ON THE LIMIT BEHAVIOR OF EXTENDED NEGATIVE QUADRANT DEPENDENCE

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Abstract. We discuss in this paper the notions of extended negative quadrant dependence and its properties. We study a class of bivariate uniform distributions having extended negative quadrant dependence, which is derived by generalizing the uniform representation of a well-known Farlie-Gumbel-Morgenstern distribution. Finally, we also study the limit behavior on the extended negative quadrant dependence.

1 Introduction

The sequences of negatively dependent random variables are widely used in the various statistical analysis fields. In statistical analysis, we usually have to assume that random variables are independent. This assumption is seldom valid in practice. For example, lifetime of two components is usually dependent. This will make two lifetime random variables to be negatively dependent, so it is significant to investigate the properties of those sequences.

Lamperti(1963) introduced the concept of the following proposition. If $\{A_n, \ n \geq 1\}$ is a sequence of events such that

$$(1.1) \ \sum_{n=1}^{\infty} P(A_n) = \infty \text{ and } P(A_i \cap A_j) \leq CP(A_i)P(A_j)$$

for all $i, \ j > N$, $i \neq j$ and some constants C and N, then $P(\limsup A_n) = 0$.

Petrov(2002) presented a more general and precise theorem of this type as follows; Let $\{A_n,\ n \geq 1\}$ be a sequence of events satisfying

$$(1.2)\ \sum_{n=1}^{\infty} P(A_n) = \infty \text{ and } P(A_i \cap A_j) \leq CP(A_i)P(A_j) \text{ for all } i,\ j > L$$

such that $i \neq j$ and some constants $C \geq 1$ and L. Then, $P(\limsup A_n) \geq \dfrac{1}{C}$.

Consider $A_n = [X_n \leq x_n]$(or $A_n = [X_n > x_n]$), where $\{X_n,\ n \geq 1\}$ is a sequence of random variables and x_n's are real numbers. From (1.1) we can consider a new negative dependence $P(X_i \leq x_i,\ X_j \leq x_j) \leq CP(X_i \leq x_i)P(X_j \leq x_j)$ which is weaker than negative quadrant dependence $P(X_i \leq x_i,\ X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j)$. Obviously negative quadrant dependence must be extended negative quadrant dependence. On the other hand, for some negative quadrant dependence it is possible to find a positive constant $M \geq 1$ such that $P(X \leq x,\ Y \leq y) \leq MP(X \leq x)P(Y \leq y)$. Therefore, the extended negative quadrant dependence structure is a substantially more comprehensive than negative dependence structure in that it can reflect not only a negative dependence structure but also a positive one, to some extent.

We consider in this paper the notions of extended negative quadrant dependence and its properties. We study a class of bivariate uniform distributions having extended negative quadrant dependence, which is derived by generalizing the uniform representation of a well-known Farlie-Gumbel-Morgenstern distribution. Finally, we also study the limit behavior on the extended negative quadrant dependence

2 Properties of extended negative quadrant dependence

Definition 2.1 Two random variables X and Y are said to be extended negative quadrant dependent if there exists a constant $M \geq 1$ such that
(2.1) $P(X \leq x,\ Y \leq y) \leq MP(X \leq x)P(Y \leq y)$,
for all x and y.

Theorem 2.2 (2.1) holds if and only if there exists a constant $M' \geq 1$ such that
(2.2) $P(X > x,\ Y > y) \leq M'P(X > x)P(Y > y)$, for all x and y.

Proof *Necessity.* When $P(X > x,\ Y > y) = 0$ obviously (2.2) holds. Now assume $P(X > x,\ Y > y) \neq 0$. Then

$$
\begin{aligned}
& P(X > x,\ Y > y) \\
=\ & P(X > x) - P(X > x,\ Y \leq y) \\
=\ & P(X > x) - P(Y \leq y) + P(X \leq x,\ Y \leq y) \\
\leq\ & P(X > x) + P(Y > y) - 1 + MP(X \leq x)P(Y \leq y) \\
=\ & P(X > x) + P(Y > y) - 1 \\
& + M(1 - P(X > x)(1 - P(Y > y)) \\
\leq\ & M - 1 + MP(X > x)P(Y > y) \\
=\ & \left(M + \frac{M - 1}{P(X > x)P(Y > y)}\right) P(X > x)P(Y > y) \\
=\ & M' P(X > x)P(Y > y) \text{ for all } x,\ y,
\end{aligned}
$$

