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DEDUCTIVE SYSTEMS IN COMMUTATIVE PRE-LOGICS

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Abstract. The notions of commutative pre-logics and terminal sections are introduced. Characterizations of a commutative pre-logic are provided. Properties of deductive systems in pre-logics which are upper semilattices are considered.

1. Introduction

I. Chajda and R. Halas [1] introduced the concept of a pre-logic which is an algebra weaker than a Hilbert algebra (an algebraic counterpart of intuitionistic logic) but strong enough to have deductive systems. They also studied algebraic properties of pre-logics and of lattices of their deductive systems.

In this paper, we introduced the notion of commutative pre-logics and terminal sections, and give some characterizations of commutative pre-logics in terms of lattices order relations, and terminal sections. We also study properties of deductive systems in pre-logics which are upper semilattices.

2. Preliminaries

We recall some definitions and results (see [1]).

Definition 2.1. By a *pre-logic*, we mean a triple $(X; \cdot, 1)$ where X is a non-empty set, $\cdot$ is a binary operation on X and $1 \in X$ is a nullary operation such that the following identities hold:

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(P1) $(\forall x \in X)\ (x \cdot x = 1)$,
(P2) $(\forall x \in X)\ (1 \cdot x = x)$,
(P3) $(\forall x \in X)\ (x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z))$,
(P4) $(\forall x, y, z \in X)\ (x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

Lemma 2.2. *Let $(X; \cdot, 1)$ be a pre-logic. Then the following hold:*

(a) $(\forall x \in X)\ (x \cdot 1 = 1)$;
(b) $(\forall x, y \in X)\ (x \cdot (y \cdot x) = 1)$;
(c) *a binary operation $\leq$ on A defined by*

$$(\forall x, y \in X)\,(x \leq y \text{ if and only if } x \cdot y = 1)$$

is a quasiorder on X (i.e., a reflexive and transitive binary relation on X);
(d) *$1 \leq x$ for all $x \in X$ implies $x = 1$.*

Remark 2.3. The quasiorder $\leq$ of Lemma 2(c) is called the *induced quasiorder of a pre-logic X*.

Lemma 2.4. *Let $\leq$ be the induced quasiorder of a pre-logic $X = (X; \cdot .1)$ and $x, y, z \in X$. If $x \leq y$, then $z \cdot x \leq z \cdot y$ and $y \cdot z \leq x \cdot z$.*

Definition 2.5. Let $X = (X; \cdot, 1)$ be a pre-logic. A non-empty subset D of X is called a *deductive system* of X if the following conditions hold:

(d1) $1 \in D$;
(d2) if $x \in D$ and $x \cdot y \in D$, then $y \in D$.

Lemma 2.6. *Let $X = (X; \cdot, 1)$ be a pre-logic and $\leq$ its induced quasi order. The the following hold:*

(a) $(\forall x, y \in X)\ (x \cdot ((x \cdot y) \cdot y) = 1)$,
(b) $(\forall x, y, z \in X)\ ((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(c) *if D is a deductive system of X and $a \in D$, $a \leq b$, then $b \in D$.*

Denote by $\mathcal{D}(X)$ the set of all deductive systems of X. It is clear that $\mathcal{D}(X)$ is non-empty.

Lemma 2.7. *Let $(X; *, 1)$ be a pre-logic and $D_i \in \mathcal{D}(X)$ for each $i \in I$. Then $D \in \mathcal{D}(X)$ for $D = \cap\{D_i | i \in I\}$.*

It implies that the set $\mathcal{D}(X)$ forms a closure operator on the power set of X. For every subset $A \subseteq X$ there exists the least deductive system on X containing A. Denote it by $\langle A \rangle$ and call the *deductive systems generated by A*. If $A = \{a\}$, we will denote $\langle\{a\}\rangle$ briefly by $\langle a \rangle$ and call it the *principal deductive system*. Hence, if A, B are subsets of X, then

(1) $A \subseteq \langle A \rangle$,
(2) $A \subseteq B$ implies $\langle A \rangle \subseteq \langle B \rangle$,
(3) $\langle \langle A \rangle \rangle = \langle A \rangle$.

If $A = \{x_1, \cdots, x_n\}$, then we denoted by $\langle x_1, \cdots, x_n \rangle = \langle \{x_1, \cdots, x_n\} \rangle$.

