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THE GENERALIZED ANALOGUE OF WIENER MEASURE SPACE AND ITS PROPERTIES

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Abstract. In this note, we introduce the definition of the generalized analogue of Wiener measure on the space $C[a, b]$ of all real-valued continuous functions on the closed interval $[a, b]$, give several examples of it and investigate some important properties of it - the Fernique theorem and the existence theorem of scale-invariant measurable subsets on $C[a, b]$.

1. Preliminaries

In 1923, Wiener proved the existence theorem of the meaningly measure on the space $C_0[a, b]$, the space of all real-valued continuous functions on a closed bounded interval $[a, b]$ which vanish at a, the so-called Wiener space in [9]. This is based on the properties of Brownian motion of a single small particle. In 2002, the author and Dr. Im presented the definition and properties of analogue of Wiener measure on the space $C[a, b]$, the space of all real-valued continuous functions on $[a, b]$ in [3]. This is the theory of many particles, moving along the law of Brownian motion.

In this note, we introduce the definition of the generalized analogue of Wiener measure space, which is more generalized concept of Wiener measure space and we give several examples. Furthermore, we investigate important theorems - the Fernique theorem on $C[a, b]$ and the existence theorem of scale-invariant measurable subsets on $C[a, b]$.

In this section, we give some notations, definitions and facts which are needed to understand the subsequent sections.

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Let a and b be two real numbers with $a < b$. Let $\alpha, \beta : [a,b] \to \mathbb{R}$ be two continuous functions such that β is strictly increasing. Let φ be a positive finite Borel measure on $\mathbb{R}$.

For $\vec{t} = (t_0, t_1, t_2, \cdots, t_n)$ with $a = t_0 < t_1 < t_2 < \cdots < t_n \le b$, let

$$\begin{aligned} &W(\vec{t}, \vec{u}; \alpha, \beta) \\ &= \frac{1}{\sqrt{(2\pi)^n \prod_{j=1}^{n} (\beta(t_j) - \beta(t_{j-1}))}} \\ &\exp\{\frac{-1}{2} \sum_{j=1}^{n} \frac{(u_j - \alpha(t_j) - (u_{j-1} - \alpha(t_{j-1})))^2}{\beta(t_j) - \beta(t_{j-1})}\} \end{aligned}$$

and let $J_{\vec{t}} : C[a,b] \to \mathbb{R}^{n+1}$ be a function with $J_{\vec{t}}(x) = (x(t_0), x(t_1), x(t_2), \cdots, x(t_n))$. For Borel subsets $B_0, B_1, \cdots, B_n$ of $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[a,b]$ is called an interval. Let $\mathcal{I}$ be the set of all intervals. We let

$$\begin{aligned} &m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) \\ &= \int_{B_0} [\int_{\prod_{j=1}^{n} B_j} W(\vec{t}, \vec{u}; \alpha, \beta) \, d \prod_{j=1}^{n} m_L(u_1, u_2, \cdots, u_n)] d\varphi(u_0) \end{aligned}$$

where m_L is the Lebesgue measure on $\mathbb{R}$.

Using the Chapman-Kolmogorov equation in [2], we can easily prove the following theorem.

Theorem 1.1. *m_φ is well-defined on $\mathcal{I}$.*

By Theorem 2.1 and Theorem 5.1 in [6], the set $\mathcal{B}(C[a,b])$ of all Borel subsets in $C[a,b]$ with the superemum norm, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive measure ω_φ on $(C[a,b], \mathcal{B}(C[a,b]))$ such that $\omega_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$. Here, ω_φ is called the generalized analogue of Wiener measure on $(C[a,b], \mathcal{B}(C[a,b]))$ associated with φ.

From the change of variable formula, we have the following theorem.

Theorem 1.2. *(The Wiener integration formula for ω_φ) If $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is a Borel measurable function, then the following equality holds ;*

$$\int_{C[a,b]} f(x(t_0),x(t_1),\cdots,x(t_n))\ d\omega_\varphi(x)$$
$$\overset{*}{=}\int_{\mathbb{R}^{n+1}} f(u_0,u_1,\cdots,u_n)W(\vec{t},\vec{u};\alpha,\beta)\ d(\prod_{j=1}^{n} m_L\times\varphi)((u_1,u_2,\cdots,u_n),$$
$$u_0),$$

where $\overset{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

Now, we give several examples for our definitions.

