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YANG-MILLS CONNECTIONS ON CLOSED LIE GROUPS

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Abstract. In this paper, we obtain a necessary and sufficient condition for a left invariant connection in the tangent bundle over a closed Lie group with a left invariant metric to be a Yang-Mills connection. Moreover, we have a necessary and sufficient condition for a left invariant connection with a torsion-free Weyl structure in the tangent bundle over $SU(2)$ with a left invariant Riemannian metric g to be a Yang-Mills connection.

§1. Introduction

The problem of finding metrics and connections which are critical points of some functional plays an important role in global analysis and Riemannian geometry. A Yang-Mills connection is a critical point of the Yang-Mills functional

$$\mathcal{YM}(D) = \frac{1}{2}\int_M \|R^D\|^2 \, v_g \tag{1.1}$$

on the space $\mathfrak{C}_E$ of all connections in a smooth vector bundle E over a closed (compact and connected) Riemannian manifold (M, g), where R^D is the curvature of $D \in \mathfrak{C}_E$. Equivalently, D is a Yang-Mills connection if it satisfies the Yang-Mills equation (cf. [1, 7, 15])

$$\delta_D R^D = 0 \tag{1.2}$$

(the Euler-Lagrange equations of the variational principle associated with (1.1)).

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The purpose of this paper is to obtain a necessary and sufficient condition for a left invariant connection in the tangent bundle over a compact connected semisimple Lie group to be a Yang-Mills connection.

If D is a connection in a vector bundle E with bundle metric h over a Riemannian manifold (M, g), then the connection D^* given by

(1.3) $h(D^*{}_X s, t) = X(h(s,t)) - h(s, D_X t), \quad (X \in \mathfrak{X}(M), \;\; s, t \in \Gamma(E))$

is referred to *conjugate* (cf. [1, 9, 13, 14, 15]) to D.

Recently using the concept of conjugate connection, Park obtained the following.

Theorem 1.1. [12] *A connection D in a vector bundle E over a closed Riemannian manifold (M, g) is a Yang-Mills connection if and only if the conjugate connection D^* is a Yang-Mills connection.*

The theory of Einstein-Weyl structures (cf. [3, 15, 16]) in the tangent bundle over a closed Riemannian manifold (M, g) is a conformally invariant generalization of the theory of Einstein structures. By virtue of the above theorem, Park obtained the following.

Theorem 1.2. [13] *Let D be a Yang-Mills connection, not necessarily torsion free, with a Weyl structure (D, g, ω) in the tangent bundle TM over a closed Riemannian manifold (M, g). Then $d\omega = 0$.*

The following lemma is well known.

Lemma 1.3. [5] *A p-form ω on G is left invariant if and only if, for any choice of p left invariant vector fields $X_1, X_2, \cdots, X_p$, the function $\omega(X_1, X_2, \cdots, X_p)$ on G is a constant.*

Using this lemma, Park obtained the following.

Theorem 1.4. [14] *Let D be a left invariant Yang-Mills connection with (not necessarily torsion-free) Weyl structure (D, g, ω) in the tangent bundle over a closed Lie group G with a left invariant Riemannian metric g. Then, the 1-form ω is also left invariant.*

Theorem 1.5. [14] *Let G be an n-dimensional $(n \geq 3)$ closed semisimple Lie group, g the canonical metric on G and (D, g, ω) a torsion-free Weyl structure. Then D is a Yang-Mills connection if and only*

(i) $\omega = 0$, *or*

(ii) $d\omega = 0$ *and* $\|\omega\|_g^2 = \frac{1}{n-2}$.

In the above theorem, the canonical metric (cf. [2, 4, 6, 8, 17]) on a compact, connected and semisimple Lie group G is minus the Killing form of the Lie algebra $\mathfrak{g}$ of the group G.

In this paper, we obtain a necessary and sufficient condition for a left invariant metric connection in the tangent bundle over a closed Lie group with a left invariant metric to be a Yang-Mills connection (cf. Theorem 3.3). And then, we get a necessary and sufficient condition for a left invariant connection with a torsion-free Weyl structure in the tangent bundle over $SU(2)$ with a left invariant Riemannian metric to be a Yang-Mills connection (cf. Theorem 4.5).

