Honam Mathematical J. **32** (2010), No. 4, pp. 643–649

ON SUBMAXIMAL AND QUASI-SUBMAXIMAL SPACES

Seung Woo Lee, Mi Ae Moon, and Myung Hyun Cho*

Abstract. The purpose of this paper is to study some properties of quasi-submaximal spaces and related examples. More precisely, we prove that if X is a quasi-submaximal and nodec space, then X is submaximal. As properties of quasi-submaximality, we show that if X is a quasi-submaximal space, then

(a) for every dense $D \subset X$, $Int(D)$ is dense in X, and

(b) there are no disjoint dense subsets.

Also, we illustrate some basic facts and examples giving the relationships among the properties mentioned in this paper.

1. Introduction

All spaces are assumed to satisfy the T_1-axiom throughout this paper, though this is not always essential. We shall denote by A° or $Int(A)$ the interior of a subset A of a space X, and by A^d its derived set, i.e., the set of all limits points of A, and by $\overline{A}$ the closure of A. Also, we denote the boundary of A by $Fr(A) = \overline{A} \backslash Int(A)$.

We recall that a subset A of a space X is *locally closed* if A is open in its closure in X or, equivalently, is the intersection of an open subset and a closed subset of X. We shall say that A is *co-locally closed* if A is the union of an open subset and a closed subset of X. A space X is said to be a *submaximal space* ([4]) if every subset of X is locally closed.

One of the reasons to consider submaximal spaces is provided by the theory of maximal spaces. A space X is called *maximal* if it is dense-in-itself (i.e., $X \subseteq X^d$) and no larger topology on the set X is dense-in-itself. It is well-known ([14]) that a space is maximal if and only

Received August 31, 2010. Accepted November 18, 2010.

2000 AMS subject classification: 54A10, 54F65.

Keywords and phrases: maximal spaces, submaximal spaces, quasi-submaximal spaces, digital planes, digital lines.

* Corresponding author.

This paper was supported by Wonkwang University in 2009.

if it is an extremally disconnected submaximal space without isolated points. Secondly, any connected Hausdorff space which does not admit a larger connected topology is submaximal (see ([9])). Third, submaximal spaces were characterized by Bourbaki as spaces that does not admit a larger topology with the same semi-regularization ([4], p. 139). Fourth, nonempty maximal spaces are not decomposable into two nonempty dense complementary subspaces just because they are submaximal—obviously, dense open subsets cannot be disjoint.

Nevertheless, in all most all articles in which we find references to submaximal spaces, the main effort and interest were directed towards the study of different types of maximal spaces.

Theorem 1.1. ([3]) *The following statements about a space X are equivalent:*

(a) X is a submaximal space,
(b) every subset of X is co-locally closed,
(c) every subset A of X, for which A° is empty, is closed,
(d) every subset A of X, for which A° is empty, is discrete,
(e) $\overline{A}\backslash A$ is closed, for every subset A of X,
(f) $\overline{A}\backslash A$ is discrete, for every subset A of X,
(g) every dense subset of X is open.

According to Kelley ([10]), a topological space X is said to be a *door space* if every subset of X is either closed or open.

Fact 1.2. (a) The discrete space is a door space.
(b) A T_2 door space has at most one accumulation point ([10]).
(c) In a T_2 door space, if x is not an accumulation point, then $\{x\}$ is open ([10]).

Theorem 1.3. ([5]) *Every door space X is submaximal.*

In general, the converse of above Theorem 1.3 is not true (see examples in [1]).

Recall that a space X is *nodec* ([15]) if all nowhere dense subsets of X are closed.

Different equivalent conditions for a space to be submaximal are given in Theorem 1.1 (or ([3], Theorem 1.2)), and the ones for a space to be nodec in [15] (Fact 1.14) and [13] (Corollary to Proposition 4). In particular, they imply that every submaximal space is nodec. The converse is not true: any trivial topology on a set with more than two elements is nodec, but not submaximal. This example shows that there exist nodec spaces that are not T_0. On the other hand, we have the following fact.

Fact 1.4. Every submaximal space is a T_0-space.

Indeed, suppose that X is not a T_0-space. Then there exist two distinct points x and y in X such that every open neighborhood of x (respectively, y) contains y (respectively, x). Let $A = \{x\}$. Then $IntA = \emptyset$ and $y \in \overline{A}$. Hence A is not closed. Therefore X is not submaximal by Theorem 1.1 ($(a) \Longleftrightarrow (c)$).

There is also an example of a nodec T_1-space which is not submaximal.

Example 1.5. Every cofinite topology on an infinite set X is a nodec space which is not submaximal.

