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# COMMON FIXED POINT FOR COMPATIBLE MAPPINGS OF TYPE($\alpha$) ON INTUITIONISTIC FUZZY METRIC SPACE WITH IMPLICIT RELATIONS

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**Abstract.** In this paper, we will establish common fixed point for compatible mappings of type($\alpha$) for four self mappings defined on intuitionistic fuzzy metric space with implicit relations.

## 1. Introduction

Several authors([4], [5]) have introduced the basic concepts on fuzzy metric spaces and fuzzy topological spaces induced by fuzzy metrics with different ways. Grabiec[2] obtained the Banach contraction principle in setting of fuzzy metric spaces. Also, I. Altun and D. Turkoglu[1] proved some fixed theorems using implicit relations in fuzzy metric spaces.

Recently, Park et.al.[11] defined the intuitionistic fuzzy metric space, and Park et.al.[7] proved a fixed point theorem of Banach for the contractive mapping of a complete intuitionistic fuzzy metric space, and Park and Kim[10] established common fixed point theorem for four self maps in intuitionistic fuzzy metric space.

In this paper, we will obtain a unique common fixed point theorem for compatible mappings of type($\alpha$) defined on intuitionistic fuzzy metric space under implicit relations.

## 2. Preliminaries

We will give some definitions, properties of the intuitionistic fuzzy metric space $X$ as following :

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Let us recall (see [12]) that a continuous $t-$norm is a binary operation $*: [0,1]\times[0,1]\to[0,1]$ which satisfies the following conditions:(a)$*$ is commutative and associative; (b)$*$ is continuous; (c)$a*1=a$ for all $a\in[0,1]$; (d)$a*b\le c*d$ whenever $a\le c$ and $b\le d$ $(a,b,c,d\in[0,1])$.

Similarly, a continuous $t-$conorm is a binary operation $\diamond: [0,1]\times[0,1]\to[0,1]$ which satisfies the following conditions: (a)$\diamond$ is commutative and associative; (b)$\diamond$ is continuous; (c)$a\diamond 0=a$ for all $a\in[0,1]$; (d)$a\diamond b\ge c\diamond d$ whenever $a\le c$ and $b\le d$ $(a,b,c,d\in[0,1])$.

**Definition 2.1.** ([6])The $5-$tuple $(X,M,N,*,\diamond)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and $M,N$ are fuzzy sets on $X^2\times(0,\infty)$ satisfying the following conditions; for all $x,y,z\in X$, such that

(a)$M(x,y,t)>0$,

(b)$M(x,y,t)=1\Longleftrightarrow x=y$,

(c)$M(x,y,t)=M(y,x,t)$,

(d)$M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,

(e)$M(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous,

(f)$N(x,y,t)>0$,

(g)$N(x,y,t)=0\Longleftrightarrow x=y$,

(h)$N(x,y,t)=N(y,x,t)$,

(i)$N(x,y,t)\diamond N(y,z,s)\ge N(x,z,t+s)$,

(j)$N(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous.

Note that $(M,N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

**Lemma 2.2.** ([8])*For all $x,y\in X$, $M(x,y,\cdot)$ is nondecreasing on $(0,\infty)$ and $N(x,y,\cdot)$ is nonincreasing on $(0,\infty)$.*

**Definition 2.3.** ([10]) Let $X$ be an intuitionistic fuzzy metric space.

(a) $\{x_n\}$ is said to be convergent to a point $x\in X$ if, for any $0<\epsilon<1$ and $t>0$, there exists $n_0\in N$ such that $M(x_n,x,t)>1-\epsilon$, $N(x_n,x,t)<\epsilon$ for all $n\ge n_0$.

(b) $\{x_n\}$ is called a Cauchy sequence if for any $0<\epsilon<1$ and $t>0$, there exists $n_0\in N$ such that $M(x_n,x_m,t)>1-\epsilon$, $N(x_n,x_m,t)<\epsilon$ for all $m,n\ge n_0$.

(c) $X$ is complete if every Cauchy sequence converges in $X$.