§2. Yang-Mills connections in vector bundles over a Riemannian manifold

Let E be a vector bundle, with bundle metric h, over an n-dimensional closed Riemannian manifold (M, g). Let us have $D \in \mathfrak{C}_E$ and ∇ the Levi-Civita connection of (M, g). The pair (D, ∇) induces a connection in product bundles $\bigwedge^p TM^* \otimes E$, also denoted by D. Set $A^p(E) := \Gamma(\bigwedge^p TM^* \otimes E)$. We consider the differential operator

$$
\begin{aligned}
&d_D : A^p(E) \longrightarrow A^{p+1}(E),\\
&(d_D\varphi)(X_1, X_2, \cdots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} (D_{X_i}\varphi)(X_1, \cdots, \widehat{X}_i, \cdots, X_{p+1}),\\
&\varphi \in A^p(E),\ X_i \in \mathfrak{X}(M)\ (i = 1, 2, \cdots, p+1),
\end{aligned}
$$

defined by

$$
\begin{aligned}
d_D(\omega \otimes \xi) &:= d\omega \otimes \xi + (-1)^p \omega \wedge D\xi,\\
D_X(\omega \otimes \xi) &:= (\nabla_X \omega) \otimes \xi + \omega \otimes D_X \xi
\end{aligned}
$$

for $\omega \in \Gamma(\bigwedge^p TM^*),\ \xi \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$.

Let δ_D be the formal adjoint of d_D with respect to the L^2-inner product

$$
(\varphi, \psi) = \int_M < \varphi, \psi > v_g
$$

for $\varphi, \psi \in A^p(E)$. Here $< \, , \, >$ is the bundle metric in $\bigwedge^p TM^* \otimes E$ induced by the pair (g, h) and v_g is the canonical volume form on (M, g). The following identity is elementary, yet crucial (cf. [1, 2])

$$\delta_D \varphi = (-1)^{p+1}(*^{-1} \cdot d_{D^*} \cdot *)(\varphi) = (-1)^{np+1}(* \cdot d_{D^*} \cdot *)(\varphi) \tag{2.1}$$

for any $\varphi \in A^{p+1}(E)$. Here, $* : A^q(E) \longrightarrow A^{n-q}(E), (0 \leq q \leq n)$, is the Hodge operator with respect to g. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame on (M, g). Note that (2.1) may also be written as (cf. [1, 2])

$$(\delta_D \varphi)(X_1, \cdots, X_p) = -\sum_{i=1}^{n} (D^*_{e_i} \varphi)(e_i, X_1, \cdots, X_p). \tag{2.2}$$

The connections D, $D^* \in \mathfrak{C}_E$ naturally induce connections, denoted by the same symbols, in $\mathrm{End}(E)$ $(:= E \otimes E^*)$. Then, a straightforward argument shows that D, $D^* \in \mathfrak{C}_{\mathrm{End}(E)}$ are conjugate connections. Thus, we find from (1.2) and (2.2) that the connection D in E is a Yang-Mills connection if and only if (cf. [1, 12, 13])

$$(\delta_D R^D)(X)s = -\sum_{i=1}^{n} (D^*_{e_i} R^D)((e_i, X), s) = 0 \tag{2.3}$$

for arbitrary given $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$.

§3. Yang-Mills connections on compact connected Lie groups

Let G be an n-dimensional closed (compact and connected) Lie group, g a left invariant Riemannian metric on G and $\{Y_i\}_{i=1}^n$ an orthonormal frame with respect to g such that each Y_i is a left invariant vector field on G. Let D be a left invariant connection in the tangent bundle over (G, g) and ∇ the Levi-Civita connection for g. Then D is a Yang-Mills connection if and only if

$$(\delta_D R^D)(Y_j)Y_k = -\sum_{i=1}^{n} (D^*_{Y_i} R^D)(Y_i, Y_j)Y_k = 0 \tag{3.1}$$

for each $j, k = 1, 2, \cdots, n$. If we put

$$\begin{aligned} D_{Y_i} Y_j &=: \sum_k D_{ij}^k Y_k, \quad D^*_{Y_i} Y_j =: \sum_k D_{ij}^{*k} Y_k, \\ \nabla_{Y_i} Y_j &=: \sum_k \Gamma_{ij}^k Y_k, \quad [Y_i, Y_j] =: \sum_k C_{ij}^k Y_k, \end{aligned} \tag{3.2}$$

then we get

$$D^{*k}_{ij} = -D^{j}_{ik}, \quad \Gamma^k_{ij} - \Gamma^k_{ji} = C^k_{ij} = -C^k_{ji}, \quad \Gamma^k_{ij} = -\Gamma^j_{ik}. \tag{3.3}$$