Example 1.3. *Let φ be a positive finite Borel measure on $\mathbb{R}$.*
(1) It is not hard to show that ω_φ has no atom.
(2) $\omega_\varphi(C[a,b])=\varphi(\mathbb{R})$.
(3) For $a\le t\le b$, let $J_t: C[a,b]\to\mathbb{R}$ be a function with $J_t(x)=x(t)$. Then for any Borel subset E in $\mathbb{R}$,

$$\omega_\varphi(J_{\vec{t}}^{-1}(E))$$
$$=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\chi_{A_{u_0}}(u_1)e^{-\frac{u_1^2}{2}}\ dm_L(u_1)\Big)d\varphi(u_0)$$

where $A_{u_0}=\sqrt{\beta(t)-\beta(a)}E+\alpha(t)-\alpha(a)+u_0$ and $\chi_{A_{u_0}}$ is the characteristic function with respect to A_{u_0}.
(4) Suppose that $f(u)=u$ is φ-integrable. Then for $a\le t\le b$,

$$\int_{C[a,b]} x(t)d\omega_\varphi(x)=(\alpha(t)-\alpha(a))\varphi(\mathbb{R})+\int_{\mathbb{R}} ud\varphi(u)$$

If $\varphi=\delta_p$, the Dirac measure at p, $\int_{C[a,b]}x(t)d\omega_\varphi(x)=\alpha(t)-\alpha(a)+p$ and if φ has a normal distribution with mean m and variance σ^2, $\int_{C[a,b]}x(t)d\omega_\varphi(x)=\alpha(t)-\alpha(a)+m$.
(5) Suppose that $f(u)=u^2$ is φ-integrable. Then for $a\le t\le b$,

$$\int_{C[a,b]} x(t)^2 d\omega_\varphi(x)$$
$$=(\beta(t_1)-\beta(a)+(\alpha(t)-\alpha(a))^2)\varphi(\mathbb{R})+2(\alpha(t)-\alpha(a))\int_{\mathbb{R}} ud\varphi(u)$$
$$+\int_{\mathbb{R}} u^2 d\varphi(u).$$

If $\varphi = \delta_p$ then

$$\int_{C[a,b]} x(t)^2 d\omega_\varphi(x)$$
$$= \beta(t) - \beta(a) + (\alpha(t) - \alpha(a))^2 + 2(\alpha(t) - \alpha(a))p + p^2$$

and if φ has a normal distribution with mean m and variance σ^2, then

$$\int_{C[a,b]} x(t)^2 d\omega_\varphi(x)$$
$$= \beta(t) - \beta(a) + (\alpha(t) - \alpha(a))^2 + 2m(\alpha(t) - \alpha(a)) + m^2 + \sigma^2.$$

(6) Suppose that $f(u) = u^2$ is φ-integrable and $a \le t_1 \le t_2 \le b$. Then

$$\int_{C[a,b]} x(t_1)x(t_2) d\omega_\varphi(x)$$
$$= (\beta(t) - \beta(a) + (\alpha(t_2) - \alpha(t_1))(\alpha(t_1) - \alpha(a)) + (\alpha(t_1) - \alpha(a))^2)\varphi(\mathbb{R})$$
$$+(\alpha(t_2) + \alpha(t_1) - 2\alpha(a)) \int_{\mathbb{R}} u d\varphi(u) + \int_{\mathbb{R}} u^2 d\varphi(u).$$