Theorem 2.8. *The lattice $\mathcal{D}(X)$ of all deductive systems of a pre-logic X is an algebraic lattice whose compact elements are just finitely generated deductive systems. Let $A \subseteq X$. If $A = \emptyset$, then $\langle \emptyset \rangle = \{1\}$; if $A \neq \emptyset$, then*

$$\langle A \rangle = \{a \in X | x_1 \cdot (x_2 \cdot (\cdots (x_n \cdot a) \cdots)) = 1 \text{ for some } x_1, x_2, \cdots x_n \in A\}.$$

Definition 2.9 ([2]). A dual BCK-algebra is an algebra $(X; *, 1)$ of type (2,0) satisfying (P1), lemma 2.2(a), and the following axioms:
(dBCK1) $x * y = y * x = 1 \Rightarrow x = y$,
(dBCK2) $(x * y) * ((y * z) * (x * z)) = 1$,
(dBCK3) $x * ((x * y) * y) = 1$.

Proposition 2.10 ([2]). *Let $(X; *, 0)$ be a dual BCK-algebra and $x, y, z \in X$. Then*
(1) $x * (y * z) = y * (x * z)$,
(2) $1 * x = x$.

3. Commutative pre-logics

Proposition 3.1. *Let X be a pre-logic and let $x, y, z \in X$. If $z \leq x \cdot y$ and $z \leq x$, then $z \leq y$.*

Proof. Assume that $z \leq x \cdot y$ and $z \leq x$ for any $x, y, z \in X$. Then $x \cdot y \in \langle z \rangle$ and $x \in \langle z \rangle$. Since $\langle z \rangle$ is a deductive system, it follows from (d2) that $y \in \langle z \rangle$. Hence $z \leq y$. □

We now give an equivalent condition of a deductive system.

Theorem 3.2. *Let D be a non-empty subset of a pre-logic X. Then D is a deductive system of X if and only if for any x and y in D, $x \leq y \cdot z$ implies $z \in D$.*

Proof. Let D be a deductive system of X and let $x, y \in D$. If $x \leq y \cdot z$, then $x \cdot (y \cdot z) = 1$. Using (d2), we have $y \cdot z \in D$. Using (d2) again, $z \in D$.

Conversely, assume that $x \leq y \cdot z$ implies $z \in D$ for all $x, y \in D$ and $z \in X$. Since $D \neq \emptyset$, we may assume $x \in D$. We note from Lemma

TABLE 1. $\cdot$ -operation

$\cdot$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	1	1	c	d
c	1	1	1	1	d
d	1	1	b	c	1

2.2(a) that $x \leq x \cdot 1$ so that $1 \in D$ by assumption. Let $x \in D$ and $x \cdot y \in D$. Combining Lemma 2.6(a) and assumption, we get $y \in D$. Hence D is a deductive system of X. This completes the proof. □

Theorem 3.3. *For any deductive system D of a pre-logic X and any $a \in X$, the set $D_a := \{x \in X | a \cdot x \in D\}$ is the least deductive system of X containing D and a.*

Proof. It follows from Lemma 2.2(a) that $a \cdot 1 = 1$ for any $a \in X$. Hence $1 \in D_a$. Using (P1), we have $a \cdot a = 1 \in D$ for any $a \in D$ and so $a \in D_a$. Let $x \in D_a$ and $x \cdot y \in D_a$. Then $a \cdot x \in D$ and $a \cdot (x \cdot y) \in D$. Since $a \cdot (x \cdot y) = (a \cdot x) \cdot (a \cdot y) \in D$ and $a \cdot x \in D$, we obtain $a \cdot y \in D$. Hence $y \in D_a$. Thus D_a is a deductive system. Let $x \in D$. Since $x \cdot (a \cdot x) = 1 \in D$ and D is a deductive system of X, we get $a \cdot x \in D$. Hence $x \in D_a$. Let H be any deductive system of X containing D and a. Let $x \in D_a$. Then $a \cdot x \in D \subseteq H$. Since $a \in H$ and H is a deductive system of X, we have $x \in H$. Therefore $D_a \subseteq H$. Thus D_a is the least deductive system of X containing D and a. □

For any x, y in a pre-logic X , we define $x \vee y$ as $(y \cdot x) \cdot x$. Under this definition, using Lemma 2.2(a) and (P4), we have

$$\begin{aligned} x \cdot (x \vee y) =& x \cdot ((y \cdot x) \cdot x) \\ =& (y \cdot x) \cdot (x \cdot x) \\ =& (y \cdot x) \cdot 1 = 1, \end{aligned}$$

i.e., $x \leq x \vee y$. From Lemma 2.6(a), it follows that $y \leq x \vee y$. Hence $x \vee y$ is an upper bound of x and y. As easily seen, we have
(c_1) $x \vee x = x$ and $x \vee 1 = 1 \vee x = 1$.

Example 3.4. Let $X := \{1, a, b, c, d\}$ be a pre-logic with the $\cdot$ -operation given by Table 1. Then $a \vee d = a \neq 1 = d \vee a$ and a is the least upper bound of a and d. Hence, in general, $x \vee y \neq y \vee x$ and $x \vee y$ may not be the least upper bound of x and y.

