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INTERVAL-VALUED FUZZY SUBGROUPS AND RINGS

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Abstract. We introduce the concepts of interval-valued fuzzy subgroups [resp. normal subgroups, rings and ideals] and investigate some of it's properties.

1. Introduction

In 1986, Atanassuv[1] introduced the concept of intuitionistic fuzzy sets as a generalitation of fuzzy sets introduced by Zadeh[13], After then, Banerjee and Basnet[3], and Hur et. al[8, 9] applied it to algebra. Çoker[5, 6] studied intuitionistic fuzzy topological spaces.

In 1975, Zadeh[14] suggested the notion of interval-valued fuzzy sets as another generalization of fuzzy sets. After that time, Biswas[4] applied it to group theory, and Gorzalczany[7] suggested a method of inference in approximate reasoning by using interval-valued fuzzy sets. Moreover Montal and Samanta[12] introduced the concept of topology of interval-valued fuzzy sets and investigate some of it's properties. Recently, Hur et. al[10] studies interval-valued fuzzy relations in the sense of a lattice theory. In this paper, we introduce the concept of interval-valued fuzzy subgroups [resp.normal subgroup, rings and ideals] and investigate some of it's properties.

2. Preliminaries

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In this section, we list some concepts and results related to interval-valued fuzzy set theory and needed in next sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $[0,1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0,0]$, $\mathbf{1}$ $= [1,1]$, and $\boldsymbol{a} = [a,a]$ for every $a \in (0,1)$, We also note that

(i) $(\forall M, N \in D(I))$ $(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))$ $(M = N \leq M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^C, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$(See[12]).

Definition 2.1[7,14]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set*(is short, *IVFS*) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^L \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$[resp $A^U(x)$] is called the *lower*[resp *upper*] *end point of x to A*. For any $[a,b] \in D(I)$, the interval-valued fuzzy A in X defined by $A(x) = [A^L(x), A^U(x)] = [a,b]$ for each $x \in X$ is denoted by $\widetilde{[a,b]}$ and if $a = b$, then the IVFS $\widetilde{[a,b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVFSs in X as $D(I)^X$ .It is clear that set $A = [A, A] \in D(I)^X$ for each $A \in I^X$.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : Y \to Z$ are mappings.

Definition 2.1′ [1,9]. Let X be a set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called *a intuitionistic fuzzy set*(in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$for each $x \in X$, where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership(namely $\mu_A(x)$) and the degree of nonmembership(namely $\nu_A(x)$) of each $x \in X$ to A, respectively. in particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denoted the set of all the IFSs in X as IFS(X).

Result 2.A[2, Lemma 1]. We define two mappings $f : D(I)^X \to \mathrm{IFS(X)}$ and $g : \mathrm{IFS(X)} \to \mathrm{D(I)^X}$ as follows, respectively:

(i) $f(A) = (A^L, 1 - A^U)$, $\forall A \in D(I)^X$,

(ii) $g(B) = [\mu_B, 1 - \nu_B]$, $\forall B \in \mathrm{IFS(X)}$.

In this case , we write as $f(A) = A_*$ and $g(B) = B^*$, respectively. Then

(a) $g \circ f = 1_{D(I)^X}$, *i.e.*, $g(f(A)) = A$, $\forall A \in D(I)^X$.

(a) $f \circ g = 1_{\mathrm{IFS(X)}}$, *i.e.*, $f(g(B)) = B$, $\forall B \in \mathrm{IFS}(X)$.

Definition 2.2[7]. An IVFS A is called an *interval-valued fuzzy point*(in short, *IVFP*) *in* X *with the support* $x \in X$ and *the value* $[a, b] \in D(I)$ with $b > 0$, denoted by $A = x_{[a,b]}$, if for each $y \in X$

$$A(y) = \begin{cases} [a, b] & \text{if} \quad y = x, \\ 0 & \text{otherwise} \end{cases}$$

In particular, if $b = a$, then $x_{[a,b]}$ is denoted by $x_{\boldsymbol{a}}$.

We will denote the set of all IVFPs in X as $\mathrm{IVF_P(X)}$.

Definition 2.3 [7]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^C = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha\in\Gamma} A_\alpha = [\bigvee_{\alpha\in\Gamma} A^L_\alpha, \bigvee_{\alpha\in\Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

Result 2.B[7, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(a) $\widetilde{0} \subset \text{A} \subset \widetilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma}(A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma}(A \cup A_\alpha)$.

(g) $(\widetilde{0})^c = \widetilde{1}$, $(\widetilde{1})^c = \widetilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A_\alpha^c$, $(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A_\alpha^c$.

Definition 2.4[7]. Let $A \in D(I)^X$ and let $x_M \in \text{IVF}_\text{P}(\text{X})$. Then:

(i) The set $\{x \in X : A^U(x) > 0\}$ is called the *support* of A and is denoted by $S(A)$.

(ii) x_M said to *belong to* A, denoted by $x_M \in A$, if $M^L \leq A^L(x)$ and $M^U \leq A^U(x)$ for each $x \in X$.

It is obvious that $A = \bigcup_{x_M \in A} x_M$ and $x_M \in A$ if and only if $x_{M^L} \in A^L$ and $x_{M^U} \in A^U$.

Definition 2.5[7]. Let $f : X \to Y$ be a mapping, let $A \in D(I)^X$ and let $B \in D(I)^Y$. Then:

(i) the *image of A under f*, denoted by $f(A)$, is an IVFS in Y defined as follows: For each $y \in Y$,

$$f(A)^L(y) = \begin{cases} \bigvee_{y=f(x)} A^L(x) & \text{if} \quad f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f(A)^U(y) = \begin{cases} \bigvee_{y=f(x)} A^U(x) & \text{if} \quad f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) the *preimage of B under f*, denoted by $f^{-1}(B)$, is an IVFS in Y defined as follows: For each $y \in Y$,

$$f^{-1}(B)^L(y) = (B^L \circ f)(x) = B^L(f(x))$$

and

$$f^{-1}(B)^U(y) = (B^U \circ f)(x) = B^U(f(x))$$

.

It can be easily seen that $f(A) = [f(A^L), f(A^U)]$ and $f^{-1}(B) = [f^{-1}(B^L), f^{-1}(B^U)]$.

Result 2.C[7, Theorem 2]. Let $f : X \to Y$ be a mapping and $g : Y \to Z$ be a mapping. Then:
(a) $f^{-1}(B^c) = [f^{-1}(B)]^c$, $\forall B \in D(I)^Y$.
(b) $[f(A)]^c \subset f(A^c)$, $\forall A \in D(I)^Y$.
(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$, where $B_1, B_2 \in D(I)^Y$.
(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$, where $A_1, A_2 \in D(I)^X$.
(e) $f(f^{-1}(B)) \subset B$, $\forall B \in D(I)^Y$.
(f) $A \subset f(f^{-1}(A))$, $\forall A \in D(I)^Y$.
(g) $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, $\forall C \in D(I)^Z$.
(h) $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.
(h) $f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha) = \bigcap_{\alpha\in\Gamma} f^{-1}B_\alpha$, where $\{B_\alpha\}_{\alpha\in\Gamma} \in D(I)^Y$.