**Lemma 2.4.** ([9])*Let $X$ be an intuitionistic fuzzy metric space. If there exists a number $k\in(0,1)$ such that for all $x,y\in X$ and $t>0$,*

$$M(x,y,kt)\ge M(x,y,t),\ N(x,y,kt)\le N(x,y,t),$$

*then* $x = y$.

**Definition 2.5.** ([9])Let $A, B$ be mappings from intuitionistic fuzzy metric space $X$ into itself. The mappings are said to be compatible of type($\alpha$) if

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1,$$
$$\lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0$$

for all $t > 0$, whenever $\{x_n\} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some $x \in X$.

Implicit relations on fuzzy metric spaces have been used in many articles([1], [3] etc). Let $\Psi = \{\phi_M, \psi_N\}$, $I = [0, 1]$, $*, \diamond$ be a continuous t-norm, t-conorm and $\phi_M, \psi_N : I^6 \to R$ be continuous functions. Now, we consider the following conditions:

(I)$\phi_M$ is decreasing and $\psi_N$ is increasing in sixth variables.

(II)If, for some $k \in (0, 1)$, we have

$$\phi_M(u(kt), v(t), v(t), u(t), 1, u(\frac{t}{2}) * v(\frac{t}{2})) \geq 1,$$
$$\psi_N(x(kt), y(t), y(t), x(t), 0, x(\frac{t}{2}) \diamond y(\frac{t}{2})) \leq 1$$
$$\text{or} \qquad \phi_M(u(kt), v(t), u(t), v(t), u(\frac{t}{2}) * v(\frac{t}{2}), 1) \geq 1,$$
$$\psi_N(x(kt), y(t), x(t), y(t), x(\frac{t}{2}) \diamond y(\frac{t}{2}), 0) \leq 1$$

for any fixed $t > 0$, any nondecreasing functions $u, v : (0, \infty) \to I$ with $0 < u(t), v(t) \leq 1$, and any nonincreasing functions $x, y : (0, \infty) \to I$ with $0 < x(t), y(t) \leq 1$, then there exists $h \in (0, 1)$ with $u(ht) \geq v(t) * u(t)$, $x(ht) \leq y(t) \diamond x(t)$.

(III)If, for some $k \in (0, 1)$, we have $\phi_M(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$ for any fixed $t > 0$ and any nondecreasing function $u : (0, \infty) \to I$, then $u(kt) \geq u(t)$. Also, if, for some $k \in (0, 1)$, we have $\psi_N(x(kt), x(t), 0, 0, x(t), x(t)) \leq 1$ for any fixed $t > 0$ and any nonincreasing function $x : (0, \infty) \to I$, then $x(kt) \leq x(t)$.

**Example 2.6.** Let $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$,

$$\phi_M(u_1, \cdots, u_6) = \frac{u_1}{\min\{u_2, \cdots, u_6\}}, \ \psi_N(x_1, \cdots, x_6) = \frac{x_1}{\max\{x_2, \cdots, x_6\}}.$$

Also, let $t > 0$, $0 < u(t), v(t), x(t), y(t) \leq 1$, $k \in (0, \frac{1}{2})$ where $u, v : [0, \infty) \to I$ are nondecreasing functions and $x, y : [0, \infty) \to I$ are nonincreasing functions. Now, suppose that

$$\phi_M(u(kt), v(t), v(t), u(t), 1, u(\frac{t}{2}) * v(\frac{t}{2})) \geq 1,$$
$$\psi_N(x(kt), y(t), y(t), x(t), 0, x(\frac{t}{2}) \diamond y(\frac{t}{2})) \leq 1,$$

then

$$\phi_M(u(kt), v(t), v(t), u(t), 1, u(\frac{t}{2}) * v(\frac{t}{2})) = \frac{u(kt)}{\min\{u(\frac{t}{2}), v(\frac{t}{2}))} \geq 1,$$
$$\psi_N(x(kt), y(t), y(t), x(t), 0, x(\frac{t}{2}) \diamond y(\frac{t}{2})) = \frac{x(kt)}{\max\{x(\frac{t}{2}), y(\frac{t}{2}))} \leq 1.$$