(7) For $a \le t_1 \le t_2 \le t_3 \le t_4 \le b$ and k_1 and k_2 in $\mathbb{R}$, using the change of variable formula,

$$\varphi(\mathbb{R})\omega_\varphi(\{x \text{ in } C[a,b] \mid x(t_2) - x(t_1) \le k_1 \text{ and } x(t_4) - x(t_3) \le k_2\})$$
$$= \omega_\varphi(\{x \text{ in } C[a,b] \mid x(t_2) - x(t_1) \le k_1\})$$
$$\times \omega_\varphi(\{x \text{ in } C[a,b] \mid x(t_4) - x(t_3) \le k_2\}).$$

Hence, if φ is a probability measure on $\mathbb{R}$, then $x(t_2) - x(t_1)$ and $x(t_4) - x(t_3)$ are stochastically independent.
(8) Let $\mathcal{F}(\varphi)$ be the Fourier transform of a measure φ on $\mathbb{R}$, that is, $[\mathcal{F}(\varphi)](\xi) = \int_{\mathbb{R}} \exp\{i\xi u\} d\varphi(u)$. Then for $a \le t \le b$,

$$\int_{C[a,b]} \exp\{i\xi x(t)\} d\omega_\varphi(x)$$
$$= \exp\{-\frac{1}{2}\xi^2(\beta(t) - \beta(a)) + i\xi(\alpha(t) - \alpha(a))\}[\mathcal{F}(\varphi)](\xi).$$

If φ has a normal distribution with mean m and variance σ^2 then

$$\int_{C[a,b]} \exp\{i\xi x(t)\} d\omega_\varphi(x)$$
$$= \exp\{-\frac{1}{2}\xi^2(\beta(t) - \beta(a) + \sigma^2) + i(\alpha(t) - \alpha(a) + m)\},$$

so $J_t(x) = x(t)$ has a normal distribution with mean $\alpha(t) - \alpha(a) + m$ and variance $\beta(t) - \beta(a) + \sigma^2$.

2. Fernique's Theorem for the generalized analogue of Wiener measure

In 1970, Skorokhod proved that there is a positive real number p_1 such that $\int_{\mathbb{B}} \exp\{p_1\|x\|\}d\omega(x)$ is finite in [5] and at the same time, Fernique showed independently that there is a positive real number p_2 such that $\int_{\mathbb{B}} \exp\{p_2\|x\|^2\}d\omega(x)$ is finite in [4], where $(\mathbb{B}, \omega)$ is an abstract Wiener measure space and $\|\cdot\|$ is a measurable norm in $(\mathbb{B}, \omega)$. These two theorems play a very important role in the theory of abstract Wiener space. In 2009, the author proved that for $1 \le d \le 2$, there is a positive real number p such that $\int_{C[0,1]} \exp\{p\|x\|^d\}d\omega_\varphi(x)$ is finite where ω_φ is the analogue of Wiener measure on $C[0,1]$ in [7].

In this section, we will prove that for $1 \le d \le 2$, there is a positive real number p such that $\int_{C[a,b]} \exp\{p\|x\|_\infty^d\}d\omega_\varphi(x)$ is finite under the certain conditions where ω_φ is the generalized analogue of Wiener measure on $C[a,b]$.

Throughout in this section, let φ be a probability Borel measure on $\mathbb{R}$ and let α be non-increasing.

From the definition of median, directly we obtain the following lemma.

Lemma 2.1. *Let $X(x) = x(t) - x(s)$ for x in $C[a,b]$ where $a \le s < t \le b$. Then the median $med(X)$ of X is $\alpha(t) - \alpha(s)$, that is, $\omega_\varphi(\{x \text{ in } C[a,b] \mid X(x) \ge \alpha(t)-\alpha(s)\}) \ge \frac{1}{2}$ and $\omega_\varphi(\{x \text{ in } C[a,b] \mid X(x) \le \alpha(t) - \alpha(s)\}) \ge \frac{1}{2}$.*

Now, we prove the theorem, which is a key role theorem in this section.