3. Interval-valued fuzzy subgroupoids

Definition 3.1. Let $(X, \cdot)$ be a groupoid and let $A, B \in D(I)^X$. Then the *interval-valued fuzzy product of A and B*, denoted by $A \circ B$, is an IVFS in X defined as follows : For each $x \in X$,

$$(A \circ B)(x) = \begin{cases} [\bigvee_{yz=x} [A^L(y) \wedge B^L(z)], \bigvee_{yz=x} [A^U(y) \wedge B^U(z)]] & \text{if} \quad yz = x, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.1′[8]. Let $X, \circ$ be geoupoid and let $A, B \in IFS(X)$. Then the *intuitionistic fuzzy product* of A and B, $A \circ B$, is defined as follow : For any $x \in X$,

$$\mu_{A\circ B}(x)=\begin{cases}\bigvee_{yz=x}[\mu_A(y)\wedge\mu_B(z)]] & \text{if} \quad \exists(y,z)\in X\times X \text{with} \quad yz=x,\\ 0 & \text{otherwise.}\end{cases}$$

and

$$\nu_{A\circ B}(x)=\begin{cases}\bigwedge_{yz=x}[\nu_A(y)\vee\nu_B(z)] & \text{if} \quad \exists(y,z)\in X\times X \text{with} \quad yz=x,\\ 1 & \text{otherwise.}\end{cases}$$

Remark 3.1. By Result 2.A, Definition 3.1 is reduced to Definition 3.1′ and the reverse holds.

Proposition 3.2. Let "$\circ$" be same as above, let $x_M, y_N \in$ IVFp(X) and let $A, B \in D(I)^X$. Then:

(a) $x_M \circ y_N = (xy)_{M\cap N}$.

(a) $A\circ B = \bigcup_{x_M\in A, y_N\in B} x_M \circ y_N$.

Proof. (a) Let $z\in X$. Then

$$(x_M\circ y_N)(z)=\begin{cases}[\bigvee_{z=x'y'}(x_M^L(x')\wedge y_N^L(y')), \bigvee_{z=x'y'}(x_M^U(x')\wedge y_N^U(y'))] \\ \qquad\qquad\qquad\qquad \text{if} \quad x'y'=z,\\ 0 \qquad\qquad\qquad\qquad \text{otherwise.}\end{cases}$$

$$=\begin{cases}[M^L\wedge N^L, M^U\wedge N^U] & \text{if} \quad z=xy,\\ 0 & \text{otherwise.}\end{cases}$$

$$=(xy)_{M\cap N}$$

.

(b) Let $C=\bigcup_{x_M\in A, y_N\in B} x_M\circ y_N$, *i.e.*,

$$C=[\bigwedge_{x_{M^L}\in A^L, y_{N^L}\in B^L}(x_{M^L}\circ y_{N^L}), \bigwedge_{x_{M^U}\in A^U, y_{N^U}\in B^U}(x_{M^U}\circ y_{N^U})].$$

For each $z \in X$, we may assume that $\exists u, v \in X$ such that $uv = z$, $x_M(u) \neq \mathbf{0}$ and $y_N(v) \neq \mathbf{0}$, without loss of generality. Then

$$
\begin{aligned}
&(A \circ B)^L(z) = \bigvee_{z=uv} [A^L(u) \wedge B^L(v)] \\
&\geq \bigvee_{z=uv} \Big(\bigvee_{x_{M^L} \in A^L, y_{N^L} \in B^L} [x_{M^L}(u) \wedge y_{N^L}(v)] \Big) \\
&= \Big(\bigcup_{x_{M^L} \in A^L, y_{N^L} \in B^L} x_{M^L} \circ y_{N^L} \Big) \\
&= C^L(z).
\end{aligned}
$$

Since $u_{A(u)} \in A$ and $v_{B(v)} \in B$,

$$
\begin{aligned}
C^L(z) \;&=\; \bigvee_{x_{M^L} \in A^L, y_{N^L} \in B^L} \Big(\bigvee_{z=uv} [x_{M^L}(u) \wedge y_{N^L}(v)] \Big) \\
&= \bigvee_{z=uv} \Big(\bigvee_{x_{M^L} \in A^L, y_{N^L} \in B^L} [x_{M^L}(u) \wedge y_{N^L}(v)] \Big) \\
&\geq \bigvee_{z=uv} [u_{A^L(u)}(u) \wedge v_{B^L(v)}(v)] \\
&= \bigvee_{z=uv} [A^L(u) \wedge B^L(v)] \\
&= (A \circ B)^L(z).
\end{aligned}
$$

Thus $(A \circ B)^L = C^L$. By the similar arguments, we have $(A \circ B)^U = C^U$. Hence

$$A \circ B = \bigcup_{x_{M^L} \in A^L, y_{N^L} \in B^L} x_{M^L} \circ y_{N^L}. \qquad \blacksquare$$

The following is the immediate result of Definition 3.1.

Proposition 3.3. Let $(X, \circ)$ be a groupoid, and let "$\circ$" be same as above.

(a) if "$\circ$" is associative[resp. commutative] in X, the so is "$\circ$" in $D(I)^X$.

(b) if "$\circ$" is has an identity $e \in X$, then $e_{\mathbf{1}} \in \mathrm{IVFp}(X)$ is an identity of "$\circ$" in $D(I)^X$, *i.e.*, $A \circ e_{\mathbf{1}} = A = e_{\mathbf{1}} \circ A$ for each $A \in D(I)^X$.

Definition 3.4. Let $(G, \cdot)$ be a groupoid and let $\widetilde{0} = A \in D(I)^X$. Then A is called an *interval-valued fuzzy groupoid* (in short, *IVGP*) in G if

$A \circ A \subset A$, $i.e.$, $A^L \circ A^L \subset A^L$ and $A^U \circ A^U \subset A^U$.

We will denote the IVGPs in G as IVGP(G).

Remark 3.4. (a) If A is a fuzzy groupoid in a group G in the sense of Liu[11], then $A = [A, A] \in$ IVGP(G).

(b)If $A \in$ IVGP(G), then $A^L, A^U \in$ FGP(G) and $A_* \in$ IFGP(G), where FGP(G)[resp. IFGP(G)] denoted the set of all fuzzy groupoids in the sense of Liu[resp. the set of all intuitionistic fuzzy groupiods in the sense of Hur et al.].

The followings are the immediate results of Definitions 3.1 and 3.4.

Proposition 3.5. Let $(G, \cdot)$ be a groupoid and let $\widetilde{0} \neq A \in D(I)^X$. Then the followings are equivalent:

(a) $A \in$ IVGP(G).

(b) For any $x_M, y_N \in A$, $x_M \circ y_N \in A$, $i.e.$, $(A, \circ)$ is a groupoid.

(c) For any $x, y \in G$, $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.

Proposition 3.6. Let $\widetilde{0} \neq A \in D(I)^X$. Then the followings are equivalent:

(a) If "$\circ$" is associative in G, then so is "$\circ$ " in A, $i.e.$, for any $x_L, y_M, z_N \in A$,

$x_L \circ (y_M \circ z_N) = (x_L \circ y_M) \circ z_N$.

(b) If "$\circ$" is commutative in G, then so is "$\circ$" in A, $i.e.$, for any $x_L, y_M \in A$,

$x_L \circ y_M = y_M \circ x_L$.

