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CALCULATING ZEROS OF THE TWISTED (h, q)-EXTENSION OF GENOCCHI POLYNOMIALS

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Abstract. In this paper we introduce the new analogs of Genocchi numbers and polynomials. Finally, we investigate the zeros of the twisted (h, q)-extension of Genocchi polynomials $G^{(h)}_{n,q,w}(x)$.

1. Introduction

Many mathematicians have studied Bernoulli numbers and polynomials, Euler numbers and polynomials, and Genocchi numbers and Genocchi polynomials (see [1-12]). These numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In this paper we consider the new analogs of Genocchi numbers and polynomials. In the 21st century, the computing environment would make more and more rapid progress. Using computer, a realistic study for new analogs of Genocchi numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the twisted (h, q)-extension of Genocchi polynomials $G^{(h)}_{n,q,w}(x)$. The outline of this paper is as follows. In Section 2, we construct the twisted (h, q)-extension of Genocchi polynomials $G^{(h)}_{n,q,w}(x)$. In Section 3, we describe the beautiful zeros of the twisted (h, q)-extension polynomials $G^{(h)}_{n,q,w}(x)$ using a numerical investigation. We also display distribution and structure of the zeros of the twisted (h, q)-extension of Genocchi polynomials $G^{(h)}_{n,q,w}(x)$ by using computer. By using the results of our paper the readers can observe the regular behaviour of the roots of the twisted (h, q)-extension of Genocchi polynomials $G^{(h)}_{n,q,w}(x)$. Finally, we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex

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roots of the twisted (h,q)-extension of Genocchi polynomials $G_{n,q,w}^{(h)}(x)$. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$.

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}, \text{ cf. [3,4,5].}$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$\begin{aligned} X &= \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. For any positive integer N,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

is known to be a distribution on X, cf. [1,3,4,5]. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by [3,4,5]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{0\le x<p^N} g(x)q^x.$$

Note that

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{0\le x<p^N} g(x)$$

(see [3,4,5]). Now, we consider the case $q \in (-1,0)$ corresponding to q-deformed fermionic certain and annihilation operators and the literature given therein [4,5]. The expression for the $I_q(g)$ remains same, so it is tempting to consider the limit $q \to -1$. That is,

$$I_{-1}(g) = \lim_{q \to -1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \le x < p^N} g(x)(-1)^x. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

From (1.2), we obtain

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l),$$

where $g_n(x) = g(x+n)$. First, we introduce the Genocchi numbers and Genocchi polynomials. The Genocchi numbers G_n are defined by the generating function:

$$F(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, (|t| < \pi), \text{ cf. [1,3,4,5]} \tag{1.3}$$

where we use the technique method notation by replacing G^n by $G_n (n \ge 0)$ symbolically. For $x \in \mathbb{C}$, we consider the Genocchi polynomials $G_n(x)$ as follows:

$$F(x, t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{1.4}$$

Note that $G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}$. In the special case $x = 0$, we define $G_n(0) = G_n$.

2. The twisted (h, q)-extension of Genocchi polynomials

In this section, we introduce the twisted (h, q)-extension of Genocchi numbers $G_{n,q,w}^{(h)}$ and Genocchi polynomials $G_{n,q,w}^{(h)}(x)$, and investigate their properties. Let $T_p = \cup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$.

In [7], we introduced the (h, q)-extension of Genocchi numbers $G_{n,q}^{(h)}$ and polynomials $G_{n,q}^{(h)}(x)$ as follows:

$$F_q(t) = \frac{2(h \log q + t)}{q^h e^t + 1} = \sum_{n=0}^{\infty} G_{n,q}^{(h)} \frac{t^n}{n!},$$

$$F_q(x,t)=\frac{2(h\log q+t)}{q^he^t+1}e^{xt}=\sum_{n=0}^{\infty}G_{n,q}^{(h)}(x)\frac{t^n}{n!}.$$

In [8], we introduced the twisted Genocchi numbers and polynomials. We defined the twisted Genocchi numbers $G_{n,w}$ and polynomials $G_{n,w}(x)$ as follows:

$$F_w(t)=\frac{2t}{we^t+1}=\sum_{n=0}^{\infty}G_{n,w}\frac{t^n}{n!},$$

$$F_w(x,t)=\frac{2t}{we^t+1}e^{xt}=\sum_{n=0}^{\infty}G_{n,w}(x)\frac{t^n}{n!}.$$