Theorem 2.2. *For $M > 0$,*

$$\begin{aligned} &\omega_\varphi(\{x \text{ in } C[a,b] \mid \|x - x(a)\|_\infty \ge M + \alpha(a) - \alpha(b)\}) \\ &\le 2\sqrt{\frac{2}{\pi}} \frac{\sqrt{\beta(b) - \beta(a)}}{M + \alpha(a) - \alpha(b)} \exp\{-\frac{(M + \alpha(a) - \alpha(b))^2}{2(\beta(b) - \beta(a))}\}. \end{aligned}$$

Proof. Let $\langle t_n\rangle$ be a dense sequence in $[a,b]$. For a natural number n, let $S_n(x) = \max\{x(t_k) - x(a) | 1 \le k \le n\}$ and $S(x) = \sup\{x(t_n) - x(a) \mid n$ is a natural number $\}$ for x in $C[a,b]$. Then $\langle S_n\rangle$ converges to S ω_φ-*a.e.*. Let us relabel $t_1, t_2, \cdots, t_n$ as $\tau_{n,1}, \tau_{n,2}, \cdots, \tau_{n,n}$ such that $\tau_{n,0} = a <$

$\tau_{n,1} < \tau_{n,2} < \cdots < \tau_{n,n} \le b$. For two natural numbers n and j with $1 \le j \le n$, let $X_{n,j}(x) = x(\tau_{n,j}) - x(\tau_{n,j-1})$ and $S_{n,j}(x) = \sum_{k=1}^{j} X_{n,k}(x)$ for x in $C[a,b]$. Then $S_{n,j}(x) = x(\tau_{n,j}) - x(a)$ for x in $C[a,b]$ and from (7) in Example 1.3, $X_{n,1}, X_{n,2}, \cdots, X_{n,j}$ are stochastically independent. By Lemma 2.1, $med(S_{n,j} - S_{n,n}) = \alpha(\tau_{n,j}) - \alpha(\tau_{n,n})$ and $\max\{S_{n,k}(x) - med(S_{n,k}(x) - S_{n,n}(x)) | 1 \le k \le n\} \ge \max\{S_{n,k}(x) | 1 \le k \le n\} - \alpha(a) + \alpha(b)$ for x in $C[a,b]$. So, for $M > 0$,

$$
\begin{aligned}
&\omega_\varphi(\{x \ \text{ in } \ C[a,b] \ | \ \sup\{x(t) - x(a) | a \le t \le b\} \ge M + \alpha(a) - \alpha(b)\}) \\
&\overset{(1)}{=} \lim_{n\to\infty} \omega_\varphi(\{x \ \text{ in } \ C[a,b] \ | \ \max\{S_{n,k}(x) | 1 \le k \le n\} \ge M + \alpha(a) \\
&-\alpha(b)\}) \\
&\overset{(2)}{\le} \overline{\lim_{n\to\infty}} \omega_\varphi(\{x \ \text{ in } \ C[a,b] \ | \ \max\{S_{n,k}(x) - med(S_{n,k}(x) - S_{n,n}(x))\} \\
&\ge M\}) \\
&\overset{(3)}{\le} 2\overline{\lim_{n\to\infty}} \omega_\varphi(\{x \ \text{ in } \ C[a,b] | S_{n,n}(x)) \ge M\}) \\
&\overset{(4)}{=} 2\overline{\lim_{n\to\infty}} \int_{\mathbb{R}} \int_{u_0+M}^{\infty} [2\pi(\beta(\tau_{n,n}) - \beta(a))]^{-\frac{1}{2}} \\
&\quad \exp\{-\frac{(u_1 - \alpha(\tau_{n,n}) - (u_0 - \alpha(a)))^2}{2(\beta(\tau_{n,n}) - \beta(a))}\} dm_L(u_1) d\varphi(u_0) \\
&\overset{(5)}{=} \sqrt{\frac{2}{\pi}} \overline{\lim_{n\to\infty}} \int_{\mathbb{R}} \int_{A_n}^{\infty} \exp\{-\frac{v^2}{2}\} dm_L(v) d\varphi(u_0) \\
&\overset{(6)}{=} \sqrt{\frac{2}{\pi}} \int_{A}^{\infty} \exp\{-\frac{v^2}{2}\} dm_L(v) \\
&\overset{(7)}{=} \sqrt{\frac{2}{\pi}} A^{-1} \exp\{-\frac{A^2}{2}\}.
\end{aligned}
$$