Thus, $u(ht) \geq v(t) * u(t)$, $x(ht) \leq y(t) \diamond x(t)$. Suppose that $t > 0$ is fixed, $u : (0, \infty) \to I$ is a nondecreasing, $x : (0, \infty) \to I$ nonincreasing function and

$$\phi_M(u(kt), u(t), 1, 1, u(t), u(t)) = \frac{u(kt)}{u(t)} \geq 1,$$
$$\psi_N(x(kt), x(t), 0, 0, x(t), x(t)) = \frac{x(kt)}{x(t)} \leq 1$$

for $k \in (0, 1)$. Then we have $u(kt) \geq u(t)$ and $x(kt) \leq x(t)$. Hence $\phi_M, \psi_N \in \Psi$.

## 3. Main Results

Now, we will prove some common fixed point theorem for four mappings on complete intuitionistic fuzzy metric space as follows:

**Theorem 3.1.** *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in I$ and $A, B, S$ and $T$ be mappings from $X$ into itself satisfying the conditions:*
*(a)$S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$,*
*(b)one of the mappings $A, B, S, T$ is continuous,*
*(c)$A$ and $S$ as well as $B$ and $T$ are compatible of type($\alpha$)*

*(d)there exist $k \in (0,1)$ and $\phi_M, \psi_N \in \Psi$ such that*

$$\phi_M\left(\begin{array}{l} M(Sx,Ty,kt), M(Ax,By,t), M(Sx,Ax,t), \\ M(Ty,By,t), M(Sx,By,t), M(Ty,Ax,t) \end{array}\right) \geq 1,$$
$$\psi_N\left(\begin{array}{l} N(Sx,Ty,kt), N(Ax,By,t), N(Sx,Ax,t), \\ N(Ty,By,t), N(Sx,By,t), N(Ty,Ax,t) \end{array}\right) \leq 1,$$

*for all $x, y \in X$ and $t > 0$.*

*Then $A, B, S$ and $T$ have a unique common fixed point in $X$.*

*Proof.* Let $x_0$ be an arbitrary point of $X$. From (a), we can construct a sequence a sequence $\{y_n\} \subset X$ as follows: $y_{2n+1} = Sx_{2n} = Bx_{2n+1}$ and $y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}$ for all $n = 0, 1, 2, \cdots$. Then, by (d), we have, for any $t > 0$,

$$\phi_M\left(\begin{array}{l} M(Sx_{2n},Tx_{2n+1},kt), M(Ax_{2n},Bx_{2n+1},t), M(Sx_{2n},Ax_{2n},t), \\ M(Tx_{2n+1},Bx_{2n+1},t), M(Sx_{2n},Bx_{2n+1},t), M(Tx_{2n+1},Ax_{2n},t) \end{array}\right) \geq 1,$$
$$\psi_N\left(\begin{array}{l} N(Sx_{2n},Tx_{2n+1},kt), N(Ax_{2n},Bx_{2n+1},t), N(Sx_{2n},Ax_{2n},t), \\ N(Tx_{2n+1},Bx_{2n+1},t), N(Sx_{2n},Bx_{2n+1},t), N(Tx_{2n+1},Ax_{2n},t) \end{array}\right) \leq 1,$$

and so

$$\phi_M\left(\begin{array}{l} M(Sx_{2n},Tx_{2n+1},kt), M(Tx_{2n-1},Sx_{2n},t), M(Sx_{2n},Tx_{2n-1},t), \\ M(Tx_{2n+1},Sx_{2n},t), 1, M(Tx_{2n+1},Sx_{2n},\frac{t}{2}) * M(Sx_{2n},Tx_{2n-1},\frac{t}{2}) \end{array}\right) \geq 1,$$
$$\psi_N\left(\begin{array}{l} N(Sx_{2n},Tx_{2n+1},kt), N(Tx_{2n-1},Sx_{2n},t), N(Sx_{2n},Tx_{2n-1},t), \\ N(Tx_{2n+1},Sx_{2n},t), 0, N(Tx_{2n+1},Sx_{2n},\frac{t}{2}) \diamond N(Sx_{2n},Tx_{2n-1},\frac{t}{2}) \end{array}\right) \leq 1,$$