Lemma 3.1(Green's Theorem). *Let (M, g) be an n-dimensional compact connected Riemannian manifold and $X \in \mathfrak{X}(M)$. Then*

$$\int_M trace \nabla X v_g (= \int_M \sum_{i=1}^n g(\nabla_{e_i} X, e_i) v_g) = 0,$$

where $\{e_i\}_{i=1}^n$ is an (locally defined) orthonormal frame on (M, g).

From Lemma 3.1 and (3.3), we get

$$\sum_j \nabla_{Y_j} Y_j = 0. \tag{3.4}$$

From (3.3), we have

$$\begin{aligned} R^D(Y_i, Y_j)Y_k :&= ([D_{Y_i}, D_{Y_j}] - D_{[Y_i, Y_j]})(Y_k) \\ &= \sum_{l,s} (D^l_{jk} D^s_{il} - D^l_{ik} D^s_{jl} - C^l_{ij} D^s_{lk}) Y_s. \end{aligned} \tag{3.5}$$

From (3.3)-(3.5), we obtain

(3.6)

$$\sum_{i=1}^n D^*_{Y_i}(R^D(Y_i, Y_j)Y_k) = -\sum_{i,l,s,t} D^t_{is}(D^l_{jk} D^t_{il} - D^l_{ik} D^t_{jl} - C^l_{ij} D^t_{lk}) Y_s,$$

$$\sum_{i=1}^n R^D(\nabla_{Y_i} Y_i, Y_j) Y_k = 0,$$

$$\sum_{i=1}^n R^D(Y_i, \nabla_{Y_i} Y_j) Y_k = \sum_{i,l,s,t} \Gamma^t_{ij} (D^l_{tk} D^s_{il}, -D^l_{ik} D^s_{tl} - C^l_{it} D^s_{lk}) Y_s,$$

$$\sum_{i=1}^n R^D(Y_i, Y_j) D^*_{Y_i} Y_k = -\sum_{i,l,s,t} D^k_{it}(D^l_{jt} D^s_{il} - D^l_{it} D^s_{jl} - C^l_{ij} D^s_{lt}) Y_s.$$

By virtue of (3.1) and (3.6), we have

$$\begin{aligned} (\delta_D R^D)(Y_j)Y_k = \sum_{i,l,s,t} \{ & D^t_{is}(D^l_{jk} D^t_{il} - D^l_{ik} D^t_{jl} - C^l_{ij} D^t_{lk}) \\ & + \Gamma^t_{ij}(D^l_{tk} D^s_{il} - D^l_{ik} D^s_{tl} - C^l_{it} D^s_{lk}) \\ & - D^k_{it}(D^l_{jt} D^s_{il} - D^l_{it} D^s_{jl} - C^l_{ij} D^s_{lt}) \} Y_s. \end{aligned} \tag{3.7}$$

From (3.1) and (3.7), we obtain the following.

Proposition 3.2. *Let D be a left invariant connection in the tangent bundle over the Riemannian manifold (G, g). Then D is a Yang-Mills connection if and only if*

$$\sum_{i,l,t=1}^{n} \{D_{is}^{t}(D_{jk}^{l}D_{il}^{t} - D_{ik}^{l}D_{jl}^{t} - C_{ij}^{l}D_{lk}^{t}) + \Gamma_{ij}^{t}(D_{tk}^{l}D_{il}^{s} - D_{ik}^{l}D_{tl}^{s} - C_{it}^{l}D_{lk}^{s}) - D_{it}^{k}(D_{jt}^{l}D_{il}^{s} - D_{it}^{l}D_{jl}^{s} - C_{ij}^{l}D_{lt}^{s})\} = 0.$$

Suppose D is a left invariant metric connection in the tangent bundle over (G, g). Then we have

$$D_{ij}^{k} = -D_{ik}^{j} \tag{3.8}$$

for each i, j and k.

From (3.8) and Proposition 3.2, we obtain the following.

