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QUASI GENERALIZED OPEN SETS AND QUASI GENERALIZED CONTINUITY ON BIGENERALIZED TOPOLOGICAL SPACES

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Abstract. We introduce the notions of bigeneralized topological spaces and quasi generalized open sets, and study some basic properties for the sets. We also introduce the notion of quasi generalized continuity on bigeneralized topological spaces, and investigate characterizations for the continuity.

1. Introduction

In [1], Császár introduced the notions of generalized neighborhood systems and generalized topological spaces. He also introduced the notions of continuous functions and associated interior and closure operators on generalized topological spaces in [1]. The concept of bitopological spaces was introduced by Kelly [3]. A set equipped with two topologies is called a bitopological space. Datta [2] has states that a subset S of bitopological space (X, P, Q) is *quasiopen* if for every $x \in S$, there exists a P-open set U such that $x \in U \subseteq S$, or a Q-open set V such that $x \in V \subseteq S$.

The purpose of this paper is to introduce and investigate the notion of quasi generalized open sets defined by using two generalized topologies instead of two topologies. We also introduce the notion of quasi generalized continuity between bigeneralized topological spaces, and investigate characterizations for the continuity.

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2. Preliminaries

We recall some notions and notations defined in [1]. Let X be a nonempty set and ψ be a collection of subsets of X. Then ψ is called a *generalized topology* (briefly GT) on X iff $\emptyset \in \psi$ and $G_i \in \psi$ for $i \in I \neq \emptyset$ implies $G = \cup_{i\in I} G_i \in \psi$. We call the pair (X, ψ) a *generalized topological space* (briefly GTS) on X. The elements of ψ are called ψ*-open* sets and the complements are called ψ*-closed* sets.

If ψ is a generalized topology on X and $A \subseteq X$, the *interior* of A (denoted by $i_\psi(A)$) is the union of all $G \subseteq A$, $G \in \psi$, and the *closure* of A (denoted by $c_\psi(A)$) is the intersection of all ψ-closed sets containing A.

Let ψ and $\psi^{'}$ be generalized topologies on X and Y, respectively. Then a function $f : (X, \psi) \to (Y, \psi^{'})$ is said to be $(\psi, \psi^{'})$*-continuous* [1] if $G^{'} \in \psi^{'}$ implies that $f^{-1}(G^{'}) \in \psi$.

Theorem 2.1. ([1]) Let (X, ψ) be generalized topological space. Then
(1) $c_\psi(A) = X - i_\psi(X - A)$;
(2) $i_\psi(A) = X - c_\psi(X - A)$.

3. Quasi generalized open sets in bigeneralized topological spaces

Definition 3.1. Let X be a nonemptyset and ψ_1, ψ_2 generalized topologies on X. A triple (X, ψ_1, ψ_2) is called a *bigeneralized topological space* (briefly *biGTS*).

Throughout the present paper, (X, ψ_1, ψ_2), (Y, μ_1, μ_2) and $(Z, \lambda_1, \lambda_2)$ denote bigeneralized topological spaces.

Remark 3.2. In a bigeneralized topological space (X, ψ_1, ψ_2), if the generalized topologies ψ_1, ψ_2 are topologies on X, then the bigeneralized topological space (X, ψ_1, ψ_2) is a bitopological space introduced by Kelly in [3].

Definition 3.3. Let (X, ψ_1, ψ_2) be a biGTS. A subset A of X is said to be *quasi* (ψ_1, ψ_2)*-open* (briefly *quasi* q_ψ*-open*) if for every $x \in A$, there exists a ψ_1-open set U such that $x \in U \subseteq A$, or a ψ_2-open set V such that $x \in V \subseteq A$. A subset A of X is said to be *quasi* q_ψ*-closed* if the complement of A is quasi q_ψ-open.

In a given bigeneralized topological space (X, ψ_1, ψ_2), a quasi q_ψ-open set need not be ψ_1-open or ψ_2-open as shown in the next example.

Example 3.4. Let $X = \{a, b, c, d\}$. Consider two generalized topologies $\psi_1 = \{\emptyset, \{a, b\}\}$ and $\psi_2 = \{\emptyset, \{a, c\}\}$ on X. Then $\{a, b, c\}$ is a quasi q_ψ-open set but it is neither ψ_1-open nor ψ_2-open

Lemma 3.5. Let (X, ψ_1, ψ_2) be a biGTS and A a subset of X. Then

(1) A is quasi q_ψ-open if and only if A is a union of a ψ_1-open set and a ψ_1-open set.

(2) A is quasi q_ψ-closed if and only if A is a intersection of a ψ_1-closed set and a ψ_1-closed set.

(3) Any union of quasi q_ψ-open sets is quasi q_ψ-open.

Proof. Obvious. □

Remark 3.6. In a given bigeneralized topological space, the intersection of two quasi q_ψ-open sets may not be quasi g-open as shown in the next example.