where $M' = M + \dfrac{M - 1}{P(X > x)P(Y > y)} \geq M \geq 1.$

Sufficiency. When $P(X \leq x,\ Y \leq y) = 0$ obviously (2.1) holds. Now assume $P(X \leq x,\ Y \leq y) \neq 0$. Then

$$
\begin{aligned}
P(X \leq x,\ Y \leq y) =\ & P(X \leq x) - P(X \leq x,\ Y > y) \\
=\ & P(X \leq x) - P(Y > y) + P(X > x,\ Y > y) \\
\leq\ & P(X \leq x) - P(Y > y) + M' P(X > x)P(Y > y) \\
=\ & P(X \leq x) - 1 + P(Y \leq y) \\
& + M'(1 - P(X \leq x))(1 - P(Y \leq y)) \\
\leq\ & M' - 1 + M' P(X \leq x)P(Y \leq y) \\
=\ & \left(M' + \frac{M' - 1}{P(X \leq x)P(Y \leq y)}\right) P(X \leq x)P(Y \leq y) \\
=\ & M'' P(X \leq x)P(Y \leq y) \text{ for all } x \text{ and } y,
\end{aligned}
$$

where $M'' = M' + \dfrac{M' - 1}{P(X \leq x)P(Y \leq y)} \geq M' \geq 1.$

The proof is complete.

Lemma 2.3 The set of inequalities in (2.1) and (2.2) is equivalent to that obtained by replacing one or both of the inequalities $X \leq x$ or $Y \leq y$ by the corresponding $X > x$ or $Y > y$.

Proof To see that implies the corresponding inequalities with one or

both of the equal signs omitted, replace x by $x - \dfrac{1}{n}$ (y by $y - \dfrac{1}{n}$) . To go in the other direction, replace x by $x + \dfrac{1}{n}$ (y by $y + \dfrac{1}{n}$) , and the proof is complete.

Lemma 2.4(a) If $(X,\ Y)$ is independent, then $(X,\ Y)$ is extended negative quadrant dependent.
(b) $(X,\ X)$ is extended negative quadrant dependent for all x.
Proof $P(X \le x,\ X \le x) \le MP(X \le x)P(X \le x)$ for some $M \ge 1$ and for all x, and the proof is complete.

Theorem 2.5 Two random variables X and Y are extended negative quadrant dependent if and only if two random variables $-X$ and $-Y$ are extended negative quadrant dependent.
Proof Since $-(-X) = X$, it is enough to show the case that if X and Y are extended negative quadrant dependent, then $-X$ and $-Y$ are extended negative quadrant dependent.

$$
\begin{aligned}
P(-X \le x ,\ -Y \le y) &= P(X \ge -x ,\ Y \ge -y) \\
&= \lim_{n \to \infty} P(X > -x - \tfrac{1}{n} ,\ Y > -y - \tfrac{1}{n}) \\
&\le M \lim_{n \to \infty} P(X > -x - \tfrac{1}{n})P(Y > -y - \tfrac{1}{n}) \\
&= MP(X \ge -x)P(Y \ge -y) \\
&= MP(-X \le x)P(-Y \le y)
\end{aligned}
$$

Hence, two random variables $-X$ and $-Y$ are extended negative quadrant dependent, and the proof is complete.

Theorem 2.6 If $(X,\ Y)$ is extended negative quadrant dependent, then $(f(X),\ g(Y))$ is also extended negative quadrant dependent for nondecreasing functions f and g.
Proof

$$
\begin{aligned}
&P(f(X) \le f(x),\ g(Y) \le g(y)) \\
&= P(X \le y,\ Y \le y) \\
&\le MP(X \le x)P(Y \le y) \\
&= MP(f(X) \le f(y))P(g(Y) \le g(y))
\end{aligned}
$$

Theorem 2.7 Every convex combination of two extended negative quadrant dependent bivariate distribution having fixed marginal distribution $F(x)$ and $G(y)$ is still extended negative quadrant dependent.

