteq B \Leftrightarrow (\forall x \in X)([A(x)]^L \subseteq [B(x)]^L$ and $[A(x)]^U \subseteq [B(x)]^U)$.

The complement A^c of A is defined by

$$[A^c(x)]^L = 1 - [A(x)]^U \text{ and } [A^c(x)]^U = 1 - [A(x)]^L$$

for all $x \in X$.

For a family of IVF sets $\{A_i : i \in J\}$ where J is an index set, the union $G = \cup_{i \in J} A_i$ and $F = \cap_{i \in J} A_i$ are defined by

$(\forall x \in X)$ $([G(x)]^L = \sup_{i \in J}[A_i(x)]^L, [G(x)]^U = \sup_{i \in J}[A_i(x)]^U)$,

$(\forall x \in X)$ $([F(x)]^L = \inf_{i \in J}[A_i(x)]^L, [F(x)]^U = \inf_{i \in J}[A_i(x)]^U)$, respectively.

Let $f : X \to Y$ be a mapping and let A be an IVF set in X. Then the image of A under f, denoted by $f(A)$, is defined as follows

$$[f(A)(y)]^L = \begin{cases} \sup_{f(x)=y}[A(x)]^L, & \text{if } f^{-1}(y) \neq \emptyset,\ y \in Y \\ 0, & \text{otherwise ,} \end{cases}$$

$$[f(A)(y)]^U = \begin{cases} \sup_{f(x)=y}[A(x)]^U, & \text{if } f^{-1}(y) \neq \emptyset,\ y \in Y \\ 0, & \text{otherwise ,} \end{cases}$$

for all $y \in Y$.

Let B be an IVF set in Y. Then the inverse image of B under f, denoted by $f^{-1}(B)$, is defined as follows

$$(\forall x \in X)([f^{-1}(B)(x)]^L = [B(f(x))]^L, [f^{-1}(B)(x)]^U = [B(f(x))]^U).$$

Definition 2.1 ([3]). A family τ of IVF sets in X is called an *interval-valued fuzzy topology* (simply, IVFT) on X if it satisfies:
(1) $\mathbf{0}, \mathbf{1} \in \tau$.
(2) $A, B \in \tau \Rightarrow A \cap B \in \tau$.
(3) For $i \in J$, $A_i \in \tau \Rightarrow \cup_{i \in J} A_i \in \tau$.
Every member of τ is called an IVF open set. An IVF set A is called an IVF closed set if the complement of A is an IVF open set. And (X, τ) is called an *interval-valued fuzzy topological space* (simply, IVFTS).

In an IVFTS (X, τ), for an IVF set A in X, the IVF closure and the IVF interior of A, denoted by $cl(A)$ and $int(A)$, respectively, are defined as

$$cl(A) = \cap\{B \in D^X : B^c \in \tau \text{ and } A \subseteq B\},$$

$$int(A) = \cup\{B \in D^X : B \in \tau \text{ and } B \subseteq A\},$$

respectively [3].

Definition 2.2 ([2]). Let A be an IVF set in an IVFTS (X, τ). Then A is said to be *IVF semiopen* if $A \subseteq cl(int(A))$. A is said to be *IVF semiclosed* if the complement of A is IVF semiopen. Denote the set of all IVF semiopen (resp., IVF semiclosed) sets by $IVFSO(X)$ (resp., $IVFSC(X)$).

Definition 2.3 ([2, 3]). Let (X, τ_1) and (Y, τ_2) be two IVFTS's. Then $f : X \to Y$ is said to be *continuous* (resp., IVF semicontinuous) if for every $B \in \tau_2$, $f^{-1}(B)$ is IVF open (resp., *IVF semiopen*).

Definition 2.4 ([2]). Let (X, τ_1) and (Y, τ_2) be two IVFTS's. Then $f : X \to Y$ is said to be *IVF open* (resp., *IVF semiopen*) if for every $A \in \tau_1$, $f(A)$ is IVF open (resp., IVF semiopen).

3. IVF Irresolute Mappings

Definition 3.1. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then f is said to be *IVF irresolute* if for every IVF semiopen set U of Y, $f^{-1}(U)$ is IVF semiopen.

Every IVF irresolute mapping is IVF semicontinuous mapping but the converse need not be true as seen in the next example.