We now consider construction the twisted (h,q)-extension of Genocchi numbers and polynomials. By using p-adic q-integral, we now consider construction the twisted (h,q)-extension of Genocchi numbers and polynomials. In (1.2), if we take $g(x)=\phi_w(x)q^{hx}e^{xt}, h\in\mathbb{Z}$, then we see that

$$I_{-1}(t\phi_w(x)q^{hx}e^{xt})=t\int_{\mathbb{Z}_p}\phi_w(x)q^{hx}e^{xt}d\mu_{-1}(x)=\frac{2t}{wq^he^t+1}.$$

Let us define the twisted (h,q)-extension of Genocchi numbers $G_{n,q,w}^{(h)}$ and polynomials $G_{n,q,w}^{(h)}(x)$ as follows:

$$I_{-1}((h\log q+t)\phi_w(y)q^{hy}e^{yt})=\sum_{n=0}^{\infty}G_{n,q,w}^{(h)}\frac{t^n}{n!}, \tag{2.1}$$

$$I_{-1}((h\log q+t)\phi_w(y)q^{hy}e^{(y+x)t})=\sum_{n=0}^{\infty}G_{n,q,w}^{(h)}(x)\frac{t^n}{n!}. \tag{2.2}$$

By simple calculation, we have

$$(h\log q+t)\int_{\mathbb{Z}_p}\phi_w(x)q^{hx}e^{xt}d\mu_{-1}(x)=\frac{2(h\log q+t)}{wq^he^t+1}.$$

Hence we obtain

$$\frac{2(h\log q+t)}{wq^he^t+1}=(h\log q+t)\int_{\mathbb{Z}_p}\phi_w(y)q^{hy}e^{yt}d\mu_{-1}(y)=\sum_{n=0}^{\infty}G_{n,q,w}^{(h)}\frac{t^n}{n!},$$

$$\begin{aligned}\frac{2(h\log q+t)}{wq^he^t+1}e^{xt}&=(h\log q+t)\int_{\mathbb{Z}_p}\phi_w(y)q^{hy}e^{(x+y)t}d\mu_{-1}(y)\\&=\sum_{n=0}^{\infty}G_{n,q,w}^{(h)}(x)\frac{t^n}{n!}.\end{aligned}$$

By (2.1) and (2.2), we obtain the following Witt's formula.

Theorem 1. For $w \in T_p$, we have

$$n \int_{\mathbb{Z}_p} \phi_w(x) q^{hx} x^{n-1} d\mu_{-1}(x) + h \log q \int_{\mathbb{Z}_p} \phi_w(x) q^{hx} x^n d\mu_{-1}(x) = G_{n,q,w}^{(h)},$$

$$n \int_{\mathbb{Z}_p} \phi_w(y) q^{hy} (x+y)^{n-1} d\mu_{-1}(y) + h \log q \int_{\mathbb{Z}_p} \phi_w(y) q^{hy} (x+y)^n d\mu_{-1}(y)$$
$$= G_{n,q,w}^{(h)}(x).$$

Let q be a complex number with $|q| < 1$ and w be the p^N-th root of unity. By the meaning of (1.3) and (1.4), let us define the twisted (h, q)-extension of Genocchi numbers $G_{n,q,w}^{(h)}$ and polynomials $G_{n,q,w}^{(h)}(x)$ as follows:

$$F_{q,w}^{(h)}(t) = \frac{2(h \log q + t)}{wq^h e^t + 1} = \sum_{n=0}^{\infty} G_{n,q,w}^{(h)} \frac{t^n}{n!}, \tag{2.3}$$

$$F_{q,w}^{(h)}(x,t) = \frac{2(h \log q + t)}{wq^h e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{2.4}$$

We have the following remark.

Remark. Note that

(1) $G_{n,q,w}^{(h)}(0) = G_{n,q,w}^{(h)}$.

(2) If $w = 1$, then $G_{n,q,w}^{(h)}(x) = G_n(x), G_{n,q,w}^{(h)} = G_{n,w}$.

(3) If $w = 1$, then $F_{q,w}^{(h)}(x,t) = F_q(x,t), F_{q,w}^{(h)}(t) = F_q(t)$.

(4) If $q \to 1$, then $G_{n,q,w}^{(h)}(x) = G_{n,w}(x), G_{n,q,w}^{(h)} = G_{n,w}$.

(5) If $q \to 1$, then $F_{q,w}^{(h)}(x,t) = F_w(x,t), F_{q,w}^{(h)}(t) = F_w(t)$.