Theorem 3.3. *Let D be a left invariant metric connection in the tangent bundle over the Riemannian manifold (G, g). Then, a necessary and sufficient condition for D to be a Yang-Mills connection is*

$$\sum_{i,l,t=1}^{n} \{D_{is}^{t}(D_{jk}^{l}D_{il}^{t} - C_{ij}^{l}D_{lk}^{t}) - 2D_{is}^{t}D_{ik}^{l}D_{jl}^{t} + \Gamma_{ij}^{t}(D_{tk}^{l}D_{il}^{s} - D_{ik}^{l}D_{tl}^{s} - C_{it}^{l}D_{lk}^{s}) - D_{it}^{k}(D_{it}^{l}D_{jl}^{s} - C_{ij}^{l}D_{lt}^{s})\} = 0.$$

If a torsion-free affine connection D in the tangent bundle over a Riemannian manifold (M, g) satisfies $Dg = \omega \otimes g$ for a 1-form ω on M, then (D, g, ω) is called a *torsion-free Weyl structure.* Recently, Park obtained the following.

Theorem 3.4. [14] *Let G be an n-dimensional ($n \geq 3$) closed semi-simple Lie group, g a left invariant Riemannian metric induced from the Killing form of the Lie algebra $\mathfrak{g}$ of G and (D, g, ω) a torsion-free Weyl structure. Then D is a Yang-Mills connection if and only*

(i) $\omega = 0$, *or*

(ii) $d\omega = 0$ *and* $\|\omega\|_g^2 = \frac{1}{n-2}$.

§4. Yang-Mills connections on $SU(2)$

Let $\mathfrak{su}(2)$ be the Lie algebra of all left invariant vector fields on $SU(2)$. The Killing form B of the simple Lie algebra $\mathfrak{su}(2)$ satisfies

$$B(X,Y) = 4\, Trace(XY), \quad (X,Y \in \mathfrak{su}(2)).$$

We define an inner product $< \, , \, >_0$ on $\mathfrak{su}(2)$ by

$$< X, Y >_0 := -B(X,Y), \quad (X,Y \in \mathfrak{su}(2)).$$

Then the inner product $< \, , \, >_0$ determines a left invariant metric g_0 on M. The following lemma is well known (cf. [10, 18, 19]).

Lemma 4.1. *Let g be an arbitrary left invariant Riemannian metric on $SU(2)$ and let $< \, , \, >$ be an inner product on $\mathfrak{su}(2)$ defined by*

$$< X, Y > := g_e(X_e, Y_e), \quad (X,Y \in \mathfrak{su}(2)),$$

where e is the identity matrix of $SU(2)$. Then there exists an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $< \, , \, >_0$ such that

$$\left\{ \begin{array}{ll} [X_1, X_2] = \frac{1}{\sqrt{2}} X_3, & [X_2, X_3] = \frac{1}{\sqrt{2}} X_1, \\ [X_3, X_1] = \frac{1}{\sqrt{2}} X_2, & < X_i, X_j >= \delta_{ij} a_i, \end{array} \right. \tag{4.1}$$

where $a_i (i = 1, 2, 3)$ are positive constant real numbers determined by the given left invariant Riemannian metric g on $SU(2)$.

We fix an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to g_0 with the property (4.1) in Lemma 4.1 and denote by $g_{(a_1,a_2,a_3)}$ the left invariant Riemannian metric on $SU(2)$ which is determined by positive real numbers a_1, a_2, a_3 in Lemma 4.1. Moreover, we normalize left invariant Riemannian metrics on $SU(2)$ by putting $a_3 = 1$. We denote by $g_{(a_1,a_2,1)}$ or simply $g_{(a_1,a_2)}$, the left invariant Riemannian metric which is determined by positive real numbers $a_1, a_2, a_3 = 1$.