Example 3.2. Let $X = [0, 1]$ and let A, B and C be IVF sets defined as follows

$$A(x) = \begin{cases} [\frac{1}{3}x, \frac{2}{3}x], & 0 \le x \le \frac{1}{2}, \\ [-\frac{1}{2}x + \frac{5}{12}, -\frac{2}{3}x + \frac{2}{3}], & \frac{1}{2} \le x \le \frac{5}{6}, \\ [0, -\frac{2}{3}x + \frac{2}{3}], & \frac{5}{6} \le x \le 1; \end{cases}$$

$B(x) = [\frac{1}{3}x, \frac{1}{2}], 0 \le x \le 1;$

$$C(x) = \begin{cases} [\frac{5}{6}x + \frac{1}{6}, x + \frac{1}{6}], & 0 \le x \le \frac{5}{6}, \\ [\frac{5}{6}x + \frac{1}{6}, 1], & \frac{5}{6} \le x \le 1. \end{cases}$$

Define IVF topologies τ_1 and τ_2 on X as follows

$$\tau_1 = \{\mathbf{0}, A, B, \mathbf{1}\}; \quad \tau_2 = \{\mathbf{0}, A, \mathbf{1}\}$$

Note that the IVF set C is IVF semiopen in (X, τ_2) but it is not an IVF semiopen set in the IVFTS (X, τ_1). Hence we know that obviously the identity mapping $f : (X, \tau_1) \to (X, \tau_2)$ is an IVF semicontinuous mapping but it is not IVF irresolute.

IVF continuous $\Rightarrow$ IVF semicontinuous $\Leftarrow$ IVF irresolute

In an IVFTS (X, τ), for an IVF set A in X, the IVF semi-closure and the IVF semi-interior of A, denoted by $scl(A)$ and $sint(A)$, respectively, are defined as

$$scl(A) = \cap\{B \in D^X : B^c \in IVFSC(X) \text{ and } A \subseteq B\};$$

$$sint(A) = \cup\{B \in D^X : B \in IVFSO(X) \text{ and } B \subseteq A\}.$$

Lemma 3.3. Let (X, τ) be an IVFTS and $A \in D^X$.
(1) $\mathbf{1} - sint(A) = scl(\mathbf{1} - A)$.
(2) $\mathbf{1} - scl(A) = sint(\mathbf{1} - A)$.

Theorem 3.4. Let $f : X \to Y$ be a mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then the following statements are equivalent:
(1) f is IVF irresolute.
(2) $f^{-1}(B)$ is IVF semiclosed for each IVF semiclosed set B of Y.
(3) $f(scl(A)) \subseteq scl(f(A))$ for each $A \in D^Y$.
(4) $scl(f^{-1}(B)) \subseteq f^{-1}(scl(B))$ for each $B \in D^Y$.
(5) $f^{-1}(sint(B)) \subseteq sint(f^{-1}(B))$ for each $B \in D^Y$.

Proof. (1) $\Leftrightarrow$ (2) From Definition 2.2, it is obvious.

(2) $\Rightarrow$ (3) Let A be any IVF set in X. Since $scl(f(A))$ is an IVF semiclosed set containing $f(A)$, by (2), $f^{-1}(scl(f(A)))$ is IVF semiclosed and $A \subseteq f^{-1}(scl(f(A)))$. So $scl(A) \subseteq scl(f^{-1}(scl(f(A))) = f^{-1}(scl(f(A)))$. It implies $f(scl(A)) \subseteq scl(f(A))$.

(3) $\Rightarrow$ (4) Let B be any IVF set in Y. From (3), it follows that $f(scl(f^{-1}(B))) \subseteq scl(f(f^{-1}(B))) \subseteq scl(B)$. Hence $scl(f^{-1}(B)) \subseteq f^{-1}(scl(B))$.

(4) $\Rightarrow$ (5) For any IVF set B of Y, from (4), it follows

$$\begin{aligned} f^{-1}(sint(B)) &= \mathbf{1} - (f^{-1}(scl(\mathbf{1} - B))) \\ &\subseteq \mathbf{1} - scl(f^{-1}(\mathbf{1} - B)) \\ &= sint(f^{-1}(B)). \end{aligned}$$

Hence $f^{-1}(sint(B)) \subseteq sint(f^{-1}(B))$.

(5) $\Rightarrow$ (1) Let V be any IVF semiopen set in Y. By (5),

$$f^{-1}(V) = f^{-1}(sint(V)) \subseteq sint(f^{-1}(V)).$$

So $f^{-1}(V)$ is an IVF semiopen set, and hence f is IVF irresolute.

□

Theorem 3.5. Let $f : X \to Y$ be a bijective mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then f is IVF irresolute if and only if $sint(f(A)) \subseteq f(sint(A))$ for each $A \in D^X$.