Step (1) come from the property of measure. By the inequality $\max\{S_{n,k}(x) - med(S_{n,k}(x) - S_{n,n}(x)) | 1 \le k \le n\} \ge \max\{S_{n,k}(x) | 1 \le k \le n\} - \alpha(a) + \alpha(b)$ for x in $C[a,b]$, we obtain Step (2). We have Step (3) by Levy's inequality in [8]. Using Theorem 1.2 in above, we obtain Step (4). Putting $A_n = \frac{M - \alpha(\tau_{n,n}) + \alpha(a)}{\sqrt{\beta(\tau_{n,n}) - \beta(a)}}$ and $v = \frac{u_1 - \alpha(\tau_{n,n}) - (u_0 - \alpha(a))}{\sqrt{\beta(\tau_{n,n}) - \beta(a)}}$, we have Step (5). Letting $A = \frac{M - \alpha(b) + \alpha(a)}{\sqrt{\beta(b) - \beta(a)}}$, Step (6) results from the continuity of α and β. Step (7) holds because $A > 0$ and $\int_A^\infty \exp\{-\frac{v^2}{2}\} dm_L(v) \le A^{-1} \int_A^\infty v \exp\{-\frac{v^2}{2}\} dm_L(v) = A^{-1} \exp\{-\frac{A^2}{2}\}$. In the essentially similar manner, one can prove that $\omega_\varphi(\{x \ \text{ in } \ C[a,b] \ | \ \inf\{x(t) - x(a) | a \le$

$t \leq b\} \leq -(M + \alpha(a) - \alpha(b))\}) \leq \sqrt{\frac{2}{\pi}} A^{-1} \exp\{-\frac{A^2}{2}\}$, so, from two our inequalities, we can obtain a given inequality. □

Theorem 2.3. *If $0 < d < 2$, then for all positive real number p, $\int_{C[a,b]} \exp\{p\|x - x(a)\|_\infty^d\} d\omega_\varphi(x)$ is finite and for $0 < p < \frac{1}{2(\beta(b)-\beta(a))}$, $\int_{C[a,b]} \exp\{p\|x - x(a)\|_\infty^2\}\, d\omega_\varphi(x)$ is finite.*

Proof. For any non-negative integer n, let $A_n = \{x \text{ in } C[a,b] \mid n \leq \|x - x(a)\|_\infty < n+1\}$. Let N be a natural number with $N > \alpha(a) - \alpha(b)$. Then, by Theorem 2.2,

$$\begin{aligned}
&\int_{C[a,b]} \exp\{p\|x - x(a)\|_\infty^d\} d\omega_\varphi(x) \\
&\leq \sum_{n=0}^{N-1} \exp\{p(n+1)^d\} \\
&\quad + \sum_{n=N}^{\infty} \exp\{p(n+1)^d\} \omega_\varphi(\{x \text{ in } C[a,b] \mid \|x - x(a)\| > n\}) \\
&\leq \sum_{n=0}^{N-1} \exp\{p(n+1)^d\} \\
&\quad + 2\sqrt{\frac{2(\beta(b) - \beta(a))}{\pi}} \sum_{n=N}^{\infty} \exp\{p(n+1)^d - \frac{n^2}{2(\beta(b) - \beta(a))}\}.
\end{aligned}$$

From the root test, if $0 < d < 2$ then for all positive real number p, the right-side term in above converges and if $d = 2$ then for $0 < p < \frac{1}{2(\beta(b)-\beta(a))}$, the right-side term in above converges. □

Using the inequalities $(|a| + |b|)^d \leq 2^{d-1}(|a|^d + |b|^d)$ for $1 \leq d < 2$ and $(|a| + |b|)^2 \leq 2(|a|^2 + |b|^2)$, we can prove the following theorem which is a main theorem in this section.