(6) If $q \to 1, w = 1$, then $G_{n,q,w}^{(h)}(x) = G_n(x), G_{n,q,w}^{(h)} = G_n$.

(7) If $q \to 1, w = 1$, then $F_{q,w}^{(h)}(x,t) = F(x,t), F_{q,w}^{(h)}(t) = F(t)$.

By above definition, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} G_{l,q,w}^{(h)}(x)\frac{t^l}{l!} &= \frac{2(h\log q+t)}{wq^h e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_{n,q,w}^{(h)}\frac{t^n}{n!}\sum_{m=0}^{\infty} x^m\frac{t^m}{m!}\\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} G_{n,q,w}^{(h)}\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right)\\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n} G_{n,q,w}^{(h)}x^{l-n}\right)\frac{t^l}{l!}.\end{aligned}$$

By using comparing coefficients of $\frac{t^l}{l!}$, we have the following theorem.

Theorem 2. For any positive integer n, we have

$$G_{n,q,w}^{(h)}(x) = \sum_{k=0}^{n}\binom{n}{k} G_{k,q,w}^{(h)}x^{n-k}.$$

For q-Genocchi numbers, Ryoo, Kim, and Agarwal constructed q-Genocchi numbers which can be uniquely determined by

$$G_{0,q} = \frac{2\log q}{1+q},\ \text{and}\ q\left(G_q+1\right)^n + G_{n,q} = \begin{cases} 2, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases}$$

with the usual convention of symbolically replacing G_q^n by $G_{n,q}$, where $G_{n,q}$ denotes the q-Genocchi numbers. For twisted (h,q)-extension of Genocchi numbers, we obtain the following theorem.

Theorem 3. The twisted (h,q)-extension of Genocchi numbers $G_{n,q,w}^{(h)}$ are defined respectively by

$$G_{0,q,w} = \frac{2h\log q}{1+wq^h},\ \text{and}\ wq^h\left(G_{q,w}^{(h)}+1\right)^n + G_{n,q,w}^{(h)} = \begin{cases} 2, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases}$$

with the usual convention about replacing $\left(G_{q,w}^{(h)}\right)^n$ by $G_{n,q,w}^{(h)}$ in the binomial expansion.

Proof. From (2.1), we obtain

$$\frac{2(h\log q+t)}{wq^h e^t+1} = \sum_{n=0}^{\infty} G_{n,q,w}^{(h)}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(G_{q,w}^{(h)}\right)^n\frac{t^n}{n!} = e^{G_{q,w}^{(h)}t}$$

which yields

$$2(h\log q+t)=(wq^he^t+1)e^{G_{q,w}^{(h)}t}=q^he^{(G_{q,w}^{(h)}+1)t}+e^{G_{q,w}^{(h)}t}.$$

Using Taylor expansion of exponential function, we have

$$\begin{aligned}2h\log q+2t&=\sum_{n=0}^{\infty}\left\{wq^h\left(G_{q,w}^{(h)}+1\right)^n+\left(G_{q,w}^{(h)}\right)^n\right\}\frac{t^n}{n!}\\&=wq^h\left(G_{q,w}^{(h)}+1\right)^0+\left(G_{q,w}^{(h)}\right)^0+wq^h\left(G_{q,w}^{(h)}+1\right)^1+\left(G_{q,w}^{(h)}\right)^1\\&\quad+\sum_{n=2}^{\infty}\left\{wq^h\left(G_{q,w}^{(h)}+1\right)^n+\left(G_{q,w}^{(h)}\right)^n\right\}\frac{t^n}{n!}.\end{aligned}$$

The result follows by comparing the coefficients.

Here is the list of the first the twisted (h,q)-extension of Genocchi numbers $G_{n,q,w}^{(h)}$.

$$\begin{aligned}G_{0,q,w}^{(h)}&=\frac{2h\log q}{1+wq^h},\\G_{1,q,w}^{(h)}&=-\frac{2(-1-wq^h+hwq^h\log q)}{(1+wq^h)^2},\\G_{2,q,w}^{(h)}&=\frac{2wq^h(-2-2wq^h-h\log q+hwq^h\log q)}{(1+wq^h)^3},\\G_{3,q,w}^{(h)}&=-\frac{2wq^h(3-3w^2q^{2h}+h\log q-4hwq^h\log q+hw^2q^{2h}\log q)}{(1+wq^h)^4},\\&\cdots\end{aligned}$$

We display the shapes of the twisted (h,q)-extension of Genocchi numbers $G_{n,q,w}^{(h)}$. For $n=1,\ldots,10$, we can draw a curve of $G_{n,q,w}^{(h)}, 1/10\leq q\leq 9/10$, respectively. This shows the ten curves combined into one. We display the shape of $G_{n,q,w}^{(h)}$ (Figure 1).

