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SOME DECOMPOSITION FORMULAS ASSOCIATED WITH THE SARAN FUNCTION F_E

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Abstract. With the help of some techniques based upon certain inverse pairs of symbolic operators initiated by Burchnall-Chaundy, the authors investigate decomposition formulas associated with Saran's function F_E in three variables. Many operator identities involving these pairs of symbolic operators are first constructed for this purpose. By employing their decomposition formulas, we also present a new group of integral representations for the Saran function F_E.

1. Introduction

A great interest in the theory of hypergeometric functions of several variables is motivated essentially by the fact that the solutions of many applied problems involving partial differential equations are obtainable with the help of such hypergeometric functions. In investigation of the boundary-value problems for these partial differential equations, we need decompositions for the hypergeometric functions of several variables in terms of simpler hypergeometric functions of the Gauss and Appell types. The familiar operator method of Burchnall and Chaundy [2, 3] has been used by them rather extensively for finding decomposition formulas for hypergeometric functions of two variables in terms of the classical Gauss hypergeometric function of one variable. In our present investigation, we construct decompositions for Saran's triple hypergeometric function F_E with the help of the Burchnall-Chaundy method. By means of the decompositions obtained by us, we also deduce some integral representations of Euler type for triple hypergeometric function F_E. Lauricella [12, 19] further generalized the four Appell fuctions F_1,

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F_2, F_3, F_4 [20, p. 53, Eq. (4) - (7)] to functions of three variables and defined his functions as follows :

$$F_A\left(a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z\right) \tag{1.1}$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b_1)_m\,(b_2)_n\,(b_3)_p}{(c_1)_m\,(c_2)_n\,(c_3)_p\,m!n!p!} x^m y^n z^p,$$

$$|x| + |y| + |z| < 1;$$

$$F_B\left(a_1, a_2, a_3, b_1, b_2, b_3; c; x, y, z\right) \tag{1.2}$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m\,(a_2)_n\,(a_3)_p\,(b_1)_m\,(b_2)_n\,(b_3)_p}{(c)_{m+n+p}\,m!n!p!} x^m y^n z^p, < 1;$$

$$max\left\{|x|, |y|, |z|\right\}$$

$$F_C\left(a, b; c_1, c_2, c_3; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{((a)_{m+n+p}\,(b)_{m+n+p}}{(c_1)_m\,(c_2)_n\,(c_3)_p\,m!n!p!} x^m y^n z^p,$$

$$\sqrt{|x|} + \sqrt{|y|} + \sqrt{|z|} < 1; \tag{1.3}$$

$$F_D\left(a, b_1, b_2, b_3; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}\,(b_1)_m\,(b_2)_n\,(b_3)_p}{(c)_{m+n+p}\,m!n!p!} x^m y^n z^p,$$

$$max\left\{|x|, |y|, |z|\right\} < 1. \tag{1.4}$$

Saran [16, 19, p. 68, Eq. (34)] initiated a systematic study of these ten triple hypergeometric functions of Lauricella's set and defined the function F_E by the following triple series :

$$F_E\left(a_1, a_2, b_1, b_2, b_3; c; x, y, z\right) \tag{1.5}$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m\,(a_2)_{n+p}\,(b_1)_m\,(b_2)_n\,(b_3)_p}{(c)_{m+n+p}\,m!n!p!} x^m y^n z^p,$$

$$|x| < r,\ |y| < s, |z| < t, r + s = rs, s = t.$$

2. The main pairs of symbolic operators

Burchnall and Chaundy [2, 3] and Chaundy [4] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators :

$$\nabla_{x,y}(h) := \frac{\Gamma(h)\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\Gamma(\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k (-\delta_2)_k}{(h)_k\, k!}, \tag{2.1}$$

$$\begin{aligned} \Delta_{x,y}(h) &:= \frac{\Gamma(\delta_1+h)\Gamma(\delta_2+h)}{\Gamma(h)\Gamma(\delta_1+\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k (-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k\, k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_{2k} (-\delta_1)_k (-\delta_2)_k}{(h+k-1)_k (h+\delta_1)_k (h+\delta_2)_k\, k!}, \\ &\left(\delta_1 := x\frac{\partial}{\partial x};\ \delta_2 := y\frac{\partial}{\partial y}\right). \end{aligned} \tag{2.2}$$