(c) If "$\circ$" has an identity $e \in G$, then

$e_{\mathbf{1}} \circ x_L = x_L = x_L \circ e_{\mathbf{1}}\ \forall x_L \in A$.

From Proposition 3.5, we can define an IVGP in G as follows.

Definition 3.4′. An interval-valued fuzzy set A in G is called an *interval-valued fuzzy subgroupoid*(in short, *IVGP*) in G if

$A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$, $\forall x, y \in G$.

It is clear that $\widetilde{0}, \widetilde{1} \in$ IVGP(G).

The following is the immediate result of Definition 3.4′.

Proposition 3.7. Let T $\in$ P(G), where P(G) denoted the set of all subsets of G. Then $A = [\chi_T, \chi_T] \in$ IVGP(G) if and only if T is a subgroupoid of G, where χ_{T} is the charecteristic function of T.

Proposition 3.8. If $\{A_\alpha\}_{\alpha\in\Gamma} \subset$ IVGP(G), then $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVGP(G).

Proof. Let $A = \bigcap_{\alpha\in\Gamma} A_\alpha$ and let $x, y \in G$. Then

$$A^L(xy) = \bigwedge_{\alpha\in\Gamma} A^L_\alpha(xy)$$
$$\geq \bigwedge_{\alpha\in\Gamma} [A^L_\alpha(x) \wedge A^L_\alpha(y)] \text{ [Since } A_\alpha \in \text{IVGP}(G)]$$
$$= (\bigwedge_{\alpha\in\Gamma} A^L_\alpha(x)) \wedge (\bigwedge_{\alpha\in\Gamma} A^L_\alpha(y))$$
$$= (\bigcap_{\alpha\in\Gamma} A^L_\alpha)(x) \wedge (\bigcap_{\alpha\in\Gamma} A^L_\alpha)(y)$$
$$= A^L(x) \wedge A^L(y).$$

Similarly, we can see that $A^U(xy) \geq A^U(x) \wedge A^U(y)$. Hence $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVGP(G). ■

Proposition 3.9. Let $f : G \to G'$ be a groupoid homomorphism, let $A \in D(I)^X$ and let $B \in D(I)^Y$.

(a) $f(x_M \circ y_N) = f(x)_M \circ f(y)_N, \forall x_M, y_N \in$ IVFp(G).

(b) If f is surjective and $A \in$ IVGP(G), then $f(A) \in$ IVGP(G').

(c) If $B \in$ IVGP(G'), then $f^{-1}(B) \in$ IVGP(G).

Proof. (a) Let $x_M, y_N \in$ IVP(G) and let $z \in G'$. Then

$$f(x_M \circ y_N)^L(z) = f((xy)_{M^L \wedge N^L})(z) \text{ [By Proposition 3.2]}$$
$$= \bigvee_{z'=f(z)} (xy)_{M^L \wedge N^L}(z')$$
$$= \begin{cases} M^L \wedge N^L & \text{if } \quad z' = f(xy), \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$(f(x)_M \circ f(y)_N)^L(z)$$

$$= \begin{cases} \bigvee_{z=uv} [f(x)_{M^L}(u) \wedge f(y)_{N^L}(v)] & \text{for} \quad (u,v) \in G' \times G' \text{with} \quad z = \mu\nu, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} M^L \wedge N^L & \text{if} \quad z = f(x)f(y), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $f(x_M \circ y_N)^L(z) = (f(x)_M \circ f(y)_N)^L(z)$. Similarly, we can see that $f(x_M \circ y_N)^U(z) = (f(x)_M \circ f(y)_N)^U(z)$, $\forall z \in G'$. So $f(x_M \circ y_N) = f(x_M) \circ f(y_N)$.

(b) Assume that $f(A) \in \text{IVGP}(G')$. Then $\exists\, y, y' \in G'$ such that

$$f(A)^L(yy') < f(A)^L(y) \wedge f(A)^L(y')$$

or

$$f(A)^U(yy') < f(A)^U(y) \wedge f(A)^U(y').$$

Thus

$$\bigvee_{f(z)=yy'} A^L(z) < (\bigvee_{f(x)=y} A^L(x)) \wedge (\bigvee_{f(x')=y'} A^L(x'))$$

or

$$\bigvee_{f(z)=yy'} A^U(z) < (\bigvee_{f(x)=y} A^U(x)) \wedge (\bigvee_{f(x')=y'} A^U(x')).$$

Since f is surjective, $\exists\, x, x' \in G$ such that $f(x) = y,\ f(x') = y'$, and

$$\bigvee_{f(z)=yy'} A^L(z) < A^L(x) \wedge A^L(x')$$

or

$$\bigvee_{f(z)=yy'} A^U(z) < A^U(x) \wedge A^U(x').$$

So

$$A^L(xx') \leq \bigvee_{f(z)=yy'} A^L(z) < A^L(x) \wedge A^L(x')$$

or

$$A^U(xx') \leq \bigvee_{f(z)=yy'} A^U(z) < A^U(x) \wedge A^U(x').$$

This is a contradiction from the fact that $A \in \text{IVGP}(G)$.

(c) It can be easily seen that $f^{-1}(B) \in \text{IVGP}(G)$ ■

Definition 3.10[2]. $A \in D(I)^X$ is said to *have the sup-property* if for each $T \in P(X)$, $\exists t_0 \in T$ such that $A(t_0) = [\bigvee_{t \in T} A^L(t), \bigwedge_{t \in T} A^U(t)]$.

Definition 3.10′[8]. $A \in \text{IFS}(X)$ is said to *have the sup-property* if each $T \in P(X)$, $\exists t_0 \in T$ such that $A(t_0) = (\bigvee_{t\in T} \mu_A(t), \bigwedge_{t\in T} \nu_A(t))$

Remark 3.10. (a) If $A \in I^X$ has the sup-property, $A = [A, A] \in D(I)^X$[resp. $A = (A, A^c) \in IFS(X)$] has the sup-property.

(b) If $A = [A^L, A^U] \in D(I)^X$[resp. $A = (\mu_A, \nu_A) \in \text{IFS}(X)$] has the sup-property, then A^L and $A^U \in I^X$[resp. μ_A and $\nu_A{}^c \in I^X$] have the sup-property.

Proposition 3.11. Let $f : G \to G'$ be a groupoid homomorphism and let $A \in D(I)^X$ have the sup-property. If $A \in \text{IVGP}(G)$, then $f(A) \in \text{IVGP}(G')$.

proof. Let $y, y' \in G'$. Then we can consider four cases:

(i) $f^{-1}(y) \neq \varnothing$ and $f^{-1}(y') \neq \varnothing$,
(ii) $f^{-1}(y) \neq \varnothing$ and $f^{-1}(y') = \varnothing$,
(iii) $f^{-1}(y) = \varnothing$ and $f^{-1}(y') \neq \varnothing$,
(iv) $f^{-1}(y) = \varnothing$ and $f^{-1}(y') = \varnothing$.