Because

$$\frac{\partial}{\partial x}F_{q,w}^{(h)}(t,x)=tF_{q,w}^{(h)}(t,x)=\sum_{n=0}^{\infty}\frac{d}{dx}G_{n,q,w}^{(h)}(x)\frac{t^n}{n!},$$

it follows the important relation

$$\frac{d}{dx}G_{n,q,w}^{(h)}(x)=nG_{n-1,q,w}^{(h)}(x).$$

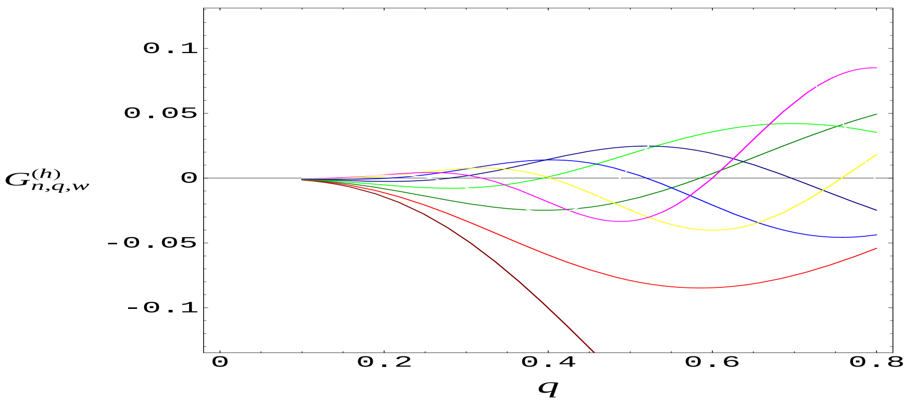

FIGURE 1. Curvers of $G^{(4)}_{n,q,w}, w = e^{\pi i}$

Here is the list of the first twisted (h, q)-extension of Genocchi numbers $G^{(h)}_{n,q,w}(x)$.

$$G^{(h)}_{0,q,w}(x) = \frac{2h \log q}{1 + wq^h},$$
$$G^{(h)}_{1,q,w}(x) = \frac{2(1 + wq^h - hwq^h \log q + hx \log q + hwq^h x \log q)}{(1 + wq^h)^2}, \cdots$$

We display the shapes of the twisted (h, q)-extension of Genocchi polynomials $G^{(h)}_{n,q,w}(x)$. For $n = 1, \ldots, 10$, we can draw a curve of $G^{(4)}_{n,q,w}(x), q = 1/2, w = e^{\pi i}, -2 \le x \le 2$, respectively. This shows the ten curves combined into one. We display the shape of $G^{(4)}_{n,q,w}(x)$ (Figure 2).

Since

$$\begin{aligned}
\sum_{l=0}^{\infty} G^{(h)}_{l,q,w}(x + y)\frac{t^l}{l!} &= \frac{2h \log q + 2t}{wq^h e^t + 1} e^{(x+y)t} = \sum_{n=0}^{\infty} G^{(h)}_{n,q,w}(x)\frac{t^n}{n!} \sum_{m=0}^{\infty} y^m \frac{t^m}{m!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} G^{(h)}_{n,q,w}(x)\frac{t^n}{n!} y^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} \binom{l}{n} G^{(h)}_{n,q,w}(x) y^{l-n}\right)\frac{t^l}{l!},
\end{aligned}$$

we have the following theorem.

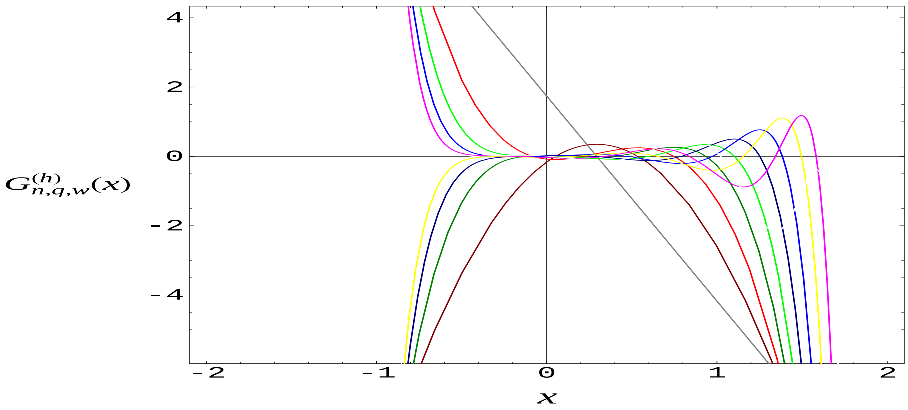

FIGURE 2. Curvers of $G^{(4)}_{n,q,w}(x), w = e^{\pi i}$

Theorem 4. The twisted (h, q)-extension of Genocchi polynomials $G^{(h)}_{n,q,w}(x)$ satisfies the following relation:

$$G^{(h)}_{l,q,w}(x+y) = \sum_{n=0}^{l} \binom{l}{n} G^{(h)}_{n,q,w}(x) y^{l-n}.$$

It is easy to see that

$$\begin{aligned}
\sum_{n=0}^{\infty} G^{(h)}_{n,q,w}(x)\frac{t^n}{n!} &= \frac{2h\log q + 2t}{wq^h e^t + 1} e^{xt} \\
&= \frac{1}{m}\sum_{a=0}^{m-1}(-1)^a w^a q^a \frac{2h\log q^m + 2mt}{w^m q^{hm} e^{mt} + 1} e^{\left(\frac{a+x}{m}\right)(mt)} \\
&= \frac{1}{m}\sum_{a=0}^{m-1}(-1)^a w^a q^{ha} \sum_{n=0}^{\infty} G^{(h)}_{n,q^m,w^m}\left(\frac{a+x}{m}\right)\frac{(mt)^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(m^{n-1}\sum_{a=0}^{m-1}(-1)^a w^a q^{ha} G^{(h)}_{n,q^m,w^m}\left(\frac{a+x}{m}\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

Hence we have the below theorem.

Theorem 5. For any positive integer m(=odd), we obtain

$$G_{n,q,w}^{(h)}(x) = m^{n-1}\sum_{a=0}^{m-1}(-1)^a w^a q^{ha} G_{n,q^m,w^m}^{(h)}\left(\frac{a+x}{m}\right), \text{ for } n \geq 0.$$

3. Distribution and Structure of the zeros

In this section, we investigate the zeros of the twisted (h,q)-extension of Genocchi polynomials $G_{n,q,w}^{(h)}(x)$ by using computer. Let $w = e^{\frac{2\pi i}{N}}$ in $\mathbb{C}$. We plot the zeros of $G_{n,q,w}^{(h)}(x), x \in \mathbb{C}$ for $N = 1,3,5,7$ (Figure 3).

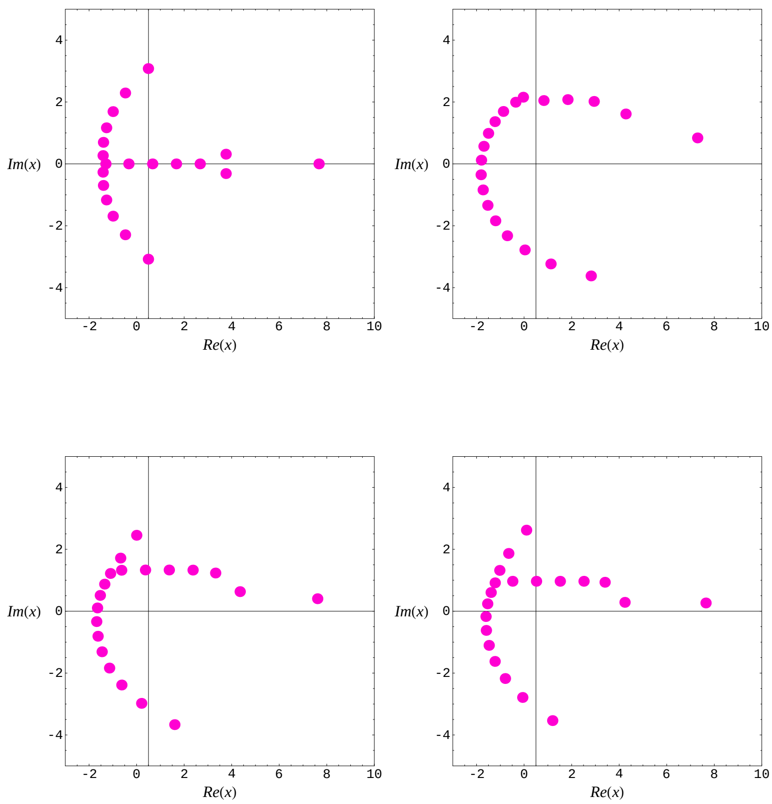

FIGURE 3. Zeros of $G_{n,q,w}^{(h)}(x)$

In Figure 3(top-left), we choose $n = 20, h = 4, q = 1/2$ and $w = e^{2\pi i}$. In Figure 3(top-right), we choose $n = 20, h = 4, q = 1/2$ and $w = e^{\frac{2\pi i}{3}}$. In