Proof Let $H_1(x,\ y)$ and $H_2(x,\ y)$ be extended negative quadrant dependent bivariate distribution;

$$
\begin{aligned}
H_1(x,\ y) &\leq M_1\ F(x)G(y) \text{ for some } M_1 \geq 1,\\
H_2(x,\ y) &\leq M_2\ F(x)G(y) \text{ for some } M_2 \geq 1
\end{aligned}
$$

Then, for $0 < \alpha < 1$,

$$
\begin{aligned}
H(x,\ y) &= \alpha H_1(x,\ y) + (1-\alpha)H_2(x,\ y)\\
&\leq (\alpha M_1 + (1-\alpha)M_2)F(x)G(y)\\
&= MF(x)G(y)
\end{aligned}
$$

where $M = \alpha M_1 + (1 - \alpha_2)M_2$.

Since $M \geq \min(M_1,\ M_2) \geq 1$, $H(x,\ y)$ is still extended negative quadrant dependent, and the proof is complete.

In the next section we consider only the continuous marginal case. But there also exists the extended negative quadrant dependent distribution such as follows. The following example is that the marginal distributions are not absolutely continuous case :

Example 2.8 Let $F(x)$ be a distribution of X, $G(y)$ a distribution of Y and $H(x,\ y)$ a joint distribution of X and Y, where

$$
F(x) = \left\{ \begin{array}{ll} 0\ , & x < 0\\ p\ , & 0 \leq x < 1\\ 1\ , & 1 \leq x\ , \end{array} \right.
$$

$$
G(y) = \left\{ \begin{array}{ll} 0\ , & y < 0\\ p\ , & 0 \leq y < 1\\ 1\ , & 1 \leq y\ , \end{array} \right.
$$

where $0 < p < 1$. Note that

$$
F(x)(1 - F(x)) = \left\{ \begin{array}{ll} p(1-p), & 0 \leq x < 1\\ 0\ , & \text{otherwise,} \end{array} \right.
$$

$$
G(y)(1 - G(y)) = \left\{ \begin{array}{ll} p(1-p), & 0 \leq y < 1\\ 0\ , & \text{otherwise.} \end{array} \right.
$$

Then, with $\alpha = [p(1-p)]^{-1} > 0$,

$$
H(x,\ y) = F(x)G(y)[1 + \alpha(1 - F(x))(1 - G(y))]
$$

corresponds to the bivariate distribution

$$P(X = 0, \ Y = 0) = p \ P(X = 1, \ Y = 1) = 1 - p,$$

even though $\alpha > 1$. Hence, $H(x, \ y)$ is extended negative quadrant dependent.

3 Conditions for extended negative quadrant dependence distribution

First, we study a continuous bivariate distributions that possesses the conditions for extended negative quadrant dependent property.

Theorem 3.1 Let $F(x, \ y)$ denote the distribution function of $(X, \ Y)$ having continuous marginal distributions $F(x)$ and $G(y)$ with marginal probability density functions $f = \dfrac{d}{dx}F$, $g = \dfrac{d}{dy}G$. Assume that $H(x, \ y)$ may be written as $H(x, \ y) = F(x)G(y) + w(x, \ y)$ with $w(x, \ y)$ satisfies the following conditions;

(3.1) $w(x, \ \infty) = 0, \ w(\infty, \ y) = 0, \ w(x, -\infty) = 0, \ w(-\infty, \ y) = 0$, for all x and y,

(3.2) $\dfrac{\partial^2 w(x,y)}{\partial x \partial y} + f_1(x)f_2(y) \geq 0$ for all x and y,

(3.3) $-F(x)G(y) < w(x, \ y) < mF(x)G(y)$ for some $m > 0$.

Then $H(x, \ y)$ is extended negative quadrant dependent.

Proof First we show that $H(x, y)$ satisfies the condition of the joint distribution.

Note that $H(x, y) \geq 0$,

$$\lim_{y \to \infty} H(x, y) = F(x), \ \lim_{x \to \infty} H(x, y) = G(y),$$

$$\lim_{y \to -\infty} H(x, y) = \lim_{x \to -\infty} H(x, y) = 0,$$

$$\frac{\partial^2 H(x, y)}{\partial x \partial y} = f(x)g(y) + \frac{\partial^2 W(x, y)}{\partial x \partial y} \geq 0 \ \text{ by (3.2)}$$

Hence, $H(x, y)$ satisfies the condition of the joint distribution of X and Y.