Example 3.7. Let $X = \{a, b, c, d\}$. Consider two generalized topologies $\psi_1 = \{\emptyset, \{d\}, \{a, b\}, \{a, b, d\}\}$ and $g_2 = \{\emptyset, \{c\}\{a, d\}, \{a, c, d\}\}$ on X. Then $\{a, b, c\}$ and $\{a, c, d\}$ are quasi q_ψ-open sets but the intersection $\{a, c\}$ is not quasi q_ψ-open set.

Definition 3.8. Let (X, ψ_1, ψ_2) be a biGTS and $A \subseteq X$. We define the *quasi q_ψ-closure* (briefly $c_{q_\psi}(A)$) and the *quasi q_ψ-interior* (briefly $i_{q_\psi}(A)$) as the following:

$$c_{q_\psi}(A) = \cap\{F : A \subseteq F \text{ for a quasi } q_\psi\text{-closed set } F\};$$
$$i_{q_\psi}(A) = \cup\{G : G \subseteq A \text{ for a quasi } q_\psi\text{-open set } G\}.$$

Let (X, ψ_1, ψ_2) be a biGTS and A a subset of X. The ψ_i-closure and ψ_i-interior of A with respect to ψ_i are denoted by $c_{\psi_i}(A)$ and $i_{\psi_i}(A)$, respectively, for $i = 1, 2$.

Theorem 3.9. Let (X, ψ_1, ψ_2) be a biGTS and $A \subseteq X$. Then

(1) $c_{q_\psi}(A) = c_{\psi_1}(A) \cap c_{\psi_2}(A)$.

(2) $i_{q_\psi}(A) = i_{\psi_1}(A) \cup i_{\psi_2}(A)$.

(3) A is quasi q_ψ-closed iff $c_{q_\psi}(A) = A$.

(4) A is quasi q_ψ-open iff $i_{q_\psi}(A) = A$.

(5) $x \in c_{q_\psi}(A)$ iff for every quasi q_ψ-open set U containing x, $A \cap U \neq \emptyset$.

(6) $x \in i_{q_\psi}(A)$ iff there exists a quasi q_ψ-open set U containing x such that $x \in U \subseteq A$.

(7) $c_{q_\psi}(A) = X - i_{q_\psi}(X - A)$; $i_{q_\psi}(A) = X - c_{q_\psi}(X - A)$

Proof. (1) From Lemma 3.5, it follows

$$\begin{aligned} c_{q_\psi}(A) &= \cap\{F : A \subseteq F \text{ for a quasi } q_\psi\text{-closed set } F\} \\ &= \cap\{F : A \subseteq F, \quad F = F_1 \cap F_2 \ \text{ for a } \psi_i\text{-closed set } F_i, i = 1, 2\} \\ &= \cap\{F_1 \cap F_2 : A \subseteq F_1 \cap F_2 \ \text{ for a } \psi_i\text{-closed set } F_i, i = 1, 2\} \\ &= (\cap\{F_1 : A \subseteq F_1 \ \text{ for a } \psi_1\text{-closed set } F_1\}) \\ &\cap (\cap\{F_2 : A \subseteq F_2 \ \text{ for a } \psi_2\text{-closed set } F_2\}) \\ &= c_{\psi_1}(A) \cap c_{\psi_2}(A). \end{aligned}$$

Similarly, (2) is obtained.

(3) If A is quasi q_ψ-closed, then $A = F_1 \cap F_2$ for a ψ_1-closed set F_1 and a ψ_2-closed set F_2, and so $c_{\psi_1}(A) \subseteq c_{\psi_1}(F_1)$ and $c_{\psi_2}(A) \subseteq c_{\psi_2}(F_2)$. And by (1),

$$c_{q_\psi}(A) = c_{\psi_1}(A) \cap c_{\psi_2}(A) \subseteq c_{\psi_1}(F_1) \cap c_{\psi_2}(F_2) \subseteq F_1 \cap F_2 = A.$$

Consequently, $c_{q_\psi}(A) = A$.

(4) The proof is similar to that of (3).

(5),(6) It can be easily proved.

(7) From Theorem 2.1, (1) and (2), it follows
$c_{q_\psi}(A) = c_{\psi_1}(A) \cap c_{\psi_2}(A) = (X - i_{\psi_1}(X - A)) \cap (X - i_{\psi_2}(X - A)) = X - (i_{\psi_1}(X - A) \cup i_{\psi_2}(X - A)) = X - i_{q_\psi}(X - A)$.
Similarly, we can show that $i_{q_\psi}(A) = X - c_{q_\psi}(X - A)$. □

Definition 3.10. Let (X, ψ_1, ψ_2) be a biGTS. X is said to be *quasi* $q_\psi T_1$ if for $x, y \in X$, $x \neq y$, there exist quasi q_ψ-open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.

Theorem 3.11. Let (X, ψ_1, ψ_2) be a biGTS. Then X is quasi $q_\psi T_1$ iff singletons are quasi q_ψ-closed.

Proof. Let X be quasi $q_\psi T_1$ and $x \in X$. For $y \in X$ with $x \neq y$, there exists a quasi q_ψ-open set U_y such that $y \in U_y$ and $x \notin U_y$. Then $X - U_y$ is a quasi q_ψ-closed set containing x and so $\cap(X - U_y) = \{x\}$. Hence the singleton set $\{x\}$ is quasi q_ψ-closed.