Figure 3(bottom-left), we choose $n=20, h=4, q=1/2$ and $w=e^{\frac{2\pi i}{5}}$. In Figure 3(bottom-right), we choose $n=20, h=4, q=1/2$ and $w=e^{\frac{2\pi i}{7}}$.

Finally, we plot the zeros of $G^{(h)}_{n,q,w}(x)$, $w=e^{\frac{2\pi i}{3}}$, $x\in\mathbb{C}$ for $n=20, q=1/2, h=3,5,7,9$ (Figure 4).

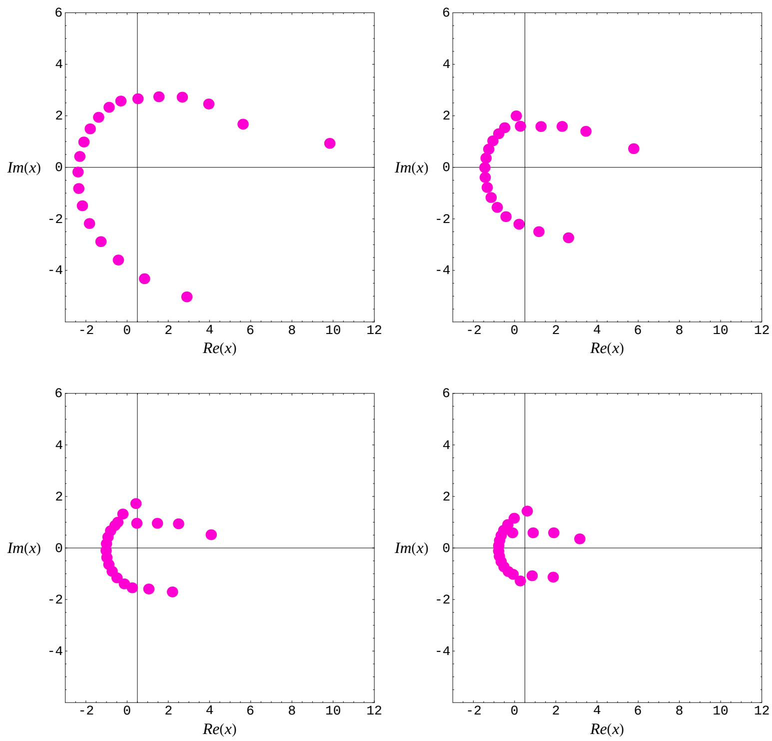

FIGURE 4. Zeros of $G^{(h)}_{n,q,w}(x)$

In Figure 4(top-left), we choose $n=20, h=3, q=1/2$ and $w=e^{\frac{2\pi i}{3}}$. In Figure 4(top-right), we choose $n=20, h=5, q=1/2$ and $w=e^{\frac{2\pi i}{3}}$. In Figure 4(bottom-left), we choose $n=20, h=7, q=1/2$ and $w=e^{\frac{2\pi i}{3}}$. In Figure 4(bottom-right), we choose $n=20, h=9, q=1/2$ and $w=e^{\frac{2\pi i}{3}}$.

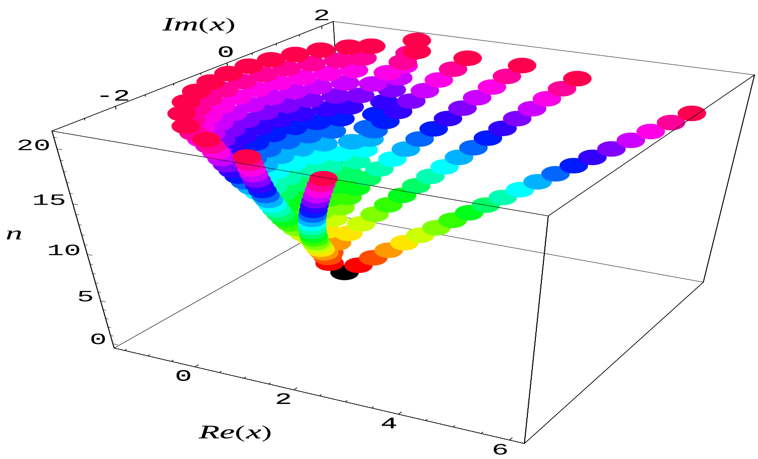

FIGURE 5. Stacks of zeros $G^{(h)}_{n,q,w}(x)$ for $1 \leq n \leq 21$

In Figure 5, we choose $h = 5, q = 1/2$ and $w = e^{\frac{2\pi i}{3}}$.