Indeed, as already observed by Srivastava and Karsson [19, p. 332-333], the aforementioned method of Burchnall and Chaundy [2, 3] was subsequently applied by Pandey [14] and Srivastava [18] in order to derive the corresponding expansion and decomposition formulas for the triple hypergeometric functions ${F_A}^{(3)}$, F_E, F_K, F_M, F_P and F_T, H_A, H_C, respectively (see, for definitions, [19, section 1.5] and [20, p. 66 et seq.]), and by Singhal and Bhati [17], Hasanov and Srivastava [9, 11] for deriving analogues multiple series expansions associated with triple hypergeometric functions. We now introduce here the following analogues of Burchnall - Chaundy symbolic operators $\Delta_{x,y}$ and $\nabla_{x,y}$ defined by (2.1) and (2.2), respectively :

$$\begin{aligned} \tilde{\nabla}_{x;yz}(h) &:= \frac{\Gamma(h)\Gamma(\delta_1+\delta_2+\delta_3+h)}{\Gamma(\delta_1+h)\Gamma(\delta_2+\delta_3+h)} \\ &= \sum_{i,j=0}^{\infty} \frac{(-\delta_1)_{i+j} (-\delta_2)_i (-\delta_3)_j}{(h)_{i+j}\, i!j!}, \end{aligned} \tag{2.3}$$

and

$$\tilde{\Delta}_{x;yz}(h) := \frac{\Gamma(\delta_1+h)\Gamma(\delta_2+\delta_3+h)}{\Gamma(h)\Gamma(\delta_1+\delta_2+\delta_3+h)}$$
$$= \sum_{i,j=0}^{\infty} \frac{(-\delta_1)_{i+j}(-\delta_2)_i(-\delta_3)_j}{(1-h-\delta_1-\delta_2-\delta_3)_k\, i!j!}$$
$$= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}(h)_{2(i+j)}(-\delta_1)_{i+j}(-\delta_2)_i(-\delta_3)_j}{(h+i+j-1)_{i+j}(h+\delta_1)_{i+j}(h+\delta_2+\delta_3)_{i+j}\, i!j!},$$
$$\left(\delta_1 := x\frac{\partial}{\partial x};\ \delta_2 := y\frac{\partial}{\partial y};\ \delta_3 := z\frac{\partial}{\partial z}\right) \tag{2.4}$$

Here we present one-dimensional inverse symbolic operators to investigate generalized hypergeometric functions :

$$H_x(\alpha,\beta) := \frac{\Gamma(\beta)\Gamma(\alpha+\delta_x)}{\Gamma(\alpha)\Gamma(\beta+\delta_x)} = \sum_{i=0}^{\infty} \frac{(\beta-\alpha)_i(-\delta_x)_i}{(\beta)_i\, i!} \tag{2.5}$$

and

$$\bar{H}_x(\alpha,\beta) := \frac{\Gamma(\alpha)\Gamma(\beta+\delta_x)}{\Gamma(\beta)\Gamma(\alpha+\delta_x)} = \sum_{i=0}^{\infty} \frac{(\beta-\alpha)_i(-\delta_x)_i}{(1-\alpha-\delta_x)_i\, i!} \tag{2.6}$$

3. A set of operator identities

By applying the pairs of symbolic operators in (2.1) to (2.5), we find the following set of operator identities involving the Gauss function ${}_2F_1$, the Appell functions F_1, F_2, F_3 and F_4 , and Lauricella functions F_A, F_B, F_C and F_D respectively :

$$F_E(a,b_1,b_2;c_1,c_2,c_2;x,y,z) = \tilde{\nabla}_{x;yz}(a)\,\nabla_{yz}(b_2)\,\nabla_{yz}(c_2)\,F(a,b_1;c_1;x) \times F_1(a,b_2,b_2;c_2;y,z); \tag{3.1}$$