We prove only the case (i) and omit the remainders. Since A has the sup-property, $\exists x_0 \in f^{-1}(y)$ and $x_0' \in f^{-1}(y')$ such that

$$A(x_0) = [\bigvee_{t\in f^{-1}(y)} A^L(t), \bigvee_{t\in f^{-1}(y)} A^U(t)]$$

and

$$A(x_0') = [\bigvee_{t'\in f^{-1}(y')} A^L(t'), \bigvee_{t'\in f^{-1}(y')} A^U(t')].$$

Then

$$f(A)^L(yy') = \bigvee_{z\in f^{-1}(yy')} A^L(z) \geq A^L(x_0x_0') \text{ [Since } f(x_0x_0') = f(x_0)f(x_0')$$

$$= yy'\text{]}$$

$$\geq A^L(x_0) \wedge A^L(x_0') \text{ [Since } A \in \text{IVGP}(G).\text{]}$$

$$= (\bigvee_{t\in f^{-1}(y)} A^L(t)) \wedge (\bigvee_{t'\in f^{-1}(y')} A^L(t'))$$

$$= f(A)^L(y) \wedge f(A)^L(y').$$

Similarly, we have $f(A)^U(yy') \geq f(A)^U(y) \wedge f(A)^U(y')$. So $f(A) \in \text{IVGP}(G')$. ■

Definition 3.12. Let $f : X \to Y$ be a mapping and let $A \in D(I)^X$. Then A is said to be *interval-valued fuzzy invariant*(in short, *IVF-invariant*) if $f(x) = f(y)$ implies $A(x) = A(y)$, *i.e.*, $A^L(x) = A^L(y)$

and $A^U(x) = A^U(y)$.

It is clear that if A is IVF-invariant, *i.e.*, $f^{-1}(f(A)) = A$.

The following is the immediate result of Definition 3.12.

Proposition 3.13. Let $f : X \to Y$ be a mapping and let $\mathcal{A} = \{A \in D(I)^X : A$ is IVF-invariant and has the sup-property$\}$. Then there is a one-to-one correspondence between $\mathcal{A}$ and $D(I)^{\mathrm{Im} f}$, where $\mathrm{Im} f$ denotes the image of f

The following is the immediatd result of Propositions 3.11 and 3.13.

Corollary 3.13. Let $f : G \to G'$ be a groupoid homomorphism and let $\mathcal{A} = \{A \in$ IVGP $(G) : A$ is IVF-invariant and has the sup-property$\}$. Then there is a one-to-one correspondence between $\mathcal{A}$ and IVGP $(\mathrm{Im} f)$.

4. Interval-value fuzzy subgroups

Definition 4.1[4]. Let A be an *IVFs* in a group G. Then A is called an *interval-valued fuzzy subgroup* (in short, *IVG*) in G if it satisfies the conditions : For any $x, y \in G$,
(i) $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$
(ii) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$

We will denote the set of all IVGS of G as IVG(G).

Example 4.1. Consider the additive group $(\mathbb{Z}, +)$. We define a mapping $A = [A^L, A^U] : \mathbb{Z} \rightharpoondown D(I)$ as follows : For each $n \in \mathbb{Z}$.
$A(0) = [A^L(0), A^U(0)] = [1, 1]$,
and

$$A(n) = [A^L(n), A^U(n)] = \begin{cases} [\frac{1}{2}, \frac{2}{3}], & \text{if n is odd,} \\ [\frac{1}{3}, \frac{4}{5}], & \text{if n is even.} \end{cases}$$

Then clearly $A \in D(I)^{\mathbb{Z}}$. Moreover, A satisfies all the conditions of Definition 4.1. So $A \in \mathrm{IVG}(\mathbb{Z})$. ■

Remark 4.1. (a) If $A \in$ FG(G), then $A = [A, A] \in$ IVG(G), where FG(G) denotes the set of all fuzzy groups in G.

(b) If $A \in$ IVG(G), then $A^L, A^U \in$ FG(G) and $(A^L, A^{U^C}) \in$ IFG(G).

(c) If $A \in$ IFG(G), then $[\mu_A, \nu_A^c] \in$ IVG(G).

The following two results can be easily proved from definition 4.1, Propositions 3.7 and 3.8.

Proposition 4.2. Let G be a group and let $H \subset G$. Then H is a subgroup of G if and only if $[\chi_H, \chi_H] \in$ IVG(G).

Proposition 4.3. Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset$ IVG(G). Then $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVG(G).

The followings can be easily seen from Definitions 3.1 and 4.1.

Proposition 4.4. Let G be group and let $A \in D(I)^G$. If $A \in$ IVG(G), then $A \circ A = A$.

Proposition 4.5. Let $A, B \in$ IVG(G). Then $A \circ B \in$ IVG(G) if and only if $A \circ B = B \circ A$.

Result 4.A [4, Proposition 3.1]. Let A be an IVG in a group G.

(a) $A(x^{-1}) = A(x), \forall x \in G$.

(b) $A^L(e) \geq A^L(x)$ and $A^U(e) \geq A^U(x), \forall x \in G$, where e is the identity of G.

Result 4.B [4, Proposition 3.2]. Let A be an IVFS in a group G. Then A is an IVG in G if and only if $A^L(xy^{-1}) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy^{-1}) \geq A^U(x) \wedge A^U(y), \forall x, y \in G$.

Proposition 4.6. If $A \in$ IVG(G), then $G_A = \{x \in G : \text{A}(x) = \text{A(e)}\}$ is a subgroup of G.

Proof. let $x, y \in G_A$. Then
$A^L(xy^{-1}) \geq A^L(x) \wedge A^L(y^{-1})$
$= A^L(x) \wedge A^L(y)$ [By Result 4.A]
$= A^L(e) \wedge A^L(e)$ [Since $x, y \in G_A$]
$= A^L(e)$.
Similarly, we have $A^U(xy^{-1}) \geq A^U(e)$, On the other hand, by Result 4.A, it is clear that $A^L(xy^{-1}) \leq A^L(e)$ and $A^U(xy^{-1}) \leq A^U(e)$, thus

$A(xy^{-1}) = A(e)$. So $xy^{-1} \in G_A$. Hence G_A is a subgroup of G. ∎

Proposition 4.7. let $A \in$ IVG(G). If $A(xy^{-1}) = A(e)$ for any $x, y \in G$, then $A(x) = A(y)$.

Proof. Let $x, y \in G$. Then
$A^L(x) = A^L((xy^{-1})y)$
$\geq A^L(xy^{-1}) \wedge A^L(y)$ [Since $A \in$ IVG(G)]
$= A^L(e) \wedge A^L(y)$ [By the hypothesis]
$= A^L(y)$. [By Result 4.A.]
On the other hard, by Result 4.A, $A^L(x^{-1}) = A^L(x)$. Then
$A^L(y) = A^L((yx^{-1})x)$
$\geq A^L(yx^{-1}) \wedge A^L(x)$
$=A^L((yx^{-1})^{-1}) \wedge A^L(x)$ [By Result 4.A.]
$=A^L(xy^{-1}) \wedge A^L(x)$
$=A^L(e) \wedge A^L(x)$
$=A^L(x)$.
Similarly, we have $A^U(x) = A^U(y)$. Hence $A(x) = A(y)$. ∎

Corollary 4.7-1. Let $A \in$ IVG(G). If G_A is a normal subgroup of G, then A is constant on each coset of G_A.