In general, the Riemannian connection ∇ for the Riemannian metric g on a Riemannian manifold (M, g) is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]) \end{aligned} \tag{4.2}$$

for $X, Y, Z \in \mathfrak{X}(M)$, and the curvature tensor field R is

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad (X,Y \in \mathfrak{X}(M)). \tag{4.3}$$

For the orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $<, >_0 = -B$ in Lemma 4.1, if we put

$$Y_1 := \frac{1}{\sqrt{a_1}} X_1, \quad Y_2 := \frac{1}{\sqrt{a_2}} X_2, \quad Y_3 := X_3,$$

then $\{Y_1, Y_2, Y_3\}$ is an orthonormal frame of $(M, g_{<,>} := g_{(a_1,a_2)})$. From (4.1) we have

$$(4.4) \quad [Y_1, Y_2] = \frac{1}{\sqrt{2a_1a_2}} Y_3, \quad [Y_2, Y_3] = \frac{\sqrt{a_1}}{\sqrt{2a_2}} Y_1, \quad [Y_3, Y_1] = \frac{\sqrt{a_2}}{\sqrt{2a_1}} Y_2.$$

Assume D is a left invariant Yang-Mills connection with torsion-free Weyl structure $(D, g_{(a_1,a_2)}, \omega)$ in the tangent bundle over $(SU(2), g_{(a_1,a_2)})$. Then $d\omega = 0$ by Theorem 1.2. We obtain from (3.2)

$$(4.5) \quad \begin{aligned} d\omega(Y_i, Y_j) &= Y_i(\omega(Y_j)) - Y_j(\omega(Y_i)) - \omega([Y_i, Y_j]) \\ &= -\sum_{k=1}^{3} C_{ij}^k \omega(Y_k) = -\sum_{k=1}^{n} C_{ij}^k \omega_k = 0. \end{aligned}$$

By virtue of (4.4) and (4.5), we get

$$\omega = 0.$$

So D is a metric connection. Moreover since D is a torsion-free Weyl structure, D is ∇ a Levi-Civita connection for the metric $g_{(a_1,a_2)}$.

Summing up these facts, we get the following.

Proposition 4.2. *Let D be a left invariant Yang-Mills connection with torsion-free Weyl structure (D, g, ω) in the tangent bundle over $(SU(2), g_{(a_1,a_2)})$. Then, if D is a Yang-Mills connection, D coincides with the Levi-Civita connection ∇ for the metric $g_{(a_1,a_2)}$.*

For ∇ the Levi-Civita connection for the metric $g_{(g_1,g_2)}$ on $SU(2)$, we obtain from (4.2) and (4.4)

(4.6)

$$\begin{cases} \nabla_{Y_1} Y_2 = c^{-1}(-a_1 + a_2 + 1) Y_3, & \nabla_{Y_2} Y_1 = c^{-1}(-a_1 + a_2 - 1) Y_3, \\ \nabla_{Y_2} Y_3 = c^{-1}(a_1 - a_2 + 1) Y_1, & \nabla_{Y_3} Y_2 = c^{-1}(-a_1 - a_2 + 1) Y_1, \\ \nabla_{Y_3} Y_1 = c^{-1}(a_1 + a_2 - 1) Y_2, & \nabla_{Y_1} Y_3 = c^{-1}(a_1 - a_2 - 1) Y_2, \\ \nabla_{Y_1} Y_1 = \nabla_{Y_2} Y_2 = \nabla_{Y_3} Y_3 = 0, & \end{cases}$$

where $c := \sqrt{8a_1a_2}$. Furthermore, by the help of (4.3), (4.4) and (4.6) we get

$$(4.7)\qquad \begin{cases} R(Y_1,Y_2)Y_1 = c^{-2}\{3-2(a_1+a_2)-(a_1-a_2)^2\}Y_2, \\ R(Y_1,Y_2)Y_2 = c^{-2}\{-3+2(a_1+a_2)+(a_1-a_2)^2\}Y_1, \\ R(Y_1,Y_3)Y_1 = c^{-2}\{3{a_2}^2-2(1+a_1)a_2-(1-a_1)^2\}Y_3, \\ R(Y_1,Y_3)Y_3 = c^{-2}\{-3{a_2}^2+2(1+a_1)a_2+(1-a_1)^2\}Y_1, \\ R(Y_2,Y_3)Y_2 = c^{-2}\{3{a_1}^2-2(a_2+1)a_1-(a_2-1)^2\}Y_3, \\ R(Y_2,Y_3)Y_3 = c^{-2}\{-3{a_1}^2+2(a_2+1)a_1+(a_2-1)^2\}Y_2, \\ R(Y_1,Y_2)Y_3 = R(Y_2,Y_3)Y_1 = R(Y_3,Y_1)Y_2 = 0. \end{cases}$$