By (II), we have

$$M(Sx_{2n},Tx_{2n+1},ht) \geq M(Sx_{2n},Tx_{2n-1},t) * M(Sx_{2n},Tx_{2n+1},t),$$
$$N(Sx_{2n},Tx_{2n+1},ht) \leq N(Sx_{2n},Tx_{2n-1},t) \diamond N(Sx_{2n},Tx_{2n+1},t)$$

and so,

$$M(y_{2n+1},y_{2n+2},ht) \geq M(y_{2n+1},y_{2n},t) * M(y_{2n+1},y_{2n+2},t),$$
$$N(y_{2n+1},y_{2n+2},ht) \leq N(y_{2n+1},y_{2n},t) \diamond N(y_{2n+1},y_{2n+2},t)$$

which implies that

$$M(y_{2n+1},y_{2n+2},ht) \geq M(y_{2n+1},y_{2n},t), \quad N(y_{2n+1},y_{2n+2},ht) \leq N(y_{2n+1},y_{2n},t)$$

Also, by (II), we have

$$M(y_{2n+1},y_{2n},ht) \geq M(y_{2n},y_{2n-1},t), \quad N(y_{2n+1},y_{2n},ht) \leq N(y_{2n},y_{2n-1},t).$$

Therefore, we have, for all $m = 1, 2, \cdots$, and $t > 0$,

$$M(y_{m+1},y_{m+2},ht) \geq M(y_m,y_{m+1},t), \quad N(y_{m+1},y_{m+2},ht) \leq N(y_m,y_{m+1},t).$$

To prove that $\{y_n\}$ is a Cauchy sequence. First, we show that, for any $0 < \lambda < 1$ and $t > 0$,

(1) $\quad M(y_{n+1}, y_{n+m+1}, t) > 1 - \lambda, \quad N(y_{n+1}, y_{n+m+1}, t) < \lambda$

for all $n \geq n_0$ and $m \in N$. Inductively, by above equation, we have, as $n \to \infty$

$$M(y_{n+1}, y_{n+2}, t) \geq M(y_n, y_{n+1}, \frac{t}{h} \geq \cdots\cdots \geq M(y_1, y_2, \frac{t}{h^n}) \to 1,$$
$$N(y_{n+1}, y_{n+2}, t) \leq N(y_n, y_{n+1}, \frac{t}{h} \leq \cdots\cdots \leq N(y_1, y_2, \frac{t}{h^n}) \to 0.$$

Hence, we can choose $n_0 \in N$ such that for all $n \geq n_0$,

$$M(y_{n+1}, y_{n+2}, t) > 1 - \lambda, \quad N(y_{n+1}, y_{n+2}, t) < \lambda.$$

Thus (3.1) is true for $m = 1$. Suppose that (3.1) is true for some $m \in N$. Then, for $m + 1 \in N$, we have

$$M(y_{n+1}, y_{n+m+2}, t) \geq M(y_{n+1}, y_{n+m+1}, \frac{t}{2}) * M(y_{n+m+1}, y_{n+m+2}, \frac{t}{2}) \geq 1 - \lambda,$$
$$N(y_{n+1}, y_{n+m+2}, t) \leq N(y_{n+1}, y_{n+m+1}, \frac{t}{2}) \diamond N(y_{n+m+1}, y_{n+m+2}, \frac{t}{2}) \leq \lambda.$$

Hence (3.1) is true for $m + 1 \in N$. Therefore $\{y_n\}$ is Cauchy sequence in $X$. Since $X$ is complete, $\{y_n\}$ converges to a point $x \in X$. Since $\{Ax_{2n+2}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\} \subset \{y_n\}$, we have

$$\lim_{n\to\infty} Ax_{2n+2} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = x.$$