Proof. Let f be IVF irresolute. Then for any IVF set A of X, $f^{-1}(sint(f(A)))$ is IVF semiopen. From Theorem 3.4 and injectivity of f, it follows $f^{-1}(sint(f(A))) \subseteq sint(f^{-1}(f(A))) = sint(A)$. Since f is surjective, $sint(f(A)) = f(f^{-1}(sint(f(A)))) \subseteq f(sint(A))$.

For the converse, let B be any IVF semiopen set of Y. From hypothesis and surjectivity of f, it follows

$$f(sint(f^{-1}(B))) \supseteq sint(f(f^{-1}(B))) = sint(B) = B.$$

Since f is injective, $sint(f^{-1}(B))) \supseteq f^{-1}(B)$, and so $f^{-1}(B)$ is IVF semiopen. Hence f is IVF irresolute. □

Definition 3.6. Let (X, τ) be an IVFTS. An IVF set A in X is said to be *IVF semicompact* if for every IVF semiopen cover $\mathcal{A} = \{A_i \in D^X : A_i \in \tau, i \in J\}$ of A, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{j \in J_0} A_j$.

Theorem 3.7. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be IVF irresolute on two IVFTS's. If A is an IVF semicompact set, then $f(A)$ is also IVF semicompact.

Proof. Let $\{B_i \in D^Y : B_i \in \tau_2,\ i \in J\}$ be an IVF semiopen cover of $f(A)$ in Y. Then $\{f^{-1}(B_i) : i \in J\}$ is an IVF semiopen cover of A in X. By definition of IVF semicompactness, there exists $J_0 = \{1, 2, \cdots, n\} \subseteq J$ such that $A \subseteq \cup_{j \in J_0}(f^{-1}(B_j))$. So

$$\begin{aligned} f(A) &\subseteq f(\cup_{j \in J_0}(f^{-1}(B_j))) \\ &= \cup_{j \in J_0} f(f^{-1}(B_j)) \\ &\subseteq \cup_{j \in J_0} B_j. \end{aligned}$$

Hence $f(A)$ is IVF semicompact. □

An IVF set A in an IVF topological space X is said to be *IVF compact* [3] if every IVF open cover $\mathcal{A} = \{A_i : i \in J\}$ of A has a finite IVF subcover.

Theorem 3.8. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be IVF semicontinuous on two IVFTS's. If A is an IVF semicompact set, then $f(A)$ is IVF compact.

Proof. It is easily proved from the definition of IVF semicontinuity and Theorem 3.7. □

Definition 3.9. Let (X, τ_1) and (Y, τ_2) be two IVFTS's. Then $f : X \to Y$ is called an *IVF irresolute open* (resp., *IVF irresolute closed* mapping if for every IVF semiopen (resp., *IVF semiclosed*) set A of X, $f(A)$ is IVF semiopen (resp., *IVF semiclosed*) in Y.

Every IVF irresolute open (resp., IVF irresolute closed) mapping is IVF semiopen (resp., IVF semiclosed) but the converse need not be true.

Example 3.10. Consider the identity mapping $f : (X, \tau_2) \to (X, \tau_1)$ in Example 3.2. Then f is an IVF semiopen mapping but not IVF irresolute open.

Theorem 3.11. Let $f : X \to Y$ be a mapping on IVFTS's (X, τ_1) and (Y, τ_2). The the following are equivalent:
(1) f is IVF irresolute open.
(2) $f(sint(A)) \subseteq sint(f(A))$ for $A \in D^X$.
(3) $sint(f^{-1}(B)) \subseteq f^{-1}(sint(B))$ for $B \in D^Y$.
(4) For $B \in D^Y$ and each IVF semiclosed set A of X with $f^{-1}(B) \subseteq A$, there exists an IVF semiclosed set C of Y such that $B \subseteq C$ and $f^{-1}(C) \subseteq A$.

Proof. (1) $\Rightarrow$ (2) For $A \in D^X$,

$$\begin{aligned} f(sint(A)) &= f(\cup\{B \in D^X : B \subseteq A, B \in IVFSO(X)\}) \\ &= \cup\{f(B) \in D^Y : f(B) \subseteq f(A), f(B) \in IVFSO(Y)\} \\ &\subseteq \cup\{U \in D^Y : U \subseteq f(A), U \in IVFSO(Y)\} \\ &= sint(f(A)). \end{aligned}$$

Hence $f(sint(A)) \subseteq sint(f(A))$.