In order to analyze

$$(4.8)\qquad \begin{aligned} (\delta_\nabla R^\nabla)(Y_j)Y_k &= -\sum_{i=1}^{3}(\nabla_{Y_i}R^\nabla)(Y_i,Y_j)Y_k \\ &= -\sum_{i=1}^{3}\{\nabla_{Y_i}(R^\nabla(Y_i,Y_j)Y_k) - R^\nabla(\nabla_{Y_i}Y_i,Y_j)Y_k \\ &\quad - R^\nabla(Y_i,\nabla_{Y_i}Y_j)Y_k - R^\nabla(Y_i,Y_j)\nabla_{Y_i}Y_k\}, \end{aligned}$$

From (4.6) and (4.7), we obtain the following.

Lemma 4.3. *On $(SU(2), g_{(g_1,g_2)})$, we have*

$$\begin{cases} (\delta_\nabla R^\nabla)(Y_i)Y_i &= 0 \quad (i=1,2,3), \\ (\delta_\nabla R^\nabla)(Y_1)Y_2 &= 4c^{-3}\{2{a_1}^3-(a_2+1){a_1}^2-{a_2}^3+{a_2}^2+a_2-1\}Y_3, \\ (\delta_\nabla R^\nabla)(Y_1)Y_3 &= -4c^{-3}\{2{a_1}^3-(a_2+1){a_1}^2-{a_2}^3+{a_2}^2+a_2-1\}Y_2, \\ (\delta_\nabla R^\nabla)(Y_2)Y_1 &= 4c^{-3}\{{a_1}^3-{a_1}^2+({a_2}^2-1)a_1-2{a_2}^3+{a_2}^2+1\}Y_3, \\ (\delta_\nabla R^\nabla)(Y_2)Y_3 &= -4c^{-3}\{{a_1}^3-{a_1}^2+({a_2}^2-1)a_1-2{a_2}^3+{a_2}^2+1\}Y_1, \\ (\delta_\nabla R^\nabla)(Y_3)Y_1 &= -4c^{-3}\{{a_1}^3-a_2{a_1}^2-({a_2}^2-1)a_1+{a_2}^3+a_2-2\}Y_2, \\ (\delta_\nabla R^\nabla)(Y_3)Y_2 &= 4c^{-3}\{{a_1}^3-a_2{a_1}^2-({a_2}^2-1)a_1+{a_2}^3+a_2-2\}Y_1, \end{cases}$$

where $c = \sqrt{8a_1a_2}$.

By virtue of (3.1), (4.8) and Lemma 4.3, we get the fact that ∇ is a Yang-Mills connections if and only if a_1 and a_2 satisfy the following two equations

$$a_1{}^3 - a_1{}^2 + (a_2{}^2 - 1)a_1 - 2a_2{}^3 + a_2{}^2 + 1 = 0 \tag{4.9}$$

and

$$a_1{}^3 - a_2a_1{}^2 - (a_2{}^2 - 1)a_1 + a_2{}^3 + a_2 - 2 = 0. \tag{4.10}$$

Subtracting (4.10) from (4.9), we obtain

$$(a_2 - 1)\{a_1{}^2 + 2(a_2 + 1)a_1 - 3a_2{}^2 - 2a_2 - 3\} = 0.$$

Hence we have

$$a_2 = 1 \quad \text{or} \quad a_1 = -a_2 - 1 + 2\sqrt{a_2{}^2 + a_2 + 1}. \tag{4.11}$$

By the help of (4.9), (4.11) and the fact that $a_1 > 0$ and $a_2 > 0$, we get

$$a_1 = a_2 = 1.$$

Thus, we obtain the following.

Proposition 4.4. *Let ∇ be the Levi-Civita connection for the metric $g_{(a_1,a_2)}$ on $SU(2)$. Then ∇ is a Yang-Mills connections if and only if $a_1 = a_2 = 1$.*

Therfore, by virtue of propositions 4.2 and 4.4, we have the following.

Theorem 4.5. *Let D be a left invariant Yang-Mills connection with torsion-free Weyl structure $(D, g_{(a_1,a_2)}, \omega)$ in the tangent bundle over $(SU(2),\ g_{(a_1,a_2)})$. Then, D is a Yang-Mills connection if and only if $D = \nabla$ and $a_1 = a_2 = 1$.*

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