Proof. Let $a \in G$ and let $x \in aG_A$. Then $\exists y \in G_A$ such that $x = ay$. Since G_A is normal, $xa^{-1} \in G_A$. Thus, by the definition of G_A, $A(xa^{-1}) = A(e)$. By proposition 4.7, $A(x) = A(a)$. So A is constant on aG_A $\forall a \in G$. Similarly, we can see that A is constant on G_Aa $\forall a \in G$. This completes the proof. ∎

Let H be a subgroup of G. Then the number of right [resp. left] cosets of H in G is called the *index of H in G* and denoted by $[G : H]$. If G is a finite group, then there can be only a finite number of distinct right [resp. left] cosets of H; hence the index $[G : H]$ is finite. If G is an infinite group, then $[G : H]$ may be either finite or infinite.

Corollary 4.7-2. Let $A \in$ IVG(G) and let G_A be normal. If G_A has a finite index, then A has the sup property.

Proof. Let $T \subset G$. Since G_A has finite index, let the index $[G : G_A] = n$, say $\mathcal{A}= \{a_1G_A, \cdots, a_nG_A\}$, where $a_i \in G(i = 1, \cdots, n)$ and $a_iG_A \cap a_jG_A = \tilde{0}$ for any $i \neq j$. Let $t \in T$. Since $G = \bigcup \mathcal{A} = \bigcup_{i=1}^{n} a_iG_i$, there exists an $i \in \{1, \cdots, n\}$ such that $t \in a_i G_A$. Since G_A is normal, by Corollary 4.7-1, $A(t) = A(a_i)$ on $a_i G_A$, say $A^L(t) = \alpha_i$ and $A^U(t) = \beta_i$, where $\alpha_i, \beta_i \in I$ and $\alpha_i \leq \beta_i$. Thus there exists a $t_0 \in T$ such that $A^L(t_0) = \bigvee_{t \in T} A^L(t) = \bigvee_{i=1}^n \alpha_i$ and $A^U(t_0) = \bigvee_{t \in T} A^U(t) = \bigvee_{i=1}^n \beta_i$. Hence A has the sup property. ■

Proposition 4.8. *A group G cannot be the union of two proper IVGs.*

Proof. Let A and B be proper IVGs of a group G such that $A \cup B = \tilde{1}$, $A \neq \tilde{1}$ and $B \neq \tilde{1}$. Since $A \cup B = (A^L \vee B^L, A^U \vee B^U)$, $A^L(x) \vee B^L(x) = 1$ and $A^U(x) \vee B^U(x) = 1$, $\forall x \in X$. Then $A^L(x) = 1$ or $B^L(x) = 1$ and $A^U(x) = 1$ or $B^U(x) = 1$. Since $A \neq \tilde{1}$ and $B \neq \tilde{1}$, $A^L(x) \neq 1$ or $A^U(x) \neq 1$ and $B^L(x) \neq 1$ or $B^U(x) \neq 1$. In either cases, this is a contradiction. This completes the proof. ■

Proposition 4.9. *If A is an IVGP of a finite group G, then $A \in$ IVG(G).*

Proof. Let $x \in G$. Since G is finite, x has the finite order, say n, Then $x^n = e$, where e is the identity of G. Thus $x^{-1} = x^{n-1}$. Since A is an IVGP of G,

$$A^L(x^{-1}) = A^L(x^{n-1}) = A^L(x^{n-2}x) \geq A^L(x)$$

and

$$A^U(x^{-1}) = A^U(x^{n-1}) = A^U(x^{n-2}x) \geq A^U(x).$$

Hence $A \in$ IVG(G). ■

Proposition 4.10. *Let A be an IVG of a group G and let $x \in G$. Then $A(xy) = A(y)$, for each $y \in G$ if and only if $A(x) = A(e)$.*

Proof. ($\Rightarrow$):Suppose $A(xy) = A(y)$ for each $y \in G$. Then clearly $A(x) = A(e)$.

($\Leftarrow$):Suppose $A(x) = A(e)$. Then, by Result 4.A, $A^L(y) \leq A^L(x)$ and $A^U(y) \leq A^U(x)$ for each $y \in G$. Since A is an IVG of G, Then $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \vee A^U(y)$. Thus $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for each $y \in G$.

On the other hand, by Result 4.A,

$$A^L(y) = A^L(x^{-1}xy) \geq A^L(x) \wedge A^L(xy)$$

and

$$A^U(y) = A^U(x^{-1}xy) \geq A^U(x) \wedge A^U(xy).$$

Since $A^L(x) \geq A^L(y)$ for each $y \in G$, $A^L(x) \wedge A^L(xy) = A^L(xy)$ and $A^U(x) \wedge A^L(xy) = A^U(xy)$. So $A^L(y) \geq A^L(xy)$ and $A^U(y) \geq A^U(xy)$

for each $y \in G$. Hence $A(xy) = A(y)$ for each $y \in G$. ■

Proposition 4.11. Let $f : G \to G'$ be a group homomorphism, let $A \in \text{IVG}(G)$ and let $B \in \text{IVG}(G')$. Then the following hold:
(a) If A has the sup property, then $f(A) \in \text{IVG}(G')$.
(b) $f^{-1}(B) \in \text{IVG}(G)$.

Proof. (a) By Proposition 3.11, since $f(A) \in \text{IVGP}(G)$, it is enough to show that $f(A)^L(y^{-1}) \geq f(A)^L(y)$ and $f(A)^U(y^{-1}) \geq f(A)^U(y)$ for each $y \in f(G)$.

Let $y \in f(G)$. Then $\phi \neq f^{-1}(y) \subset G$. Since A has the sup property, there exists an $x_0 \in f^{-1}(y)$ such that $A^L(x_0) = \bigvee_{t \in f^{-1}(y)} A^L(t)$ and $A^U(x_0) = \bigvee_{t \in f^{-1}(y)} A^U(t)$.
Thus

$$f(A)^L(y^{-1}) = \bigvee_{t \in f^{-1}(y^{-1})} A^L(t) \geq A^L(x_0^{-1}) \geq A^L(x_0) = f(A)^L(y)$$

and

$$f(A)^U(y^{-1}) = \bigvee_{t \in f^{-1}(y^{-1})} A^U(t) \geq A^U(x_0^{-1}) \geq A^U(x_0) = f(A)^U(y).$$

Hence $f(A) \in \text{IVG}(G)$.

(b) By proposition 3.9, since $f^{-1}(B) \in \text{IVGP}(G)$, it is enough to show that $f^{-1}(B)^L(x^{-1}) \geq f^{-1}(B)^L(x)$ and $f^{-1}(B)^U(x^{-1}) \geq f^{-1}(B)^U(x)$ for each $x \in G$.

Let $x \in G$. Then
$f^{-1}(B)^L(x^{-1}) = B^L(f(x^{-1})) = B^L(f(x)^{-1})$
$\geq B^L(f(x)) = f^{-1}(B)^L(x)$
and
$f^{-1}(B)^U(x^{-1}) = B^U(f(x^{-1})) = B^U(f(x)^{-1})$
$\geq B^U(f(x)) = f^{-1}(B)^U(x)$.
Thus $f^{-1}(B) \in \text{IVG}(G)$. This completes the proof. ■

Proposition 4.12. Let G_p be the cyclic group of prime order p. Then $A \in \text{IVG}(G_p)$ if and only if $A^L(x) = A^L(1) \leq A^L(0)$ and $A^U(x) = A^U(1) \leq A^U(0)$ for each $0 \neq x \in G_p$.