$$F_E(a,b_1,b_2;c_1,c_2,c_3;x,y,z) = \tilde{\nabla}_{x;yz}(a)\,\nabla_{yz}(b_2)\,F(a,b_1;c_1;x) \times F_2(a,b_2,b_2;c_2,c_3;y,z); \tag{3.2}$$

$$F_E(a,b_1,b_2;c_1,c_2,c_2;x,y,z) = \tilde{\nabla}_{x;yz}(a)\,\nabla_{yz}(a)\,\nabla_{yz}(b_2)\,\nabla_{yz}(c_2)\,F(a,b_1;c_1;x) \times F_3(a,a,b_2,b_2;c_2;y,z); \tag{3.3}$$

$$F_E(a,b_1,b_2;c_1,c_2,c_3;x,y,z) = \tilde{\nabla}_{x;yz}(a)\,F(a,b_1;c_1;x)\,F_4(a,b_2;c_2,c_3;y,z); \tag{3.4}$$

$$F_E(a,b_1,b_2;c_1,c_2,c_3;x,y,z) = \tilde{\nabla}_{x;yz}(a)\,\nabla_{yz}(a)\,\nabla_{yz}(b_2)\,F(a,b_1;c_1;x) \times F(a,b_2;c_2;y)\,F(a,b_2;c_3;z); \tag{3.5}$$

$$F_E(a,b_1,b_2;c_1,c_2,c_3;x,y,z) = H_x(b_1,c_1)\,(1-x)^{-a} \times F_4\left(a,b_2;c_2,c_3;\frac{y}{1-x},\frac{z}{1-x}\right); \tag{3.6}$$

$$(1-x)^{-a}F_4\left(a,b_2;c_2,c_3;\frac{y}{1-x},\frac{z}{1-x}\right) = \bar{H}_x(b_1,c_1) \times F_E(a,b_1,b_2;c_1,c_2,c_3;x,y,z) \tag{3.7}$$

$$F_E(a,b_1,b_2;c_1,c_2,c_3;x,y,z) = \nabla_{yz}(b_2)\,F_A(a,b_1,b_2;c_1,c_2,c_3;x,y,z); \tag{3.8}$$

$$F_E(a,b_1,b_2;c,c,c;x,y,z) = \tilde{\nabla}_{x;yz}(a)\,\nabla_{yz}(a)\,\nabla_{yz}(b_2)\,\tilde{\nabla}_{x;yz}(c)\,\nabla_{yz}(c) \times F_B(a,a,a,b_1,b_2,b_2;c;x,y,z); \tag{3.9}$$

$$F_E(a,b,b;c_1,c_2,c_3;x,y,z) = \tilde{\Delta}_{x;yz}(b)\,F_C(a,b;c_1,c_2,c_3;x,y,z); \tag{3.10}$$

$$F_E(a,b_1,b;c,c,c;x,y,z) = \nabla_{yz}(b)\,\tilde{\nabla}_{x;yz}(c)\,\nabla_{yz}(c)\,F_D(a,b_1,b;c;x,y,z). \tag{3.11}$$

In view of the known Mellin - Barnes contour integral representation for the Gauss function ${}_2F_1$ [1, p. 11], the Appell functions F_1, F_2, F_3, F_4

and Saran's function F_S, it is not difficult to give proofs of the operator identities (3.1) to (3.11) by making use of Mellin and inverse Mellin transformations [1, 13].

4. Decompositions for Saran's triple hypergeometric functions F_E

Making use of the principle of superposition of operators, from the operator identities (3.1) - (3.11), we can derive the following decomposition formulas for Saran's triple hypergeometric function F_E, respectively :

$$\begin{aligned} &F_E\left(a, b_1, b_2; c_1, c_2, c_2; x, y, z\right) \\ &= \sum_{i,j,k,l=0}^{\infty} \frac{(a)_{i+j+2k+2l}\,(b_1)_{i+j}\,(b_2)_{i+j+2k+l}}{(c_1)_{i+j}\,(c_2)_k\,(c_2)_{i+j+2k+2l}\,i!j!k!l!} x^{i+j} y^{i+k+l} z^{j+k+l} \\ &\times F\left(a+i+j+2k, b_1+i+j; c_1+i+j; x\right) \\ &\times F_1\left(\alpha, \beta, \beta; \gamma; y, z\right), \end{aligned} \tag{4.1}$$

where $\alpha = a+i+j+2k+2l$, $\beta = b_2+i+j+2k+l$, and $\gamma = c_2+i+j+2k+2l$.