TABLE 2. $\cdot$ -operation

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

Definition 3.5. A pre-logic X is said to be *commutative* if it satisfies the following identity

(C) $(y \cdot x) \cdot x = (x \cdot y) \cdot y$, i.e., $x \vee y = y \vee x$

for all $x, y \in X$.

Example 3.6. Let $X := \{1, a, b, c\}$ be a set with the $\cdot$ -operation given by Table 2. It is easy to show that $(X; \cdot, 1)$ is a commutative pre-logic.

Theorem 3.7. *If X is a commutative pre-logic X, then it is a semilattice with respect to $\vee$.*

Proof. Assume that X is a commutative pre-logic. As already seen, $x \vee y$ is an upper bound of x and y. We shall show that $x \vee y$ is the least upper bound of x and y, To do this, suppose that $x \leq z$ and $y \leq z$. Then $x \cdot z = y \cdot z = 1$. Hence by commutative we have (i): $z = 1 \cdot z = (x \cdot z) \cdot z = (z \cdot x) \cdot x$ and (ii): $z = 1 \cdot z = (y \cdot z) \cdot z = (z \cdot y) \cdot y$. Using (i) and (ii), we have (iii): $z = (z \cdot x) \cdot x = (((z \cdot y) \cdot y) \cdot x) \cdot x$. Set $u = (z \cdot y) \cdot y$. Then $z = (u \cdot x) \cdot x$ follows from (iii). Since $y \leq (z \cdot y) \cdot y = u$, by Lemma 2.4 we have $u \cdot x \leq y \cdot x$. Using Lemma 2.4, we get $(y \cdot x) \cdot x \leq (u \cdot x) \cdot x = z$. Hence we get $x \vee y \leq z$, which shows that $x \vee y$ is the least upper bound of x and y. Therefore we have the associative law with respect to $\vee$. Consequently, X is a semilattice with respect to $\vee$. The proof is complete. □

The converse of Theorem 3.7 is not true as seen in the following example.

Example 3.8. Let $X := \{1, a, b, c\}$ be a set with the $\cdot$ -operation given by Table 3. It is easy to check that X is a pre-logic with semilattice with respect to $\vee$. Since $a \vee b = a \neq 1 = b \vee a$, X is not commutative.

Theorem 3.9. *Let X be a pre-logic. If X is commutative, then the following properties hold:*

(c_2) $y \cdot (x \vee z) = (z \cdot x) \cdot (y \cdot x)$.

Table 3. $\cdot$ -operation

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	1	b	1

(c_3) *$x \leq y$ implies $x \vee y = y$.*
(c_4) *$z \leq x$ and $x \cdot z \leq y \cdot z$ imply $y \leq x$.*
(c_5) *$y \vee x = x \vee (y \vee x)$, i.e., $(x \cdot y) \cdot y = (((x \cdot y) \cdot y) \cdot x) \cdot x$. for all $x, y \in X$.*

Proof. (c_2) Using (P4), we have $y \cdot (x \vee z) = y \cdot ((z \cdot x) \cdot x) = (z \cdot x) \cdot (y \cdot x)$.
(c_3) If $x \leq y$, then $y = 1 \cdot y = (x \cdot y) \cdot y = y \vee x$. Hence by the commutativity, $x \leq y$ implies $x \vee y = y$.
(c_4) Assume that X is commutative. If $z \leq x$ and $x \cdot z \leq y \cdot z$, then $z \cdot x = 1$ and $(x \cdot z) \cdot (y \cdot z) = 1$. Using (P2), (P4) and commutative, we have

$$\begin{aligned} y \cdot x =& y \cdot (1 \cdot x) \\ =& y \cdot ((z \cdot x) \cdot x) \\ =& y \cdot ((x \cdot z) \cdot z) \\ =& (x \cdot z) \cdot (y \cdot z) \\ =& 1 \end{aligned}$$

which implies that $y \leq x$.
(c_5) Let X be a commutative pre-logic and let $x, y \in X$. We recall that $x \leq (x \cdot y) \cdot y$ means $x \leq y \vee x$. Then (c_3) yields $x \vee (y \vee x) = y \vee x$, which is ($c_5$). This completes the proof. □

Proposition 3.10. *If $(X; \cdot, 1)$ is a commutative pre-logic, then for all $x, y \in X$, $x \cdot y = 1$ and $y \cdot x = 1 \Rightarrow x = y$.*

Proof. Suppose that $x \cdot y = 1$ and $y \cdot x = 1$ for all $x, y \in X$. Then $x = 1 \cdot x = (y \cdot x) \cdot x = (x \cdot y) \cdot y = y$. □

Theorem 3.11. If $(X; \cdot, 1)$ is a commutative pre-logic, then $(X; \cdot, 1)$ is a dual BCK-algebra.