Theorem 2.4. *(Fernique's theorem for the generalized analogue of Wiener measure) If $1 \leq d < 2$ and $\int_{\mathbb{R}} \exp\{2^d p|u|^d\} d\varphi(u)$ is finite then $\int_{C[a,b]} \exp\{p\|x\|_\infty^d\} d\omega_\varphi(x)$ is finite and if $p < \frac{1}{2(\beta(b)-\beta(a))}$ and $\int_{\mathbb{R}} \exp\{4p|u|^2\}$ $d\varphi(u)$ is finite then $\int_{C[a,b]} \exp\{p\, \|x\|_\infty^2\} d\omega_\varphi(x)$ is finite.*

3. Scale-invariant measurable subsets in the generalized analogue of Wiener measure space

In 1979, Johnson and Skoug presented a nice paper related to the scale-invariant measurable subsets of the concrete Wiener space in [1]. This was contributed the big development of the theories of Wiener process.

In this section, we establish the existence theorem of scale-invariant measurable subsets of the generalized analogue of Wiener space.

Throughout in this section, let φ be a probability Borel measure on $\mathbb{R}$.

For $\vec{t}(n) = (t_0, t_1, t_2, \cdots, t_n)$ with $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, let $\|\vec{t}(n)\| = max\{t_j - t_{j-1} | j = 1, 2, \cdots, n\}$ and let $S_{\vec{t}(n)}(x) = \sum_{j=1}^{n}(x(t_j) - x(t_{j-1}))^2$ for x in $C[a,b]$. Suppose that $\lim_{n\to\infty} \|\vec{t}(n)\| = 0$.

Theorem 3.1. *There is a subsequence $\langle \vec{t}(\sigma(n))\rangle$ of $\langle \vec{t}(n)\rangle$ such that $\langle S_{\vec{t}(\sigma(n))}\rangle$ converges to $\beta(b) - \beta(a)$ $\quad \omega_\varphi$-a.e.*

Proof. Using Theorem 2.2, we have

$$\begin{aligned}
&\int_{C[a,b]} S_{\vec{t}(n)}(x) d\omega_\varphi(x) \\
&= \sum_{j=1}^{n}(\beta(t_j) - \beta(t_{j-1})) + \sum_{j=1}^{n}(\alpha(t_j) - \alpha(t_{j-1}))^2 \\
&= \beta(b) - \beta(a) + \sum_{j=1}^{n}(\alpha(t_j) - \alpha(t_{j-1}))^2.
\end{aligned}$$

Since α is Riemann integrable on $[a,b]$, $\lim_{n\to\infty}\sum_{j=1}^{n}(\alpha(t_j) - \alpha(t_{j-1}))^2 = 0$, so we have

$$\lim_{n\to\infty} \int_{C[a,b]} S_{\vec{t}(n)}(x) d\omega_\varphi(x) = \beta(b) - \beta(a).$$

Similarly,

$$\begin{aligned}
&\int_{C[a,b]} S_{\vec{t}(n)}(x)^2 d\omega_\varphi(x) \\
&= \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \int_{C[a,b]} (x(t_j)-x(t_{j-1}))^2 (x(t_k)-x(t_{k-1}))^2 d\omega_\varphi(x) \\
&\quad + \sum_{j=1}^{n} \int_{C[a,b]} (x(t_j)-x(t_{j-1}))^4 d\omega_\varphi(x) \\
&= \sum_{\substack{j,k=1 \\ j\neq k}}^{n} [(\beta(t_j)-\beta(t_{j-1})) + (\alpha(t_j)-\alpha(t_{j-1}))^2][(\beta(t_k)-\beta(t_{k-1})) \\
&\quad +(\alpha(t_k)-\alpha(t_{k-1}))^2] + \sum_{j=1}^{n} [3(\beta(t_j)-\beta(t_{j-1}))^2 - 6(\alpha(t_j)-\alpha(t_{j-1}))^6 \\
&\quad \times((\beta(t_j)-\beta(t_{j-1})) + (\alpha(t_j)-\alpha(t_{j-1}))^2) + (\alpha(t_j)-\alpha(t_{j-1}))^4] \\
&= (\beta(b)-\beta(a))^2 + 2\sum_{j=1}^{n}(\beta(t_j)-\beta(t_{j-1}))^2 + 2(\beta(b)-\beta(a)) \\
&\quad \times \sum_{j=1}^{n}(\alpha(t_j)-\alpha(t_{j-1}))^2 - 8\sum_{j=1}^{n}(\alpha(t_j)-\alpha(t_{j-1}))^2(\beta(t_j)-\beta(t_{j-1})).
\end{aligned}$$