$$\begin{aligned} &F_E\left(a, b_1, b_2; c_1, c_2, c_2; x, y, z\right) \\ &= \sum_{i,j,k=0}^{\infty} \frac{(a)_{i+j+2k}\,(b_1)_{i+j}\,(b_2)_{i+j+k}}{(c_1)_{i+j}\,(c_2)_{i+k}\,(c_3)_{j+k}\,i!j!k!} x^{i+j} y^{i+k} z^{j+k} \\ &\times F\left(a+i+j, b_1+i+j; c_1+i+j; x\right) \\ &\times F_2\left(a+i+j+2k, b_2+i+j+k, b_2+i+j+k; c_2+i+k, c_3+j+k; y, z\right); \end{aligned} \tag{4.2}$$

$$\begin{aligned} &F_E\left(a, b_1, b_2; c_1, c_2, c_2; x, y, z\right) \\ &= \sum_{i,j,k,l,q=0}^{\infty} \frac{\left[(a)_{i+j+2k+l+q}\right]^2 (b_1)_{i+j}\,(b_2)_{i+j+2k+2l+q}}{(a)_{i+j+2k+l}\,(c_1)_{i+j}\,(c_2)_k\,(c_2)_{i+j+2k+2l+2q}\,i!j!k!l!q!} x^{i+j} y^{i+k} z^{j+k} \\ &\times F\left(a+i+j+2k, b_1+i+j; c_1+i+j; x\right) \\ &\times F_3\left(\alpha, \alpha, \beta, \beta; \gamma; y, z\right), \end{aligned} \tag{4.3}$$

where $\alpha = a + i + j + 2k + l + q$, $\beta = b_2 + i + j + 2k + 2l + q$, $\gamma = c_2 + i + 2k + 2l + 2q$.

$$\begin{aligned} &F_E\left(a, b_1, b_2; c_1, c_2, c_3; x, y, z\right) \\ &= \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}\,(b_1)_{i+j}\,(b_2)_{i+j}}{(c_1)_{i+j}\,(c_2)_i\,(c_3)_j\, i!j!} x^{i+j} y^i z^j \\ &\times F\left(a+i+j, b_1+i+j; c_1+i+j; x\right) \\ &\times F_4\left(a+i+j, b_2+i+j; c_2+i, c_3+j; y, z\right); \end{aligned} \tag{4.4}$$

$$\begin{aligned} &F_E\left(a, b_1, b_2; c_1, c_2, c_2; x, y, z\right) \\ &= \sum_{i,j,k,l=0}^{\infty} \frac{(a)_{i+k+l}\,(a)_{j+k+l}\,(b_1)_{i+j}\,(b_2)_{i+j+2k+l}}{(a)_k\,(c_1)_{i+j}\,(c_2)_{i+k+l}\,(c_3)_{j+k+l}\, i!j!k!l!} x^{i+j} y^{i+k+l} z^{j+k+l} \\ &\times F\left(a+i+j+2k, b_1+i+j; c_1+i+j; x\right) \\ &\times F\left(a+i+k+l, b_2+i+j+2k+l; c_2+i+k+l; y\right) \\ &\times F\left(a+j+k+l, b_2+i+j+2k+l; c_3+j+k+l; z\right); \end{aligned} \tag{4.5}$$

$$\begin{aligned} &F_E\left(a, b_1, b_2; c_1, c_2, c_3; x, y, z\right) \\ &= (1-x)^{-a} \sum_{i=0}^{\infty} \frac{(-1)^i\,(a)_i\,(c_1-b_1)_i}{(c_1)_i\, i!} \left(\frac{x}{1-x}\right)^i \\ &F_4\left(a+i, b_2; c_2, c_3; \frac{y}{1-x}, \frac{z}{1-x}\right); \end{aligned} \tag{4.6}$$