Next we show that $H(x, y)$ is extended negative quadrant dependent. If

$$-F(x)G(y) < w(x, y) \leq 0,$$

then

$$H(x,y) \leq F(x)G(y).$$

If

$$0 < w(x,y) < mF(x)G(y)$$

then

$$
\begin{aligned}
H(x,y) &= F(x)G(y) + w(x,y) \\
&< F(x)G(y) + mF(x)G(y) \\
&< (1+m)F(x)G(y) \\
&= m'F(x)G(y), \text{ where } (1+m) = m' > 1.
\end{aligned}
$$

Both the case $H(x,y)$ is extended negative quadrant dependent, that is $H(x,y)$ is extended negative quadrant dependent.

Remark If $-F(x)G(y) < w(x, y) \leq 0$, then $H(x, y)$ is negatively quadrant dependent. If $0 \leq w(x, y)$, then $H(x, y)$ is positively quadrant dependent.

Example 3.2 Farlie(1960), Gumbel(1958) and Morgenstern(1956) have discussed families of bivariate distribution of the form
(3.4) $H(x, y) = F(x)G(y)[1 + \alpha\overline{F}(x)\overline{G}(y)]$, $-1 \leq \alpha \leq 1$
where $H(x, y)$ is the joint distribution of X and Y, $F(x)$ and $G(y)$ are marginal distributions of X and Y, respectively, $\overline{F}(x) = 1 - F(x)$ and $\overline{G}(y) = 1 - G(y)$.
It is clear that $w(x, y) = \alpha F(x)G(y)\overline{F}(x)\overline{G}(y)$ satisfies (3.1) and (3.3). If the density of $H(\cdot)$ exists then (3.4) implies

$$
\begin{aligned}
h(x, y) &= f(x)g(y) + \frac{\partial^2 w(x, y)}{\partial x \partial y} \\
&= f(x)g(y)[1 + \alpha\{1 - 2F(x)\}\{1 - 2G(y)\}],
\end{aligned}
$$

which satisfies (3.2).

Remark We have the well known symmetrical relationships

$$H(x, y) = F(x)G(y)[1 + \alpha\{1 - F(x)\}\{1 - G(y)\}]$$

$$\overline{H}(x, y) = \overline{F}(x)\overline{G}(y)[1 + \alpha\{1 - \overline{F}(x)\}\{1 - \overline{G}(y)\}].$$

Example 3.3 Consider a special case of the F-G-M system where both

marginal distributions are exponential(see for Example, Johnson and Kotzs(1972)).

Clearly, for $-1 \leq \rho \leq 1$

$$w(x, \ y) = \rho e^{\lambda_1 x - \lambda_2 y}(1 - e^{-\lambda_1 x})(1 - e^{\lambda_2 y})$$

satisfies (3.1), (3.2) and (3.3). Then for $-1 \leq \rho \leq 1$, X and Y are extended negative quadrant dependent.

Theorem 3.4 Every convex combination of two bivariate distribution functions having fixed marginal distributions $F(x)$ and $G(y)$ and satisfying conditions (3.1), (3.2) and (3.3) still satisfies (3.1), (3.2) and (3.3).

Proof Let

$$
\begin{aligned}
H_1(x, \ y) &= F(x)G(y) + w_1(x, \ y), \\
H_2(x, \ y) &= F(x)G(y) + w_2(x, \ y), \\
H(x, \ y) &= \alpha H_1(x, \ y) + (1-\alpha)H_2(x, \ y), \ 0 \leq \alpha \leq 1.
\end{aligned}
$$

Then,

$$H(x, \ y) = F(x)G(y) + w(x, \ y)$$

where $w(x, \ y) = \alpha w_1(x, \ y) + (1-\alpha)w_2(x, \ y)$.

Clearly $w(x, \ y)$ satisfies (3.1). Next we have

$$
\begin{aligned}
\frac{\partial^2 w(x, \ y)}{\partial x \partial y} + f(x)g(y) &= \alpha \frac{\partial^2 w_1(x, \ y)}{\partial x \partial y} + (1-\alpha)\frac{\partial^2 w_2(x, \ y)}{\partial x \partial y} + f(x)g(y) \\
&\geq \min\left(\frac{\partial^2 w_1(x, \ y)}{\partial x \partial y}, \ \frac{\partial^2 w_2(x, \ y)}{\partial x \partial y}\right) + f(x)g(y) \\
&\geq 0,
\end{aligned}
$$

which satisfies (3.2).