Now, suppose that $A$ is continuous. Then $\lim_{n\to\infty} ASx_{2n} = Ax$. Also, since $A$, $S$ are compatible of type($\alpha$), $\lim_{n\to\infty} SAx_{2n} = Ax$. Using (d), we have, for any $t > 0$,

$$\phi_M \left( \begin{array}{l} M(SAx_{2n}, Tx_{2n+1}, kt), M(AAx_{2n}, Bx_{2n+1}, t), M(SAx_{2n}, AAx_{2n}, t), \\ M(Tx_{2n+1}, Bx_{2n+1}, t), M(SAx_{2n}, Bx_{2n+1}, t), M(Tx_{2n+1}, AAx_{2n}, t) \end{array} \right) \geq 1,$$
$$\psi_N \left( \begin{array}{l} N(SAx_{2n}, Tx_{2n+1}, kt), N(AAx_{2n}, Bx_{2n+1}, t), N(SAx_{2n}, AAx_{2n}, t), \\ N(Tx_{2n+1}, Bx_{2n+1}, t), N(SAx_{2n}, Bx_{2n+1}, t), N(Tx_{2n+1}, AAx_{2n}, t) \end{array} \right) \leq 1$$

and then letting $n \to \infty$, $\phi_M$, $\psi_N$ are continuous, we have

$$\phi_M \left( \begin{array}{l} M(Ax, x, kt), M(Ax, x, t), M(Ax, x, t), \\ M(x, x, t), M(Ax, x, t), M(x, x, t) \end{array} \right) \geq 1,$$
$$\psi_N \left( \begin{array}{l} N(Ax, x, kt), N(Ax, x, t), N(Ax, x, t), \\ N(x, x, t), N(Ax, x, t), N(x, x, t) \end{array} \right) \leq 1.$$

Therefore, by (III), we have

$$M(Ax, x, kt) \geq M(Ax, x, t), \quad N(Ax, x, kt) \leq N(Ax, x, t).$$

Hence $Ax = x$ from Lemma 2.4. Also, we have, by (d),

$$\phi_M \begin{pmatrix} M(Sx, Tx_{2n+1}, kt), M(Ax, Bx_{2n+1}, t), M(Ax, Sx, t), \\ M(Tx_{2n+1}, Bx_{2n+1}, t), M(Sx, Bx_{2n+1}, t), M(Tx_{2n+1}, Ax, t) \end{pmatrix} \geq 1,$$
$$\psi_N \begin{pmatrix} N(Sx, Tx_{2n+1}, kt), N(Ax, Bx_{2n+1}, t), N(Ax, Sx, t), \\ N(Tx_{2n+1}, Bx_{2n+1}, t), N(Sx, Bx_{2n+1}, t), N(Tx_{2n+1}, Ax, t) \end{pmatrix} \leq 1$$

and, let $n \to \infty$, we get

$$\phi_M \begin{pmatrix} M(Sx, x, kt), 1, M(x, Sx, t), \\ 1, M(Sx, x, t), 1 \end{pmatrix} \geq 1,$$
$$\psi_N \begin{pmatrix} N(Sx, x, kt), 0, N(x, Sx, t), \\ 0, N(Sx, x, t), 0 \end{pmatrix} \leq 1.$$

On the other hand, since

$$M(Sx, x, t) \geq M(Sx, x, \frac{t}{2}) = M(Sx, x, \frac{t}{2}) * 1,$$
$$N(Sx, x, t) \leq N(Sx, x, \frac{t}{2}) = N(Sx, x, \frac{t}{2}) \diamond 0,$$