$$\begin{aligned} &(1-x)^{-a} F_4\left(a, b_2; c_2, c_3; \frac{y}{1-x}, \frac{z}{1-x}\right) \\ &= \sum_{i=0}^{\infty} \frac{(a)_i\,(c_1-b_1)_i}{(c_1)_i\, i!} x^i F_E\left(a+i, b_1, b_2; c_1+i, c_2, c_3; x, y, z\right); \end{aligned} \tag{4.7}$$

$$\begin{aligned} &F_E\left(a, b_1, b_2; c_1, c_2, c_3; x, y, z\right) \\ &= \sum_{i=0}^{\infty} \frac{(a)_{2i}\,(b_2)_i}{(c_2)_i\,(c_3)_i\, i!} y^i z^i F_A\left(a+2i, b_1, b_2+i, b_2+i; c_1, c_2+i, c_3+i; x, y, z\right); \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&F_E\left(a, b, b; c_1, c_2, c_3; x, y, z\right) \\
&=\sum_{i=0}^{\infty} \frac{(-1)^{i+j}\,(a)_{2i+2j}\,(b)_{i+j}}{(c_1)_{i+j}\,(c_2)_i\,(c_3)_j\, i! j!} x^{i+j} y^i z^j \\
&\times F_C\left(a+2i+2j, b+i+j; c_1+i+j, c_2+i, c_3+j; x, y, z\right).
\end{aligned} \tag{4.9}
$$

Our operational derivations of the decomposition formulas (4.1) to (4.9) would indeed run parallel to those presented in the earlier works [1, 19], which we have already cited in the preceding sections. In addition to the various operator expressions and operator identities listed in Section 3, we also make use of the following operator identities [15] :

$$
\left(\delta_\xi+\alpha\right)_n \left\{f(\xi)\right\}=\xi^{1-\alpha} \frac{d^n}{d\xi^n}\left\{\xi^{\alpha+n-1} f(\xi)\right\} \tag{4.10}
$$

and

$$
\left(-\delta_\xi\right)_n \left\{f(\xi)\right\}=(-1)^n \xi^n \frac{d^n}{d\xi^n}\left\{f(\xi)\right\}, \quad \left(\delta_\xi:=\xi \frac{d}{d\xi};\ \alpha \in \mathbb{C}\right) \tag{4.11}
$$

for every analytic function $f(\xi)$.

5. Integral representation for Saran function F_E

Here in this section, we observe that several integral representations of the Eulerian type can be deduced also from the corresponding decomposition formulas of Section 4. For example, using integral representations [1, p. 29, (4) and p. 142, (13)] in (4.1):

$$
F\left(a, b; c; z\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \xi^{b-1}(1-\xi)^{c-b-1}(1-z\xi)^{-a} d\xi,
$$

$\Re\left(c\right)>\Re\left(b\right)>0$, and

$$
F_1\left(a; b_1, b_2; c; x, y\right)
$$

$$
=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \int_0^1 \eta^{a-1}(1-\eta)^{c-a-1}(1-x\eta)^{-b_1}(1-y\eta)^{-b_2} d\eta,
$$

$\Re\left(c\right)>\Re\left(a\right)>0$.

we obtain, after a little simplification,

$$
\begin{aligned}
&F_E\,(a, b_1, b_2; c_1, c_2, c_2; x, y, z) = \frac{\Gamma\,(c_1)\,\Gamma\,(c_2)}{\Gamma(a)\Gamma\,(b_1)\,\Gamma\,(c_2-a)\,\Gamma\,(c_1-b_1)} \\
&\times \int_0^1\int_0^1 \xi^{b_1-1}\eta^{a-1}\,(1-\xi)^{c_1-b_1-1}\,(1-\eta)^{c_2-a-1}\,(1-x\xi)^{-a}\,\{(1-y\eta) \\
&\qquad (1-z\eta)\}^{-b_2} \times \Big[\frac{(1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2}{(1-\eta)^2(1-y\eta)(1-z\eta)}\Big]^{-b_2} \\
&\times \sum_{i,j,k=0}^{\infty} \frac{(b_2)_{i+j+2k}}{(c_2)_k\, i!j!k!}\Big\{\frac{xy\xi\eta(1-\eta)}{(1-x\xi)[(1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2]}\Big\}^i \\
&\times\Big\{\frac{xz\xi\eta(1-\eta)}{(1-x\xi)[(1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2]}\Big\}^j \\
&\times\Big\{\frac{yz\eta^2(1-\eta)^2}{(1-x\xi)^2[(1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2]^2}\Big\}^k d\xi d\eta, \\
&\Re\,(c_1) > \Re\,(b_1) > 0,\ \Re\,(c_2) > \Re\,(a) > 0.
\end{aligned} \tag{5.1}
$$