Since α and β are both Riemann integrable on $[a,b]$ and β is increasing, $\lim_{n\to\infty}\sum_{j=1}^{n}(\alpha(t_j)-\alpha(t_{j-1}))^2 = \lim_{n\to\infty}\sum_{j=1}^{n}(\beta(t_j)-\beta(t_{j-1}))^2 = \lim_{n\to\infty}\sum_{j=1}^{n}(\alpha(t_j)-\alpha(t_{j-1}))^2(\beta(t_j)-\beta(t_{j-1})) = 0$, so we have

$$\lim_{n\to\infty}\int_{C[a,b]} S_{\vec{t}(n)}(x)^2 d\omega_\varphi(x) = (\beta(b)-\beta(a))^2.$$

Hence, $\lim_{n\to\infty}\int_{C[a,b]}(S_{\vec{t}(n)}(x) - (\beta(b)-\beta(a)))^2 d\omega_\varphi(x) = 0$, that is, $\langle S_{\vec{t}(n)}\rangle$ converges to $\beta(b)-\beta(a)$ in $L^2(C[a,b],\omega_\varphi)$-sense. Thus, there is a subsequence $\langle \vec{t}(\sigma(n))\rangle$ of $\langle \vec{t}(n)\rangle$ such that $\langle S_{\vec{t}(n)}\rangle$ converges to $\beta(b)-\beta(a)$ ω_φ-*a.e.*, as desired. □

Given $\lambda > 0$ and let $C_\lambda = \{x \text{ in } C[a,b] \mid \lim_{n\to\infty} S_{\vec{t}(\sigma(n))}(x) = \lambda^2(\beta(b)-\beta(a))\}$ and let $C_0 = \{x \text{ in } C[a,b] \mid \lim_{n\to\infty} S_{\vec{t}(\sigma(n))}(x) \text{ doesn't exist}\}$. Then we have the following theorem by the similar method as in [1].

Theorem 3.2. *(1) For $\lambda \geq 0$, C_λ is Borel measurable.*
(2) $C[a,b] = \cup_{\lambda \geq 0} C_\lambda$
(3) If λ_1 and λ_2 are distinct positive real numbers then $C_{\lambda_1} \cap C_{\lambda_2} = \emptyset$.
(4) For two positive real numbers λ_1 and λ_2, $\lambda_1 C_{\lambda_2} = C_{\lambda_1 \lambda_2}$.
(5) $\omega_\varphi(C_1) = 1$.

References

[1] G. W. Johnson and D. L. Skoug, *Scale-invariant measurability in Wiener space*, Pacific J. Math., 83(1979), pp. 157-176 .
[2] G. W. Johnson and M. L. Lapidus, *The Feynman integral and Feynman's operational calculus*, Oxford Mathematical Monographs, Oxford Univ. Press, (2000).
[3] K. S. Ryu and M. K. Im, *A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula*, Trans. Amer. Math. Soc., 354(2002), pp. 4921-4951.
[4] M. X. Fernique, *Integrabilite des Vecteurs Gaussians*, Academie des Sciences, Paris Comptes Rendus, 270(1970), pp. 1698-1699.
[5] A. V. Skorokhod, *Notes on Gaussian measure in a Banach space*, Toer. Veroj. I Prim., 15(1970), pp. 517-520.
[6] K. R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York, 1967.
[7] K. S. Ryu, *The generalized Fernique's theorem for analogue of Wiener measure space*, J. Chungcheong Math. Soc., 22(2009), 743-748.
[8] H. G. Tucker, *A graduate course in probability*, Academic press, New York (1967).
[9] N. Wiener, *Differential space*, J. Math. Phys., 2(1923), pp. 131-174.

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