Proof. Proposition 3.10 yields (dBCK1). Now let $x, y, z \in X$. Applying (P4) and (C), we have

$$(y \cdot z) \cdot (x \cdot z) = x \cdot [(y \cdot z) \cdot z] = x \cdot [(z \cdot y) \cdot y] = (z \cdot y) \cdot (x \cdot y).$$

TABLE 4. $\cdot$ -operation

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Hence

$$(x \cdot y) \cdot [(y \cdot z) \cdot (x \cdot z)] = (x \cdot y) \cdot [(z \cdot y) \cdot (x \cdot y)].$$

Since $x \cdot (y \cdot x) = 1$ for any $x \in X$, we have $(x \cdot y) \cdot [(y \cdot z) \cdot (x \cdot z)] = 1$. Therefore (dBCK2) holds. Moreover, by (P1) and (P4), $x \cdot ((x \cdot y) \cdot y) = (x \cdot y) \cdot (x \cdot y) = 1$. From this we have (dBCK3). Thus X is a dual BCK-algebra. □

The converse of Theorem 3.11 is not true as seen in the following example.

Example 3.12. Let $X := \{1, a, b, c\}$ be a set with the $\cdot$ -operation given by Table 4. It is easy to check that X is a dual BCK-algebra. Since $(a \cdot a) \cdot (a \cdot b) = a \neq 1 = a \cdot (a \cdot b)$, X is not a pre-logic.

For an element a of a pre-logic X, we consider the set

$$\{x \in X | a \leq x\},$$

denoted by $H(a)$, which is called the *terminal section* of an element a. Since $1, a \in H(a)$, $H(a)$ is not empty. Using this notation, we can characterize a commutative pre-logic.

Theorem 3.13. *If a pre-logic X is commutative, then it satisfies the identity:*
(c_6) $H(a) \cap H(b) = H(a \vee b)$
for all $a, b \in X$.

Proof. Let X be a commutative pre-logic and let $a, b \in X$. If $x \in H(a) \cap H(b)$, then $a \leq x$ and $b \leq x$. Hence $a \vee b \leq x$, which implies that $x \in H(a \vee b)$. Hence $H(a) \cap H(b) \subseteq H(a \vee b)$. Now if $x \in H(a \vee b)$, then $a \vee b \leq x$. Since $a \vee b$ is an upper bound of a and b, it follows that $a \leq x$ and $b \leq x$, i.e., $x \in H(a)$ and $x \in H(b)$. Hence $x \in H(a) \cap H(b)$. Therefore (c_6) holds. □

Proposition 3.14. *Let X be a pre-logic and $x, y \in X$. Then $\langle x \vee y \rangle \subseteq \langle x \rangle \cap \langle y \rangle$ with equality in a commutative pre-logic.*

Proof. The inclusion $\langle x \vee y \rangle \subseteq \langle x \rangle \cap \langle y \rangle$ is trivial. Conversely, let X be a commutative pre-logic and $a \in \langle x \rangle \cap \langle y \rangle$. Then $\langle x \rangle = \{b \in X | x \leq b\}$ and $\langle y \rangle = \{b \in X | y \leq b\}$, whence $x \vee y \leq a$ giving $a \in \langle x \vee y \rangle$. □

Definition 3.15. A deductive system D is said to be *maximal* if $D \neq X$ and $D \subseteq D_1 \subseteq X$ implies $D = D_1$ or $D_1 = X$ for $D_1 \in \mathcal{D}(X)$.

Theorem 3.16. *Let D be a maximal deductive system of a pre-logic X. Then for any $x, y \in X$, we have $x \cdot y \in D$ or $y \cdot x \in D$.*

Proof. Let $x, y \in X$. If $x \in D$, then $x \leq y \cdot x$ implies $y \cdot x \in D$. Similarly, if $y \in D$, then $x \cdot y \in D$, since $y \leq x \cdot y$. Finally assume that $x \notin D$ and $y \notin D$ and $x \cdot y \notin D$. Then $D_{x \cdot y} = \{z \in X | (x \cdot y) \cdot z \in D\}$ is a deductive system containing D and $x \cdot y$. Since D is maximal, $D_{x \cdot y} = X$. Hence $(x \cdot y) \cdot (y \cdot x) \in D$, which implies from (P4) that $y \cdot ((x \cdot y) \cdot x) \in D$. Using (P2) and Lemma 2.2(b) and (P3), we have

$$\begin{aligned} y \cdot x &= 1 \cdot (y \cdot x) \\ &= (y \cdot (x \cdot y)) \cdot (y \cdot x) \\ &= y \cdot ((x \cdot y) \cdot x) \in D. \end{aligned}$$

which completes the proof. □

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