Our numerical results for numbers of real and complex zeros of $G^{(h)}_{n,q,w}(x)$ are displayed in Table 1.

Table 1. Numbers of real and complex zeros of $G^{(4)}_{n,1/3,w}(x)$

degree n	$w = e^{\frac{2\pi i}{3}}$		$w = e^{\frac{2\pi i}{4}}$	
	real zeros	complex zeros	real zeros	complex zeros
1	0	1	0	1
2	0	2	0	2
3	0	3	0	3
4	0	4	0	4
5	0	5	0	5
6	0	6	0	6
7	0	7	0	7
8	0	8	0	8
9	0	9	0	9
10	0	10	0	10

We calculated an approximate solution satisfying $G^{(h)}_{n,q,w}(x), N = 3, h = 4, 6, x \in \mathbb{C}$. The results are given in Table 2 and Table 3.

Table 2. Approximate solutions of $G^{(4)}_{n,1/2,w}(x) = 0, w = e^{\frac{2\pi i}{3}}$

degree n	x
1	$0.3316 + 0.0575i$
2	$-0.00271 - 0.03348i, \quad 0.6660 + 0.1485i$
3	$-0.0250 + 0.0910i, \quad 0.0046 - 0.1576i, \quad 1.0152 + 0.2391i$
4	$-0.2376 + 0.0355i, \quad 0.0573 - 0.3660i, \quad 0.1333 + 0.2412i,$ $1.3735 + 0.3193i$
5	$-0.3213 - 0.1479i, \quad -0.2556 + 0.2418i, \quad 0.1550 - 0.5836i,$ $0.3434 + 0.3885i, \quad 1.737 + 0.389i$
6	$-0.415 + 0.094i, \quad -0.353 - 0.339i, \quad -0.196 + 0.415i,$ $0.2750 - 0.8014i, \quad 0.577 + 0.528i, \quad 2.102 + 0.449i$

Table 3. Approximate solutions of $G^{(6)}_{n,1/2,w}(x) = 0, w = e^{\frac{2\pi i}{3}}$

degree n	x
1	$0.2328 + 0.01374i$
2	$0.00647 - 0.01709i, \quad 0.4591 + 0.0446i$
3	$-0.05831 + 0.03503i, \quad 0.0709 - 0.0801i, \quad 0.6857 + 0.0863i$
4	$-0.1138 - 0.0142i, \quad -0.0197 + 0.1013i, \quad 0.1486 - 0.1646i$ $0.9158 + 0.1325i$
5	$-0.1825 - 0.1223i, \quad -0.1515 + 0.1497i, \quad 0.1138 + 0.1451i,$ $0.2345 - 0.2828i, \quad 1.1495 + 0.1790i$
6	$-0.2741 + 0.0184i, \quad -0.1895 - 0.2508i, \quad -0.1468 + 0.2891i,$ $0.2792 + 0.2171i, \quad 0.3420 - 0.4150i, \quad 1.3858 + 0.2235i$

Plot of real zeros of $G^{(h)}_{n,q,w}(x)$ for $1 \le n \le 20$ are presented (Figure 6).

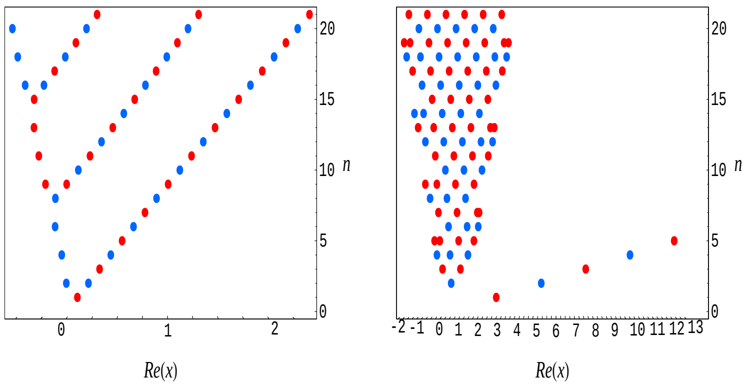

Figure 6. Real zeros of $G^{(h)}_{n,q,w}(x)$ for $1 \le n \le 20$

In Figure 6(left), we choose $h = 4, q = 1/10$ and $w = e^{2\pi i}$. In Figure 6(right), we choose $h = 4, q = 9/10$ and $w = e^{2\pi i}$.