$\phi_M$ is nonincreasing and $\psi_N$ is nondecreasing in the fifth variable, we have, for any $t > 0$,

$$\phi_M \begin{pmatrix} M(Sx, x, kt), 1, M(x, Sx, t), \\ 1, M(Sx, x, t) * 1, 1 \end{pmatrix} \geq 1,$$
$$\psi_N \begin{pmatrix} N(Sx, x, kt), 0, N(x, Sx, t), \\ 0, N(Sx, x, t) \diamond 0, 0 \end{pmatrix} \leq 1$$

which implies that $Sx = x$. Since $S(X) \subseteq B(X)$, there exists a point $y \in X$ such that $By = x$. Using (d), we have

$$\begin{aligned} &\phi_M \begin{pmatrix} M(Sx, Ty, kt), M(Ax, By, t), M(Sx, Ax, t), \\ M(Ty, By, t), M(Sx, By, t), M(Ty, Ax, t) \end{pmatrix} \\ =&\phi_M \begin{pmatrix} M(x, Ty, kt), 1, 1, \\ M(Ty, x, t), 1, M(Ty, x, t) \end{pmatrix} \geq 1, \\ &\psi_N \begin{pmatrix} N(Sx, Ty, kt), N(Ax, By, t), N(Sx, Ax, t), \\ N(Ty, By, t), N(Sx, By, t), N(Ty, Ax, t) \end{pmatrix} \\ =&\psi_N \begin{pmatrix} N(x, Ty, kt), 0, 0, \\ N(Ty, x, t), 0, N(Ty, x, t) \end{pmatrix} \leq 1 \end{aligned}$$

which implies that $x = Ty$. Since $By = Ty = x$ and $B, T$ are compatible of type($\alpha$), we have $TTy = BTy$. Hence $Tx = TTy = BTy = Bx$.

Therefore, from (d), we have, for any $t > 0$,

$$\begin{aligned}&\phi_M\begin{pmatrix} M(Sx,Tx,kt), M(Ax,Bx,t), M(Sx,Ax,t), \\ M(Tx,Bx,t), M(Sx,Bx,t), M(Tx,Ax,t)\end{pmatrix}\\ =&\phi_M\begin{pmatrix} M(x,Tx,kt), M(x,Tx,t), 1, \\ 1, M(x,Tx,t), 1, M(x,Tx,t)\end{pmatrix} \geq 1,\\ &\psi_N\begin{pmatrix} N(Sx,Tx,kt), N(Ax,Bx,t), N(Sx,Ax,t), \\ N(Tx,Bx,t), N(Sx,Bx,t), N(Tx,Ax,t)\end{pmatrix}\\ =&\psi_N\begin{pmatrix} N(x,Tx,kt), N(x,Tx,t), 0, \\ 0, N(x,Tx,t), 0, N(x,Tx,t)\end{pmatrix} \leq 1.\end{aligned}$$

From (III), we have

$$M(x,Tx,kt) \geq M(x,Tx,t), \quad N(x,Tx,kt) \leq N(x,Tx,t).$$

Therefore, we have $x = Tx = Bx$. Hence $X$ is a common fixed point of $A, B, S$ and $T$. The same result holds if we assume that $B$ is continuous insead of $A$.

Now, suppose that $S$ is continuous. Then $\lim_{n\to\infty} SAx_{2n} = Sx$. Since $A, S$ are compatible of type($\alpha$), $\lim_{n\to\infty} ASx_{2n} = Sx$. Using (d), we have for any $t > 0$,

$$\phi_M\begin{pmatrix} M(SSx_{2n},Tx_{2n+1},kt), M(ASx_{2n},Bx_{2n+1},t), M(SSx_{2n},ASx_{2n},t), \\ M(Tx_{2n+1},Bx_{2n+1},t), M(SSx_{2n},Bx_{2n+1},t), M(Tx_{2n+1},ASx_{2n},t)\end{pmatrix} \geq 1,$$

$$\psi_N\begin{pmatrix} N(SSx_{2n},Tx_{2n+1},kt), N(ASx_{2n},Bx_{2n+1},t), N(SSx_{2n},ASx_{2n},t), \\ N(Tx_{2n+1},Bx_{2n+1},t), N(SSx_{2n},Bx_{2n+1},t), N(Tx_{2n+1},ASx_{2n},t)\end{pmatrix} \leq 1,$$

and then by $n \to \infty$, since $\phi_M$, $\psi_N \in \Psi$ are continuous, we have

$$\phi_M\begin{pmatrix} M(Sx,x,kt), M(Sx,x,t), 1, \\ 1, M(Sx,x,t), M(Sx,x,t)\end{pmatrix} \geq 1,$$

$$\psi_N\begin{pmatrix} N(Sx,x,kt), N(Sx,x,t), 0, \\ 0, N(Sx,x,t), N(Sx,x,t)\end{pmatrix} \leq 1.$$