Proof. $(\Rightarrow)$: Suppose $A \in \text{IVG}(G_p)$ and let $0 \neq x \in G_p$. Then $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$ for any $x, y \in G_p$. Since G_p is the cyclic group of prime order $p, G_p = \{0, 1, 2, \cdots, p-1\}$. Since x is the sum of 1's and 1 is the sum of $x's$, $A^L(x) \geq A^L(1) \geq A^L(x)$ and $A^U(x) \geq A^U(1) \geq A^U(x)$. Thus $A^L(x) = A^L(1)$ and $A^U(x) = A^U(1)$. Since 0 is the identity element of G_p, $A^L(x) \leq A^L(0)$ and $A^U(x) \leq A^U(0)$. Hence the necessary conditions hold.

$(\Leftarrow)$: Suppose the necessary conditions hold and let $x, y \in G_p$. Then we have four cases : (i)$x \neq 0, y \neq 0$ and $x = y$, (ii) $x \neq 0, y = 0$, (iii) $x = 0, y \neq 0$, (iv) $x \neq 0, y \neq 0$ and $x \neq y$.

Case(i) Suppose $x \neq 0, y \neq 0$ and $x = y$. Then, by the hypothesis, $A^L(x) = A^L(y) = A^L(1) \leq A^L(0)$ and $A^U(x) = A^U(y) = A^U(1) \leq A^U(0)$. So $A^L(x-y) = A^L(0) \geq A^L(x) \wedge A^L(y)$ and $A^L(x-y) \geq A^U(x) \wedge A^U(y)$.

Case(ii) Suppose $x \neq 0$ and $y = 0$. Since $x - y \neq 0$, by the hypothesis, $A^L(x-y) = A^L(x) = A^L(1) \leq A^L(0) = A^L(y)$ and $A^U(x-y) = A^U(x) = A^U(1) \leq A^U(0) = A^U(y)$. So $A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y)$.

Case(iii) is the same as Case(ii).

Case(iv) Suppose $x \neq 0, y \neq 0$ and $x \neq y$. Since $x - y \neq 0$, by the hypothesis, $A^L(x-y) = A^L(x) = A^L(y) = A^L(1) \leq A^L(0)$ and $A^U(x-y) = A^U(x) = A^U(y) \leq A^U(0)$. So $A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y)$. In all, $A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \wedge A^U(y)$. Hence, by Result 4.B, $A \in \text{IFG}(G_p)$. ■

Definition 4.13. Let G be a groupoid and let $A \in \text{IVS}(G)$. Then A is called an:

(1) *interval-valued fuzzy left ideal* (in short, $IVLI$) of G if for any $x, y \in G, A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$.

(2) *interval-valued fuzzy right ideal* (in short, $IVRI$) of G if for any $x, y \in G, A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$.

(3) *interval-valued fuzzy ideal* (in short, IVI) of G if it is both an IFLI and an IFRI.

We will denote the set of all IVLIs[resp. IVRIs and IVIs] of a groupoid G as IVLI(G)[resp. IVRI(G) and IVI(G)].

It is clear that $A \in$ IVI(G) if and only if and only if for any $x, y \in G, A^L(xy) \geq A^L(x) \vee A^L(y)$ and $A^U(xy) \geq A^U(x) \vee A^U(y)$. Moreover, an IFI(resp. IFLI, IFRI) is an IVGP of G. Note that for any $A \in$ IVGP(G), we have $A^L(x^n) \geq A^L(x)$ and $A^U(x^n) \geq A^U(x)$ for each $x \in G$, where x^n is any composite of $x's$.

Proposition 4.14. The IVLIs (resp. IVLIs, IVRIs) in a group G are just the constant mappings.

Proof. Suppose A is an constant mapping and let $x, y \in G$. Then $A(xy) = A(x) = A(y)$. Thus $A \in$ IVI(G).

Now suppose $A \in$ IVLI(G). Then $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for any $x, y \in G$. In particular, $A^L(x) \geq A^L(e)$ and $A^U(x) \geq A^U(e)$ for each $x \in G$. Moreover, $A^L(e) = A^L(x^{-1}x) \geq A^L(x)$ and $A^U(e) = A^U(x^{-1}x) \geq A^U(x)$ for each $x \in G$. So $A(x) = A(e)$ for each $x \in G$. Hence A is a constant mapping. ■

Definition 4.15. Let A be an IVFS in a set X and let $\lambda, \mu \in I$ with $\lambda \leq \mu$. Then the set $A^{[\lambda,\mu]} = \{x \in X : A^L(x) \geq \lambda$ and $A^U(x) \geq \mu\}$ is called a $[\lambda, \mu]$-*level subset* of A.

Proposition 4.16. Let A be an IVG of a group G. Then, for each $(\lambda, \mu) \in I \times I$ such that $\lambda \leq \mu_A(e), \mu \leq \nu_A(e)$ and $\lambda \leq \mu$, $A^{[\lambda,\mu]}$ is a subgroup of G.

Proof. Clearly, $A^{[\lambda,\mu]} \neq \emptyset$. Let $x, y \in A^{[\lambda,\mu]}$. Then $A^L(x) \geq \lambda, A^U(y) \geq \mu$ and $A^L(y) \geq \lambda, A^U(y) \geq \mu$. Since $A \in$ IVG(G), $A^L(xy) \geq A^L(x) \wedge A^L(y) \geq \lambda$ and $A^U(xy) \geq A^U(x) \wedge A^U(y) \geq \mu$. Thus $A^L(xy) \geq \lambda$ and $A^U(xy) \geq \mu$. So $xy \in A^{[\lambda,\mu]}$. On the other hand, $A^L(x^{-1}) \geq A^L(x) \geq \lambda$ and $A^U(x^{-1}) \geq A^U(x) \geq \mu$. Thus $A^L(x^{-1})\lambda$ and $A^U(x^{-1}) \geq \mu$. So $x^{-1} \in A^{[\lambda,\mu]}$. Hence $A^{[\lambda,\mu]}$ is a subgroup of G. ■

Proposition 4.16. Let A be an IVS in a group G such that $A^{[\lambda,\mu]}$ is a subgroup of G for each $(\lambda, \mu) \in I \times I$ such that $\lambda \leq A^L(e), \mu \leq A^U(e)$ and $\lambda \leq \mu$. Then A is an IVG of G.

Proof. For any $x, y \in G$, let $A(x) = [t_1, s_1]$ and let $A(y) = [t_2, s_2]$. Then clearly, $x \in A^{[t_1,s_1]}$ and $y \in A^{[t_2,s_2]}$. Suppose $t_1 < t_2$ and $s_1 < s_2$. Then $A^{[t_2,s_2]} \subset A^{[t_1,s_1]}$. Thus $y \in A^{[t_1,s_1]}$. Since $A^{[t_1,s_1]}$ is a subgroup of $G, xy \in A^{[t_1,s_1]}$. Then $A^L(xy) \geq t_1$ and $A^U(xy) \geq s_1$. So $A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$. For each $x \in G$, let $A(xy) = [\lambda, \mu]$. Then $x \in A^{[\lambda,\mu]}$. Since $A^{[\lambda,\mu]}$ is a subgroup of $G, x^{-1} \in A^{[\lambda,\mu]}$. So $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U$. Hence $A \in$

IVG(G). ■

5. Interval-value fuzzy normal subgroups

Definition 5.1. Let $A \in$ IVG(G). Then A is called an *interval-valued fuzzy normal subgroup*(in short, $IVNG$) of G if $A(xy) = A(yx)$, for any $x, y \in G$.