4. Directions for Further Research

We shall consider the more general open problem. In general, how many roots does $G^{(h)}_{n,q,w}(x)$ have? Prove or disprove: $G^{(h)}_{n,q,w}(x)$ has n distinct solutions. Find the numbers of complex zeros $C_{G^{(h)}_{n,q,w}(x)}$ of $G^{(h)}_{n,q,w}(x)$, $\mathrm{Im}(x) \neq 0$. Prove or give a counterexample: *Conjecture*: Since n is the degree of the polynomial $G^{(h)}_{n,q,w}(x)$, the number of real zeros $R_{G^{(h)}_{n,q,w}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{G^{(h)}_{n,q,w}(x)} = n - C_{G^{(h)}_{n,q,w}(x)}$, where $C_{G^{(h)}_{n,q,w}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{G^{(h)}_{n,q,w}(x)}$ and $C_{G^{(h)}_{n,q,w}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane, and the equation of a trajectory curve running through the complex zeros on any one of the arcs. We plot the $G^{(h)}_{n,q,w}(x)$, respectively (Figures 2-6). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $G^{(h)}_{n,q,w}(x)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand.

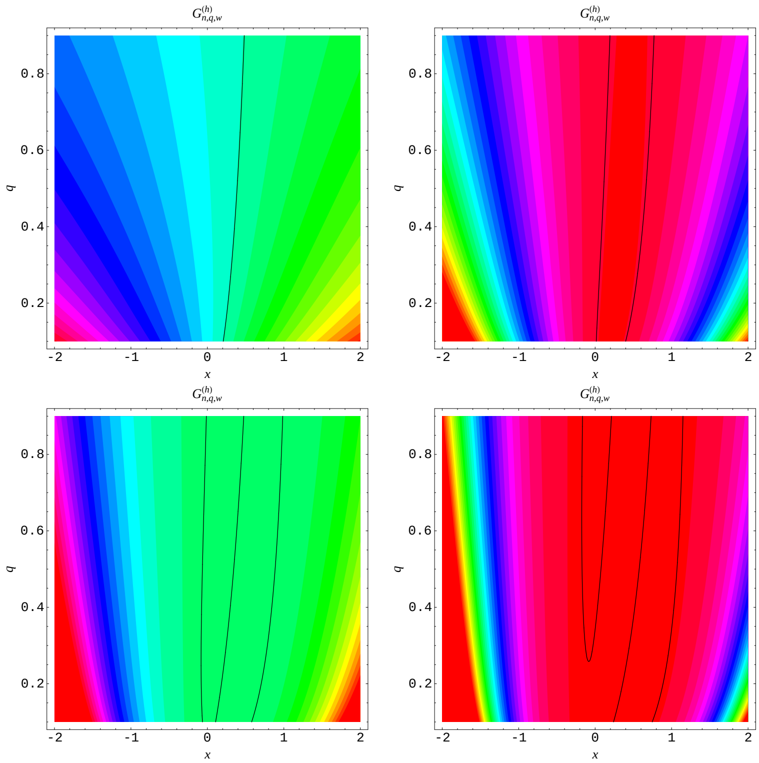

FIGURE 7. Zero contour of $G_{n,q,w}^{(h)}(x)$

The plot above shows $G_{n,q,w}^{(h)}(x)$ for $1/10 \leq q \leq 9/10$ and $-2 \leq x \leq 2$, with the zero contour indicated in black(Figure 7).

In Figure 7(top-left), we choose $n = 1, h = 2$ and $w = e^{\frac{2\pi i}{2}}$. In Figure 7(top-right), we choose $n = 2, h = 2$ and $w = e^{\frac{2\pi i}{2}}$. In Figure 7(bottom-left), we choose $n = 3, h = 2$ and $w = e^{\frac{2\pi i}{2}}$. In Figure 7(bottom-right), we choose $n = 4, h = 2$ and $w = e^{\frac{2\pi i}{2}}$.

The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the twisted (h, q)-extension of Genocchi polynomials $G_{n,q,w}^{(h)}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [6-12].

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