Proof. Suppose A is infinite in the cofinite topology on X. Then A is dense (every non-empty open set misses only finitely many elements of X) and so $Int(\overline{A}) = Int(X) = X$, and A is not nowhere dense. So every nowhere dense subset of X must be finite and thus closed. Hence X is nodec. However, it is not submaximal because every infinite set in X is dense, but only cofinite sets are open. □

However, the converse of Theorem 1.3 holds if it is also an irreducible space. A nonempty space X is *irreducible* if it satisfies the following equivalent conditions: (a) Every two nonempty open subset of X intersect. (b) X is not the union of a finite family of closed proper subsets. (c) Every nonempty open subset of X is dense. (d) Every open subset of X is connected. An irreducible space is called sometimes *hyperconnected*. Here it is interesting to note that one can meet irreducible spaces under a variety of names. In literature, all names such as D-space, semi-connected space, hyperconnected, S-connected, stand for irreducible spaces.

Theorem 1.6. ([5]) *Every irreducible submaximal space X is a door space.*

A space X is an *I-space* if its derived set X^d is closed and discrete. As mentioned in [3], the class of I-spaces includes convergent sequences, Alexandroff one-point compactification of discrete spaces, and the Mrówka-Isbell Ψ-spaces. It also includes the digital lines (see Theorem 2.1 in [8]).

Fact 1.7. *Every I-space is submaximal.*

Proof. Let X be an I-space and $A \subset X$. We claim that $\overline{A} \backslash A$ is close and discrete. Note first that $\overline{A} \backslash A = (A^d \cup A) \backslash A \subset A^d \subset X^d$. Since X is an I-space, X^d is closed and discrete. Hence $\overline{A} \backslash A$ is discrete and thus by Theorem 1.1, X is submaximal. $\square$

In 2001, Al-Nashef ([2]) introduced the concept of quasi-submaximal spaces which is weaker than submaximality. From [2], a space X is called *quasi-submaximal* if for every dense $D \subset X$, $Fr(D) = \overline{D} \backslash Int(D)$ is nowhere dense in X. He investigated some characterizations of quasi-submaximal spaces. Moreover, he gave an example of a quasi-submaximal space $(X, \mathcal{T})$ which is not submaximal, i.e., $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, X\}$. More geometric examples than the example of Al-Nashef ([2]) are known in [7]. It is proved in [7] (Theorem 1.1 and Theorem 3.4) that the digital plane and digital 3-space are typical and geometric examples of quasi-submaximal spaces.

We have the following basic diagram which exhibits the general relationships among the properties mentioned above.

$$\begin{array}{ccccc} & & \mathrm{quasi-submaximal} & & \\ & & \uparrow & & \\ \mathrm{door} & \longrightarrow & \mathrm{submaximal} & \longrightarrow & \mathrm{nodec} \\ & & \uparrow & & \\ & & \mathrm{I-space} & & \end{array}$$

The purpose of this paper is to study some properties of quasi-submaximal spaces and related examples. More precisely, we prove that if X is a quasi-submaximal and nodec space, then X is submaximal. As properties of quasi-submaximality, we show that if X is a quasi-submaximal space, then

(a) for every dense $D \subset X$, $Int(D)$ is dense in X, and

(b) there are no disjoint dense subsets.

Also, we illustrate some basic facts and examples giving the relationships among the properties mentioned above.

2. Main Results

The *digital line*, also known as the *Khalimsky line* is the set of the integers $\mathbb{Z}$, equipped with the topology κ, generated by the family $\{\{2n-1, 2n, 2n+1\} \mid n \in \mathbb{Z}\}$. Note that a single-ton set $\{2m+1\}$ is open and a subset $\{2n-1, 2n, 2n+1\}$ is the smallest open set containing $2n$, where m and n are any integers.

Let $(\mathbb{Z}^2, \kappa^2)$ be the topological product of two copies of the digital line $(\mathbb{Z}, \kappa)$, where $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $\kappa^2 = \kappa \times \kappa$. In this paper, the space $(\mathbb{Z}^2, \kappa^2)$ is called the *digital plane*.

Theorem 2.1. ([2]) (a) *The digital line* $(\mathbb{Z}, \kappa)$ *is submaximal.*
(b) *The digital plane* $(\mathbb{Z}^2, \kappa^2)$ *is not submaximal.*
(c) *The digital plane* $(\mathbb{Z}^2, \kappa^2)$ *is quasi-submaximal.*

Theorem 2.2. ([3]) $X \times Y$ *is a submaximal (nodec) space if and only if* X *and* Y *are submaximal (nodec) spaces and one of them is discrete.*

Remark 2.3. *(a) From Theorem 2.1 (ii), it follows that the digital plane is not an* I*-space.*
(b) In Theorem 2.2, we cannot drop the condition that one of them is discrete. For if $X = Y = (\mathbb{Z}, \kappa)$*, then* X *and* Y *are submaximal and clearly they are not discrete, but* $X \times Y = (\mathbb{Z}^2, \kappa^2)$ *is not submaximal by Theorem 2.1.*

It directly follows from the definition that every submaximal space is quasi-submaximal. The converse is not true in general. For example, the digital plane is quasi-submaximal but not submaximal. Then a natural question arises. Under what conditions does the converse hold ? More concretely, is every irreducible quasi-submaximal space submaximal? We have seen in section 1 that if $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, X\}$, then $(X, \mathcal{T})$ is a quasi-submaximal space which is not submaximal. This space is not irreducible. However, in the class of nodec spaces, the answer is “yes”.