(2) $\Rightarrow$ (3) For $B \in D^Y$, from (2), we have

$$f(sint(f^{-1}(B))) \subseteq sint(f(f^{-1}(B))) \subseteq sint(B).$$

Hence $sint(f^{-1}(B)) \subseteq f^{-1}(sint(B))$.

(3) $\Rightarrow$ (4) Let A be an IVF semiclosed set of X with $f^{-1}(B) \subseteq A$ for $B \in D^Y$. Since $\mathbf{1} - A \subseteq \mathbf{1} - f^{-1}(B) = f^{-1}(\mathbf{1} - B)$ and $sint(\mathbf{1} - A) = \mathbf{1} - A \subseteq sint(f^{-1}(\mathbf{1} - B))$. By (3), $\mathbf{1} - A \subseteq sint(f^{-1}(\mathbf{1} - B)) \subseteq f^{-1}(sint(\mathbf{1} - B))$. Thus $A \supseteq \mathbf{1} - (f^{-1}(sint(\mathbf{1} - B))) = f^{-1}(\mathbf{1} - sint(\mathbf{1} - B)) = f^{-1}(scl(B))$. Now set $C = scl(B)$. Then C is an IVF semiclosed set of Y such that $B \subseteq C$ and $f^{-1}(C) \subseteq A$.

(4) $\Rightarrow$ (1) Let A be an IVF semiopen set of X. Then $f^{-1}(\mathbf{1} - f(A)) = \mathbf{1} - f^{-1}(f(A)) \subseteq \mathbf{1} - A$ and $\mathbf{1} - A$ is IVF semiclosed. By (4), there exists an IVF semiclosed set $C \in D^Y$ such that $\mathbf{1} - f(A) \subseteq C$ and $f^{-1}(C) \subseteq \mathbf{1} - A$. It implies $\mathbf{1} - C \subseteq f(A)$ and $f(A) \subseteq f(\mathbf{1} - f^{-1}(C)) = f(f^{-1}(\mathbf{1} - C)) \subseteq \mathbf{1} - C$.
Hence $f(A)$ is an IVF semiopen set in Y.

□

Theorem 3.12. Let $f : X \to Y$ be a mapping on IVFTS's (X, τ_1) and (Y, τ_2). The the following are equivalent:
(1) f is IVF irresolute closed.
(2) $scl(f(A)) \subseteq f(scl(A))$ for $A \in D^X$.

Proof. (1) $\Rightarrow$ (2) For $A \in D^X$,

$$\begin{aligned} f(scl(A)) &= f(\cap\{B \in D^X : A \subseteq B, B \in IVFSC(X)\}) \\ &= \cap\{f(B) \in D^Y : f(A) \subseteq f(B), f(B) \in IVFSC(Y)\} \\ &\supseteq \cap\{F \in D^Y : f(A) \subseteq F, F \in IVFSC(Y)\} \\ &= scl(f(A)). \end{aligned}$$

Hence $scl(f(A)) \subseteq f(scl(A))$.

(2) $\Rightarrow$ (1) Let A be an IVF semiclosed set in X. Then by (2),

$$scl(f(A)) \subseteq f(scl(A)) = f(A).$$

It implies $f(A)$ is IVF semiclosed, and so f is IVF irresolute closed. □

Theorem 3.13. Let $f : X \to Y$ be a bijective mapping between IVFTS's (X, τ_1) and (Y, τ_2). Then
(1) f is IVF irresolute closed.
(2) $scl(f(A)) \subseteq f(scl(A))$ for $A \in D^X$.
(3) $f^{-1}(scl(B)) \subseteq scl(f^{-1}(B))$ for each $B \in D^Y$.

Proof. It is sufficient to show that (2) is equivalent to (3).
(2) $\Rightarrow$ (3) For $B \in D^Y$, since f is surjective,

$$scl(B) = scl(f(f^{-1}(B))) \subseteq f(scl(f^{-1}(B))).$$

From injectivity of f,

$$f^{-1}(scl(B)) \subseteq f^{-1}(f(scl(f^{-1}(B)))) = scl(f^{-1}(B)).$$

(3) $\Rightarrow$ (2) Conversely, let $A \in D^X$. Then from hypothesis and injectivity of f, $f^{-1}(scl(f(A))) \subseteq scl(f^{-1}(f(A))) = scl(A)$ Since f is surjective, $f(scl(A)) \supseteq f(f^{-1}(scl(f(A)))) = scl(f(A))$. Hence the statement (2) is obtained.

□

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