Thus, we have, from (III),

$$M(Sx,x,kt) \geq M(Sx,x,t), \quad N(Sx,x,kt) \leq N(Sx,x,t).$$

Hence $Sx = x$. Since $S(X) \subseteq B(X)$, there exists a point $z \in X$ such that $Bz = x$. Using (d), we have

$$\phi_M\begin{pmatrix} M(SSx_{2n},Tz,kt), M(ASx_{2n},Bz,t), M(SSx_{2n},ASx_{2n},t), \\ M(Tz,Bz,t), M(SSx_{2n},Bz,t), M(Tz,ASx_{2n},t)\end{pmatrix} \geq 1,$$

$$\psi_N\begin{pmatrix} N(SSx_{2n},Tz,kt), N(ASx_{2n},Bz,t), N(SSx_{2n},ASx_{2n},t), \\ N(Tz,Bz,t), N(SSx_{2n},Bz,t), N(Tz,ASx_{2n},t)\end{pmatrix} \leq 1,$$

letting $n \to \infty$, we get

$$\phi_M \begin{pmatrix} M(x, Tz, kt), 1, 1, \\ M(x, Tz, t), 1, M(x, Tz, t) \end{pmatrix} \geq 1,$$
$$\psi_N \begin{pmatrix} N(x, Tz, kt), 0, 0, \\ N(x, Tz, t), 0, N(x, Tz, t) \end{pmatrix} \leq 1$$

which implies that $x = Tz$. Since $Bz = Tz = x$ and $B, T$ are compatible of type($\alpha$), we have $TBz = BBz$ and so $Tx = TBz = BBz = Bx$. Thus, we have

$$\phi_M \begin{pmatrix} M(Sx_{2n}, Tx, kt), M(Ax_{2n}, Bx, t), M(Sx_{2n}, Ax_{2n}, t), \\ M(Tx, Bx, t), M(Sx_{2n}, Bx, t), M(Tx, Ax_{2n}, t) \end{pmatrix} \geq 1,$$
$$\psi_N \begin{pmatrix} N(Sx_{2n}, Tx, kt), N(Ax_{2n}, Bx, t), N(Sx_{2n}, Ax_{2n}, t), \\ N(Tx, Bx, t), N(Sx_{2n}, Bx, t), N(Tx, Ax_{2n}, t) \end{pmatrix} \leq 1,$$

letting $n \to \infty$,

$$\phi_M \begin{pmatrix} M(x, Tx, kt), M(x, Tx, t), 1, \\ 1, M(x, Tx, t), M(x, Tx, t) \end{pmatrix} \geq 1,$$
$$\psi_N \begin{pmatrix} N(x, Tx, kt), N(x, Tx, t), 0, \\ 0, N(x, Tx, t), N(x, Tx, t) \end{pmatrix} \leq 1.$$

Thus, $x = Tx = Bx$. Since $T(X) \subseteq A(X)$, there exists $w \in X$ such that $Aw = x$. Thus, from (d),

$$\begin{aligned} &\phi_M \begin{pmatrix} M(Sw, Tx, kt), M(Aw, Bx, t), M(Sw, Aw, t), \\ M(Tx, Bx, t), M(Sw, Bx, t), M(Tx, Aw, t) \end{pmatrix} \\ =&\phi_M \begin{pmatrix} M(Sw, x, kt), 1, M(Sw, x, t), \\ 1, M(Sw, x, t), 1 \end{pmatrix} \geq 1, \\ &\psi_N \begin{pmatrix} N(Sw, Tx, kt), N(Aw, Bx, t), N(Sw, Aw, t), \\ N(Tx, Bx, t), N(Sw, Bx, t), N(Tx, Aw, t) \end{pmatrix} \\ =&\psi_N \begin{pmatrix} N(Sw, x, kt), 0, N(Sw, x, t), \\ 0, N(Sw, x, t), 0 \end{pmatrix} \leq 1. \end{aligned}$$