The converse is obvious.

□

Let (X, ψ_1, ψ_2) be a biGTS and $A \subseteq X$. We define the *quasi q_ψ-kernel* (briefly $q_\psi ker(A)$) as the following:

$$q_\psi ker(A) = \cap\{F : A \subseteq F \text{ for a quasi } q_\psi\text{-open set } F\}.$$

Theorem 3.12. Let (X, ψ_1, ψ_2) be a biGTS. Then the following are equivalent:
(1) X is quasi $q_\psi T_1$.
(2) For every $A \subseteq X$, $A = q_\psi ker(A)$.
(3) For $x \in X$, $\{x\} = q_\psi ker(\{x\})$.

Proof. (1) ⇒ (2) Let $A \subseteq X$. For each $x \notin A$, from Theorem 3.11, $X - \{x\}$ is quasi q_ψ-open and $A \subseteq X - \{x\}$. This implies

$$A \subseteq q_\psi ker(A) \subseteq \cap_{x \notin A}(X - \{x\}) = A.$$

So $A = q_\psi ker(A)$.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) For $x, y \in X$ with $x \neq y$, by (3), $\{x\} = q_\psi ker(\{x\})$ and $\{y\} = q_\psi ker(\{y\})$. Since $y \notin q_\psi ker(\{x\})$ and $x \notin q_\psi ker(\{y\})$, there exist quasi q_ψ-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Hence X is quasi $q_\psi T_1$. □

4. quasi (q_ψ, q_μ)-continuity

Definition 4.1. Let (X, ψ_1, ψ_2) and (Y, μ_1, μ_2) be two biGTS's. Then a function $f : (X, \psi_1, \psi_2) \rightarrow (Y, \mu_1, \mu_2)$ is said to be *quasi (q_ψ, q_μ)-continuous* (or *quasi generalized continuous*) if for every quasi q_μ-open set U in (Y, μ_1, μ_2), $f^{-1}(U)$ is quasi q_ψ-open in (X, ψ_1, ψ_2).

Let (X, ψ_1, ψ_2) be a biGTS. The bigeneralized topological space X is said to be *strong* if X is quasi q_ψ-open.

Theorem 4.2. Let $f : (X, \psi_1, \psi_2) \rightarrow (Y, \mu_1, \mu_2)$ be a quasi (q_ψ, q_μ)-continuous function on two biGTS's $(X, \psi_1, \psi_2), (Y, \mu_1, \mu_2)$. If (Y, μ_1, μ_2) is strong, then (X, ψ_1, ψ_2) is also strong.

Proof. Since Y is quasi q_μ-open, from quasi (q_ψ, q_μ)-continuity of f, $f^{-1}(Y) = X$ must be quasi q_ψ-open. Thus X is strong. □

Theorem 4.3. Let (X, ψ_1, ψ_2) and (Y, μ_1, μ_2) be two biGTS's. Then the following are equivalent:
(1) f is quasi (q_ψ, q_μ)-continuous.
(2) $f^{-1}(i_{q_\mu}(B)) \subseteq i_{q_\psi}(f^{-1}(B))$ for all $B \subseteq Y$.

(3) $c_{q_\psi}(f^{-1}(B)) \subseteq f^{-1}(c_{q_\mu}(B))$ for all $B \subseteq Y$.
(4) $f(c_{q_\psi}(A)) \subseteq c_{q_\mu}(f(A))$ for all $A \subseteq X$.
(5) For every quasi q_μ-closed set F in Y, $f^{-1}(F)$ is quasi q_ψ-closed in X.
(6) For each $x \in X$ and each quasi q_μ-open set V containing $f(x)$, there exists a quasi q_ψ-open set U containing x such that $f(U) \subseteq V$.

Proof. (1) $\Rightarrow$ (2) For every $B \subseteq Y$, since $i_{q_\mu}(B)$ is quasi q_μ-open, by (1), easily we have the condition (2).

(2) $\Rightarrow$ (1) Obvious.

(2) $\Leftrightarrow$ (3) It follows from Theorem 3.9.

(3) $\Leftrightarrow$ (4) and (1) $\Leftrightarrow$ (5) Obvious.

(6) $\Rightarrow$ (1) Let V be a quasi q_μ-open set in (Y, μ_1, μ_2) and $x \in f^{-1}(V)$. Then by (6), there exists a quasi q_ψ-open set U_x containing x such that $f(U_x) \subseteq V$. So by Lemma 3.5, $f^{-1}(V) = \cup U_x$ is quasi q_ψ-open. Thus f is quasi (q_ψ, q_μ)-continuous.

(1) $\Rightarrow$ (6) Obvious.

□

References

[1] Á. Császár; Generalized Topology, Generalized Continuity, Acta Math. Hungar., 96 (2002), 351-357.
[2] M. C. Datta; Contribution to the theory of bitopological spaces, Ph. D. Cissertation, B.I.T. Pilani(India), 1971.
[3] J. C. Kelly; Bitopological spaces, Proc. London Mat. Soc.,13(3)(1963), 71-89.

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