Note that

$$w(x, \ y) = \alpha w_1(x, \ y) + (1-\alpha)w_2(x, \ y).$$

Since $-F(x)G(y) \leq w_1(x, \ y) \leq m_1 F(x)G(y)$ and
$-F(x, \ y) < w_2(x, \ y) < m_2 F(x)G(y), m_1, m_2 > 0$,

$$
\begin{aligned}
-F(x)G(y) &\leq \alpha w_1(x, \ y) + (1-\alpha)w_2(x, \ y) \\
&\leq (\alpha m_1 + (1-\alpha)m_2)F(x)G(y) \\
&\leq \max(m_1, \ m_2)F(x)G(y),
\end{aligned}
$$

which satisfies (3.3), and the proof is complete.

4 Limit behaviors under extended negative quadrant dependence

Theorem 4.1 Let $A_1, A_2, \cdots$ be a sequence of events satisfying conditions

(4.1) $\sum_{n=1}^{\infty} P(A_n) = \infty$

and

(4.2) $P(A_k A_j) \leq C P(A_k) P(A_j)$

for all $k, j > L$, such that $k \neq j$ and for some constants $C \geq 1$ and L. Then, we have

(4.3) $P(\limsup A_n) \geq \dfrac{1}{C}$.

Proof See for proof, Petrov(2002).

Lemma 4.2 Let $\{X_n, \ n \geq 1\}$ be a sequence of pairwise extended negative quadrant dependent random variables, $S_n = \sum_{i=1}^{n} X_i$, and $\{a_n, \ n \geq 1\}$ a sequence of positive numbers. If $S_n/a_n \to 0$ $a.s.$ and $\sup_{n\geq1} a_{n-1}/a_n \leq M < \infty$ for some M, then $\sum_{n=1}^{\infty} P\{|X_n| \geq a_n\} < \infty$.

Proof If $S_n/a_n \to 0$ $a.s.$, then $X_n/a_n \to 0$ $a.s.$ as $n \to \infty$. Thus putting $X_n^+ = \max\{0, \ X_n\}$, $X_n^- = \max\{0, \ -X_n\}$, we see that $X_n^+/a_n \to 0$ $a.s.$, and $X_n^-/a_n \to 0$ $a.s.$. By taking into account Theorem 2.6 we see that $\{X_n^+, \ n \geq 1\}$ and $\{X_n^-, \ n \geq 1\}$ are extended negative quadrant dependent. Now we define the following events,

$$A_n = [X_n^+ > \frac{1}{3}a_n], \ B_n = [X_n^- > \frac{1}{3}a_n], \ n \geq 1.$$

Then, for $C \geq 1$, we get

$$P(A_k \cap A_j) \leq C P(A_k) P(A_j) \text{ for } k \neq j$$

and

$$P(B_k \cap B_j) \leq C P(B_k) P(B_j) \text{ for } k \neq j.$$

By Theorem 4.1, if $\sum_{n=1}^{\infty} P[X_n^+ > \frac{1}{3}a_n]$ diverges then $P[\limsup A_n] \geq \dfrac{1}{C}$ contrary to the almost sure convergence of $X_n^+/a_n \to$ to zero. Hence, $\sum_{n=1}^{\infty} P[X_n^+ > \frac{1}{3}a_n] < \infty$. The same argument for X_n^- yields $\sum_{n=1}^{\infty} P[X_n^- \geq$

$\frac{1}{3}a_n] < \infty$. Thus

$$
\begin{aligned}
\sum_{n=1}^{\infty} P[|X_n| > a_n] &= \sum_{n=1}^{\infty} P[X_n^+ + X_n^- \geq a_n] \\
&\leq \sum_{n=1}^{\infty} P[X_n^+ > \frac{1}{3}a_n] + \sum_{n=1}^{\infty} P[X_n^- > \frac{1}{3}a_n] < \infty,
\end{aligned}
$$

and the proof is complete.

Theorem 4.3 Let $\{X_n, \ n \geq 1\}$ be a sequence of pairwise extended negative quadrant dependent random variables, with the same distribution $F(x)$. If $S_n/n \to 0$ a.s., then $E|X_1| < \infty$.

Proof Setting $a_n = n$ in Lemma 4.2 and taking into account that X_i's are equidistributed we get $\sum_{n=1}^{\infty} P[|X_n| > n] < \infty$, thus $E|X_1| < \infty$, and the proof is complete.

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