We will denote the set of all IVNGs of a group G as IVNG(G). It is clear that if G is abelian, then $A \in$ IVNG(G), $\forall A \in$ IVG(G).

Example 5.1. Consider the general linear group of degree $n, GL(n, R)$. Then clearly, $GL(n, R)$ is a non abelian group. Let us define a mapping $A : GL(n, R) \to D(I)$ as follows: for any $I_n \neq M \in GL(n, R)$, where I_n is the unit matrix,

$A(I_n) = \tilde{1}$,

$$A^L(M) = \begin{cases} \frac{1}{5} & \text{if M is not a triangular matrix,} \\ \frac{1}{3} & \text{if M is a triangular matrix} \end{cases}$$

and

$$A^U(M) = \begin{cases} \frac{2}{3} & \text{if M is not a triangular matrix,} \\ \frac{1}{2} & \text{if M is a triangular matrix} \end{cases}$$

Then we can easily see that A is an IVNG of $GL(n, R)$. ■

The following is the immediate result of Definitions 3.1 and 5.1.

Proposition 5.2. Let $A \in D(I)^G$ and let $B \in$ IVNG(G). Then $A \circ B = B \circ A$.

Proposition 5.3. Let $A \in$ IVNG(G). If $B \in$ IVG(G), then so is $B \circ A$.

Proof. By Definitions 3.1 and 3.4, it can be easily seen that $B \circ A \in$ IVGP(G). Thus it is sufficient to show that $(B \circ A)^L(x^{-1}) \geq (B \circ A)^L(x)$ and $(B \circ A)^U(x^{-1}) \geq (B \circ A)^U(x)$ for each $x \in G$.

Let $x \in G$. Then

$$(B \circ A)^L(x^{-1}) = \bigvee_{yz=x^{-1}} [B^L(y) \wedge A^L(z)]$$
$$= \bigvee_{z^{-1}y^{-1}=x} [B^L((y^{-1})^{-1}) \wedge A^L((z^{-1})^{-1})]$$
$$\geq \bigvee_{z^{-1}y^{-1}=x} [B^L(y^{-1}) \wedge A^L(z^{-1})]$$
$$= (A \circ B)^L(x) = (B \circ A)^L(x).$$

Similarly, we have $(B \circ A)^U(x^{-1}) \geq (B \circ A)^U(x)$ for each $x \in G$. Hence $B \circ A \in \mathrm{IVG}(G)$. ■

Corollary 5.3. Let $A, B \in \mathrm{IVNG}(G)$. Then $A \circ B \in \mathrm{IVNG}(G)$.

Proof. By Proposition 4.5, $A \circ B \in \mathrm{IVG}(G)$. Let $a, b \in G$. Then there exists $x, y \in G$ such that $ab = xy$. Since $b = a^{-1}xy, ba = (a^{-1}xa)(a^{-1}ya)$. Since $A, B \in \mathrm{IVNG}(G)$,

$$(A \circ B)(ab) = [(A \circ B)^L(ab), (A \circ B)^U(ab)]$$
$$= [\bigvee_{ab=xy} (A^L(x) \wedge B^L(y)), \bigvee_{ab=xy} (A^U(x) \wedge B^U(y))]$$
$$= [\bigvee_{ba=(a^{-1}xa)(a^{-1}ya)} (A^L(a^{-1}xa) \wedge B^L(a^{-1}ya)),$$
$$\bigvee_{ba=(a^{-1}xa)(a^{-1}ya)} (A^U(a^{-1}xa) \wedge B^U(a^{-1}ya)]$$
$$= [(A \circ B)^L(ba), (A \circ B)^U(ba)]$$
$$= (A \circ B)(ba).$$

Hence $A \circ B \in \mathrm{IFNG}(G)$. ■

Proposition 5.4. If $A \in \mathrm{IVNG}(G)$, then G_A is a normal subgroup of G.

Proof. By Proposition 4.6, G_A is a subgroup of G. Moreover $G_A \neq \emptyset$. Let $x \in G_A$ and let $y \in G$. Then

$$A^L(yxy^{-1}) = A^L((yx)x^{-1}) = A^L(y^{-1}(yx)) = A^L(x) = A^L(e)$$

and

$$A^U(yxy^{-1}) = A^U((yx)x^{-1}) = A^U(y^{-1}(yx)) = A^U(x) = A^U(e)$$

Thus $yxy^{-1} \in G_A$. Hence G_A is a normal subgroup of G. ■

It is clear that if A is a (usual) normal subgroup of G, then $A = [\chi_A, \chi_A] \in \mathrm{IVNG}(G)$ and $G_A = A$.

Definition 5.5. Let $A \in \text{IVNG}(G)$. Then the quotient group G/G_A is called the *interval-valued fuzzy quotient subgroup* (in short, $IVQG$) of X with respect to A.

Now let $\pi : G \to G/G_A$ be the natural projection.

Proposition 5.6. If $A \in \text{IVNG}(G)$ and $B \in D(I)^G$, then $\pi^{-1}(\pi(B)) = G_A \circ B$.

Proof. Let $x \in G$. then

$$\pi^{-1}(\pi(B))^L = \pi(b)^L(\pi(x)) = \bigvee_{\pi(y)=\pi(x)} B^L(y) = \bigvee_{xy^{-1}\in G_A} B^L(y)$$

and

$$\pi^{-1}(\pi(B))^U = \pi(b)^U(\pi(x)) = \bigvee_{\pi(y)=\pi(x)} B^U(y) = \bigvee_{xy^{-1}\in G_A} B^U(y).$$

On the other hand

$$(G_A \circ B)^L(x) = \bigvee_{xy=x} [G_A(z) \wedge B^L(y)] = \bigvee_{z=xy^{-1}\in G_A} B^L(y)$$

and

$$(G_A \circ B)^U(x) = \bigvee_{xy=x} [G_A(z) \wedge B^U(y)] = \bigvee_{z=xy^{-1}\in G_A} B^U(y).$$

Thus $\pi^{-1}(\pi(b))(x) = (G_A \circ B)(x)$ for each $x \in G$. Hence $\pi^{-1}(\pi(B)) = G_A \circ B$. ■

6. Interval-valued fuzzy subrings and ideals

Definition 6.1. Let $(R, +, \cdot)$ be a ring and let $\tilde{0} \neq A \in D(I)^R$. Then A is called an *interval-valued fuzzy subring* (in short, IVR) in R if it satisfies the following conitions:

(i) A is an IVG in R with respect to the operation "+"(in the sense of Definition 4.1).

(ii) A is an IVGP in R with respect to the operation "$\cdot$"(in the sense of Definition 3.4 or Definition 3.4$'$).

We will denote the set of all IVRs of R as IVR(R).

Example 6.1. Consider the ring $(\mathbb{Z}_2, +, \cdot)$, where $\mathbb{Z}_2 = \{0, 2\}$. We define the mapping $A : \mathbb{Z}_2 \to D(I)$ as follows: $A(0) = [0.2, 0.7]$ and $A(1) = [0.5, 0.6]$. Then we can see that $A \in \mathrm{IVR}(\mathbb{Z}_2)$. ■

Remark 6.1. (1)If A is a fuzzy subring of a ring R, then $[A, A] \in$IVR(R)
(2)If $A \in \mathrm{IVR}(R)$, then A^L and A^U are fuzzy subrings of R.