Hence we have $x = Sw = Aw$. Also, since $A, S$ are compatible of type($\alpha$), $x = Sx = SAw = AAw = Ax$. Hence $x$ is a common fixed point of $A, B, S$ and $T$. The same result holds if we assume that $T$ is continuous instead of $S$.

Finally, suppose that $A, B, S$ and $T$ have another common fixed point $u$. Then we have, for any $t > 0$,

$$\begin{aligned}
&\phi_M\begin{pmatrix} M(Sx,Tu,kt), M(Ax,Bu,t), M(Sx,Ax,t), \\ M(Tu,Bu,t), M(Sx,Bu,t), M(Tu,Ax,t) \end{pmatrix} \\
=&\phi_M\begin{pmatrix} M(x,u,kt), M(x,u,t), 1, \\ 1, M(x,u,t), M(x,u,t) \end{pmatrix} \geq 1, \\
&\psi_N\begin{pmatrix} N(Sx,Tu,kt), N(Ax,Bu,t), N(Sx,Ax,t), \\ N(Tu,Bu,t), N(Sx,Bu,t), N(Tu,Ax,t) \end{pmatrix} \\
=&\psi_N\begin{pmatrix} N(x,u,kt), N(x,u,t), 0, \\ 0, N(x,u,t), N(x,u,t) \end{pmatrix} \leq 1.
\end{aligned}$$

Therefore, from (III), $x = u$. This completes the proof. □

## References

[1] Altun I., Turkoglu D., 2008. Some fixed point theorems on fuzzy metric spaces with implicit relations. Commun. Korean Math. Soc. 23, 111–124.
[2] Grabiec, M., 1988. Fixed point in fuzzy metric spaces. Fuzzy Sets and Systems 27, 385–389.
[3] Imbad M., Kumar S., Khan M.S., 2002. Remarks on some fixed point theorems satisfying implicit relations. Rad. Math. 11, 135–143.
[4] Kramosil,J., Michalek J., 1975. Fuzzy metric and statistical metric spaces. Kybernetica 11, 326–334.
[5] Kaleva, O., Seikkala, S., 1984. On fuzzy metric spaces. Fuzzy Sets and Systems 12, 215–229.
[6] Park, J.H., Park, J.S., Kwun, Y.C., 2006. A common fixed point theorem in the intuitionistic fuzzy metric space. Advances in Natural Comput. Data Mining(Proc. 2nd ICNC and 3rd FSKD), 293–300.
[7] Park, J.H., Park, J.S., Kwun, Y.C., 2007. Fixed point theorems in intuitionistic fuzzy metric space(I). JP J. fixed point Theory & Appl. 2(1), 79–89.
[8] Park, J.S., Kim, S.Y., 1999. A fixed point theorem in a fuzzy metric space. F.J.M.S. 1(6), 927–934.
[9] Park, J.S., Park, J.H., Kwun, Y.C., 2008. On some results for five mappings using compatibility of type($\alpha$) in intuitionistic fuzzy metric space. International J. Fuzzy Logic Intelligent Systems 8(4), 299–305
[10] Park, J.S., Kim, S.Y., 2008. Common fixed point theorem and example in intuitionistic fuzzy metric space. J. Fuzzy Logic and Intelligent Systems 18(4), 524–529.
[11] Park, J.S., Kwun, Y.C., Park, J.H., 2005. A fixed point theorem in the intuitionistic fuzzy metric spaces. F.J.M.S. 16(2), 137–149.
[12] Schweizer, B., Sklar, A., 1960. Statistical metric spaces. Pacific J. Math. 10, 314–334.

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