In case of $x = 0$ in (5.1), we know

$$
\begin{aligned}
&F_E\,(a, b_1, b_2; c_1, c_2, c_2; 0, y, z) = F_4\,(a, b_2; c_2, c_2; y, z) \\
&= \frac{\Gamma\,(c_1)\,\Gamma\,(c_2)}{\Gamma(a)\Gamma\,(b_1)\,\Gamma\,(c_2-a)\,\Gamma\,(c_1-b_1)} \\
&\times \int_0^1\int_0^1 \xi^{b_1-1}\eta^{a-1}\,(1-\xi)^{c_1-b_1-1}\,(1-\eta)^{c_2-a-1} \\
&\qquad \Big(\frac{(1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2}{(1-\eta)^2}\Big)^{-b_2} \\
&\times \sum_{k=0}^{\infty} \frac{(b_2)_{2k}}{(c_2)_k\, k!}\Big\{\frac{yz\eta^2(1-\eta)^2}{[(1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2]^2}\Big\}^k d\xi d\eta, \\
&\Re\,(c_1) > \Re\,(b_1) > 0,\ \Re\,(c_2) > \Re\,(a) > 0.
\end{aligned} \tag{5.2}
$$

$$F_E\,(a, b_1, b_2; c_1, c_2, c_2; 0, y, z) = F_4\,(a, b_2; c_2, c_2; y, z)$$

$$= \frac{\Gamma\,(c_1)\,\Gamma\,(c_2)}{\Gamma(a)\Gamma\,(b_1)\,\Gamma\,(c_2-a)\,\Gamma\,(c_1-b_1)}$$

$$\times \int_0^1 \int_0^1 \xi^{b_1-1}\eta^{a-1}(1-\xi)^{c_1-b_1-1}(1-\eta)^{c_2+2b_2-a-1}$$
$$\Big((1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2\Big)^{-b_2} \tag{5.3}$$
$$\times F\left(\frac{b_2}{2},\frac{b_2}{2}+\frac{1}{2};c_2;\frac{4yz\eta^2(1-\eta)^2}{[(1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2]^2}\right)d\xi d\eta$$
$$\Re(c_1)>\Re(b_1)>0,\ \Re(c_2)>\Re(a)>0.$$

Finally using the following well-known relationship between the Beta function $B(\alpha,\beta)$ and the Gamma function Γ:

$$B(\alpha,\,\beta) := \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\ \Re(\beta)>0) \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-), \end{cases} \tag{5.4}$$

where $\mathbb{C}$ and $\mathbb{Z}_0^-$ being the set of complex numbers and non-positive integers respectively, we have the following

$$F_E(a,b_1,b_2;c_1,c_2,c_2;0,y,z)=F_4(a,b_2;c_2,c_2;y,z)=\frac{\Gamma(c_2)}{\Gamma(a)\Gamma(c_2-a)}$$
$$\times\int_0^1 \eta^{a-1}(1-\eta)^{c_2+2b_2-a-1}\Big((1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2\Big)^{-b_2}$$
$$\times F\left(\frac{b_2}{2},\frac{b_2}{2}+\frac{1}{2};c_2;\frac{4yz\eta^2(1-\eta)^2}{[(1-\eta)^2(1-y\eta)(1-z\eta)-yz\eta^2]^2}\right)d\eta$$
$$\Re(c_2)>\Re(a)>0. \tag{5.5}$$

Similarly, we can obtain other results by integral representations [1, p. 29-30, (1)-(4) and p. 142, (13)] in (4.2)-(4.6).

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