The following is the immediate result of Definition 3.4′ and Result 4.B.

Proposition 6.2. Let R be a ring and let $\tilde{0} \neq A \in D(I)^R$. Then $A \in$ IVR(R) if and only of for any $x, y \in R$,
(i)$A^L(x - y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x - y) \geq A^U(x) \wedge A^U(y)$.
(ii)$A^L(xy) \geq A^L(x) \wedge A^L(y)$ and $A^U(xy) \geq A^U(x) \wedge A^U(y)$.

The following is easily seen.

Proposition 6.3. Let R be a ring. Then A is a subring of R if and only if $[\chi_A, \chi_A] \in \mathrm{IVR}(R)$.

Definition 6.4. Let R be a ring and let $\tilde{0} \neq A \in \mathrm{IVR}(R)$. Then A is called an:
(1) *interval-valued fuzzy left ideal* (in short, *IVLI*) in R if $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for any $x, y \in R$.
(2) *interval-valued fuzzy right ideal* (in short, *IVRI*) in X if $A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$ for any $x, y \in R$.
(3) *interval-valued fuzzy ideal* (in short, *IFI*) in X if it both an IVLI and an IVRI in R.

We will denote the set of all IVLIs [resp. IVRIs and IVIs] of a ring R as IVLI(R)[resp. IVRI(R) and IVI(R)].

Example 6.4. Consider the ring $(\mathbb{Z}_4, +, \cdot)$, where $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. We define the mapping $A : \mathbb{Z}_4 \to D(I)$ as follows: $A(0) = [0.2, 0.8]$, $A(1) = [0.3, 0.6] = A(3)$, and $A(2) = [0.4, 0.5]$. Then we can easily see that $A \in \mathrm{IVI}(\mathbb{Z}_4)$. ■

Remark 6.4. (1) If A is a fuzzy [resp. left, right] ideal of a ring R, then $[A, A^c] \in \mathrm{IVI}(R)$ [resp. IVLI(R) and IVRI(R)].

(2) If $A \in \text{IVI}(R)$ [resp. $\text{IVLI}(R)$ and $\text{IVRI}(R)$], then A^L and A^U are fuzzy [resp. left and right] ideals of R.

The following can be directly verified.

Proposition 6.5. Let R be a ring and let $\tilde{0} \neq A \in D(I)^R$. Then A is an IVI[resp. IFLI and IFRI] of R if and only of for any $x, y \in R$,

(i) $A^L(x-y) \geq A^L(x) \wedge A^L(y)$ and $A^U(x-y) \geq A^U(x) \vee A^U(y)$.

(ii) $A^L(xy) \geq A^L(x) \vee A^L(y)$ and $A^U(xy) \geq A^U(x) \vee A^U(y)$[resp. $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y), A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$].

The following is easily seen.

Proposition 6.6. Let R be a ring. Then A is an ideal [resp. a left ideal and a right ideal] of R if and only if $[\chi_A, \chi_A] \in \text{IVI}(R)$ [resp. $\text{IVLI}(R)$ and $\text{IVRI}(R)$].

Proposition 6.7. Let R be a skew field (also division ring) and let $\tilde{0} \neq A \in D(I)^R$. Then A is an IFI(IFLI, IFRI) of R if and only if $A^L(x) = A^L(e) \leq A^L(0)$ and $A^U(x) = A^U(e) \geq A^U(0)$ for any $0 \neq x \in R$, where 0 is the identity of R for "+" and e is the identity of R for "$\cdot$".

Proof. ($\Rightarrow$): Suppose $A \in \text{IVLI}(R)$ and let $0 \neq x \in R$. Then

$$A^L(x) = A^L(xe) \geq A^L(e), A^L(e) = A^L(x^{-1}x) \geq A^L(x)$$

and

$$A^U(x) = A^U(xe) \geq A^U(e), A^U(e) = A^U(x^{-1}x) \geq A^U(x).$$

Thus $A(x) = A(e)$. On the other hand,

$$A^L(0) = A^L(e-e) \geq A^L(e) \wedge A^L(e) = A^L(e)$$

and

$$A^U(0) = A^U(e-e) \geq A^U(e) \wedge A^U(e) = A^U(e).$$

So $A^L(e) \leq A^L(0)$ and $A^U(e) \leq A^U(0)$. Hence the necessary conditions hold.

($\Leftarrow$): Suppose the necessary conditions hold. Let $x \in R$. Then we have four cases:

(i) $x \neq 0, y \neq 0$ and $x \neq y$ (ii) $x \neq 0, y \neq 0$ and $x = y$

(iii) $x \neq 0, y = 0$ (iv) $x = 0, y \neq 0$.

Case (i) Suppose $x \neq 0, y \neq 0$ and $x \neq y$. Then
$A^L(x-y) = A^L(e) \geq A^L(x) \wedge A^L(y)$,
$A^U(x-y) = A^U(e) \geq A^U(x) \wedge A^U(y)$
and
$A^L(xy) = A^L(e) \geq A^L(x) \vee A^L(y)$,
$A^U(xy) = A^U(e) \geq A^U(x) \vee A^U(y)$.

Case(ii): Suppose $x \neq 0, y \neq 0$ and $x = y$. Then
$A^L(x-y) = A^L(0) \geq A^L(x) \wedge A^L(y)$,
$A^U(x-y) = A^U(0) \geq A^U(x) \wedge A^U(y)$
and
$A^L(xy) = A^L(e) \geq A^L(x) \vee A^L(y)$,
$A^U(xy) = A^U(e) \geq A^U(x) \vee A^U(y)$.

Case(iii): Suppose $x \neq 0$ and $y = 0$. Then
$A^L(x-y) = A^L(x) = A^L(e) \geq A^L(x) \wedge A^L(y)$,
$A^U(x-y) = A^U(x) = A^U(0) \geq A^U(x) \wedge A^U(y)$
and
$A^L(xy) = A^L(0) \geq A^L(x) \vee A^L(y)$,
$A^U(xy) = A^U(0) \geq A^U(x) \vee A^U(y)$.

Case(iv): It is similar to case(iii).

In all, $A \in \text{IVI}(R)$. This completes the proof. ■

Remark 6.8. Proposition 6.5 shows that an IVLI(IVRI) is an IVI in a skew field.

The following gives a characteristic of a (usual) field by an IVI.

Proposition 6.9. Let R be a commutative ring with a unity e. If for $A \in \text{IVI}(R)$, $A^L(x) = A^L(e) \leq A^L(0)$ and $A^U(x) = A^U(e) \leq A^U(0)$ for each $0 \neq x \in R$, then R is a field.

Proof. Let A be an ideal of R such that $A \neq R$. Then clearly $A = [\chi_A, \chi_A] \in \text{IVI}(R)$ such that $A \neq \tilde{1}$. Thus there exists $y \in R$ such that $y \notin A$. Thus $\chi_A(y) = 0$. By the hypothesis, $\chi_A(x) = \chi_A(e) \leq \chi_A(0)$, for each $0 \neq x \in X$. So $\chi_A(0) = 1$, i.e.,$A = \{0\}$. Hence R is a field. ■

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