Theorem 2.4. *If* X *is a quasi-submaximal and nodec space, then* X *is submaximal.*

Proof. Let D be a dense subset of X. Since X is quasi-submaximal, $\overline{D} \backslash Int(D) = X \backslash Int(D)$ is nowhere dense in X. So $X \backslash D$ is nowhere dense in X. Since X is nodec, $X \backslash D$ is closed in X and hence D is open in X. Therefore, X is submaximal. □

Corollary 2.5. *The digital plane* $(\mathbb{Z}^2, \kappa^2)$ *is not a nodec space.*

Theorem 2.6. *If X is a quasi-submaximal space, then*
(a) for every dense $D \subset X$, $Int(D)$ is dense in X, and
(b) there are no disjoint dense subsets.

Proof. (a) Let D be a dense subset of X. Since X is quasi-submaximal, $Fr(D) = \overline{D}\backslash Int(D) = X\backslash Int(D)$ is nowhere dense in X. So $X\backslash\overline{X\backslash Int(D)}$ $= X\backslash(X\backslash Int(D)) = Int(D)$ is dense in X.

(b) Suppose A and B are dense subsets of X such that $A \cap B = \emptyset$. Since X is quasi-submaximal, $Fr(A)$ is nowhere dense in X. But $Fr(A) = \overline{A} \cap \overline{X\backslash A} \supset \overline{A} \cap \overline{B} = X$ is not nowhere dense in X. This is a contradiction. □

Example 2.7. *Every cofinite topology on an infinite set X is a nodec space which is not quasi-submaximal (and so not submaximal as in Example 1.5).*

Proof. It is shown in Example 1.5 that X is nodec. Let A be an infinite subset of X whose compliment is also infinite. Then $\overline{A} = \overline{X\backslash A} = X$. By Theorem 2.6, X is not quasi-submaximal. □

Example 2.8. *Let $X = \omega \cup \{p\}$ be the one-point compactification of the discrete space ω. Then the only proper dense subset is ω. $Fr(\omega) = \overline{\omega}\backslash Int(\omega) = X\backslash\omega = \{p\}$ is nowhere dense in X. Thus X is quasi-submaximal.*

References

[1] M. E. Adams, Karim Belaid, Lobna Dridi, and Othman Echi, *Submaximal and spectral spaces*, Mathematical Proceedings of the Royal Irish Academy, **108**A (2) (2008), 137-147.

[2] B. Al-Nashef , *On semipreopen sets*, Questions and Answers in General Topology, **19** (2001), 203-212.

[3] A. V. Arhangel'skii, P. J. Collins, *On submaximal spaces*, Topology and its Applications, **64** (1995), 219-241.

[4] N. Bourbaki. *General topology* : Chapter 1-4, Translated from the French, Reprint of the 1989 English translation, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998.

[5] J. Dontchev, *On door spaces*, Indian J. pure appl. Math., **26**(9) (1995), 873-881.

[6] J. Dontchev, On submaximal spaces, *Tamkang Journal of Mathematics* **26** (1995), 243-50.

[7] M. Fusimoto, H. Maki, T. Noiri and S. Takigawa , *The digital plane is quasi-submaximal*, Questions and Answers in General Topology, **22** (2004), 163-168.

[8] M. Fusimoto, S. Takigawa, J. Dontchev, T. Noiri and H. Maki, *The topological stucture and groups of digital n-spaces*, Kochi J. Math., **1** (2006), 31-55.

[9] J.A. Guthrie, H.E. Stone, and M.L. Wage, *Maximal connected Hausdorff topologies*, Topology Proc., **2** (1977), 349-353.

[10] J. L. Kelley, *General Topology*, D. Van Nostrand Company, Inc. Princeton, new Jersey, 1955.

[11] E.D Khalimsky, R. Kopperman and P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, *Topology and its applications* **36** (1990), 1-17.

[12] T.Y. Kong, R. Kopperman and P.R. Meyer, A topological approach to digital topology, *American Mathematical Monthly* **98** (1991), 901-17.

[13] O. Njastad, *On some classes of nearly open sets*, Pacific J. Math., **15**, (1965), 961-970.

[14] J. Schröder, Some answers concerning submaximal spaces, *Questions and Answers in General Topology* **17** (1999), 221-225.

[15] Eric K. van Douwen, *Applications of maximal topologies*, Topology and its Applications, **51** (2) (1993), 125-139.

Seung Woo Lee
Division of Mathematics & Informational Statistics
Wonkwang University
Iksan 570-749, Korea
E-mail: swlee@wonkwang.ac.kr

Mi Ae Moon
Division of Mathematics & Informational Statistics
Wonkwang University
Iksan 570-749, Korea
E-mail: moonmae@wonkwang.ac.kr

Myung Hyun Cho
Department of Mathematics Education
Wonkwang University
Iksan 570-749, Korea
E-mail: mhcho@wonkwang.ac.kr