Honam Mathematical J. **32** (2010), No. 4, pp. 545–561

T-INTERVAL-VALUED FUZZY SUBGROUPS AND RINGS

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Abstract. We introduce the concepts of interval-valued fuzzy subgroups [resp. normal subgroups, rings and ideals] and investigate some of it's properties.

1. Introduction

In 1975, Zadeh[8] suggested the notion of interval-valued fuzzy sets as another generalization of fuzzy sets. After that time, Biswas[1] applied it to group theory, and also Kang and Hur[4] applied it to group and ring theory. Gorzalczany[2] suggested a method of inference in approximate reasoning by using interval-valued fuzzy sets. Moreover Montal and Samanta[6] introduced the concept of topology of interval-valued fuzzy sets and investigate some of it's properties. Recently, Hur et. al[3] studies interval-valued fuzzy relations in the sense of a lattice theory. In this paper, we introduce the concept of t-interval-valued fuzzy subgroups [resp.normal subgroup, rings and ideals] and investigate some of it's properties.

2. Preliminaries

In this section, we list some concepts and results related to interval-valued fuzzy set theory and needed in next sections.

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters

Received September 14, 2010. Accepted October 23, 2010.
2000 Mathematics Subject Classification : 20N25.
Keywords and phrases. t-norm : t-interval-valued fuzzy subgroup[ring and ideal], t-interval-valued fuzzy normal subgroup.
* This paper was supported by Woosuk University in 2010.

$M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted , $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\boldsymbol{a} = [a,a]$ for every $\boldsymbol{a} \in (0,1)$, We also note that

(i) $(\forall M, N \in D(I))$ $(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,

(ii) $(\forall M, N \in D(I))$ $(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the *complement* of M, denoted by M^C, is defined by $M^C = 1 - M = [1 - M^U, 1 - M^L]$(See[6]).

Definition 2.1[6,8]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set*(is short, *IVFS*) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^L \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$[resp $A^U(x)$] is called the *lower*[resp *upper*] *end point of* x *to* A. For any $[a,b] \in D(I)$, the interval-valued fuzzy A in X defined by $A(x) = [A^L(x), A^U(x)] = [a,b]$ for each $x \in X$ is denoted by $\widetilde{[a,b]}$ and if $a = b$, then the IVFS $\widetilde{[a,b]}$ is denoted by simply $\widetilde{a}$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in X, respectively.

We will denote the set of all IVFSs in X as $D(I)^X$ .It is clear that A $= [A, A] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2[6]. An IVFS A is called an *interval-valued fuzzy point*(in short, *IVFP*) *in* X *with the support* $x \in X$ and *the value* $[a,b] \in D(I)$ with $b > 0$, denoted by $A = x_{[a,b]}$, if for each $y \in X$

$$A(y) = \begin{cases} [a,b] & \text{if} \quad y = x, \\ 0 & \text{otherwise} \end{cases}$$

In particular, if $b = a$, then $x_{[a,b]}$ is denoted by x_a.

We will denote the set of all IVFPs in X as $\mathrm{IVF_P(X)}$.

Definition 2.3 [6]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(i) $A \subset B$ iff $A^L \le B^L$ and $A^U \le B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^C = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha\in\Gamma} A_\alpha = [\bigvee_{\alpha\in\Gamma} A^L_\alpha, \bigvee_{\alpha\in\Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

Result 2.B[6, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(a) $\widetilde{0} \subset \mathrm{A} \subset \widetilde{1}$.

(b) $A \cup B = B \cup A$, $A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B$, $A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} (A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma} (A \cup A_\alpha)$.

(g) $(\widetilde{0})^c = \widetilde{1}$, $(\widetilde{1})^c = \widetilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A^c_\alpha$, $(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A^c_\alpha$.

Definition 2.4[6]. Let $A \in D(I)^X$ and let $x_M \in \mathrm{IVF_P(X)}$. Then x_M is said to *belong to* A, denoted by $x_M \in A$, if $M^L \le A^L(x)$ and $M^U \le A^U(x)$ for each $x \in X$.

It is obvious that $A = \bigcup_{x_M \in A} x_M$ and $x_M \in A$ if and only if $x_{M^L} \in A^L$ and $x_{M^U} \in A^U$.

Definition 2.5[7]. A *t-norm* is a mapping $t : I \times I \to I$ satisfing the following conditions : for any $x, y, z, u, v \in I$

(i) $t(x, y) = t(y, x), i.e., xty = ytx$.

(ii) $xt(ytz) = (xty)tz$.

(iii) If $x \le u$ and $y \le v$, then $xty \le utv$

In particular, if $y \leq v$, then $xty \leq xtv$.
(iv) $xt1 = x$ and $xt0 = 0$.

t-norms which are frequently encountered are :
(a) $xt_0y = \min\{x, y\}$ for $x, y \in I$.
(b) $xt_1y = \text{Prod}\{x, y\} = xy$ for $x, y \in I$.
(c) $xt_2y = \max\{x + y - 1, 0\}$ for $x, y \in I$.

Definition 2.6[7]. A *t-conorm* or *s-norm* is a mapping $s_t : I \to I$ defined by : for any $u, v \in I$,
$us_tv = 1 - (1 - u)t(1 - v)$.

It is clear that s_t satisfies the following conditions : for any $x, y, z, u, v \in I$.
(i) $xs_ty = ys_tx$.
(ii) $xs_t(ys_tz) = (xs_ty)s_tz$.
(iii) If $x \leq u$ and $y \leq v$, then $xs_ty \leq us_tv$
In particular, if $y \leq v$, then $xs_ty \leq xs_tv$.
(iv) $xs_t0 = x$ and $xs_t1 = 1$.

t-conorms corresponding to the above t-norms t_0, t_1, t_2 are as follows:
(a′) $xs_0y = \max\{x, y\}$ for any $x, y \in I$.
(b′) $xs_1y = x + y - xy$ for any $x, y \in I$.
(c′) $xs_2y = \min\{1, x + y\}$ for any $x, y \in I$.

3. t-interval-valued fuzzy subgroupoids

Definition 3.1. Let $(G, \cdot)$ be a groupoid and let $A, B \in D(I)^G$. Then the *interval-valued fuzzy product of A and B under t-norm t* (in short, *t-interval-valued fuzzy product of A and B*), denoted by $A \circ_t B$, is an IVFS in G defined as follows : For each $x \in G$,

$$(A \circ_t B)(x) = \begin{cases} [\bigvee_{yz=x} [A^L(y)tB^L(z)], \bigvee_{yz=x} [A^U(y)tB^U(z)]] & \text{if} \quad yz = x \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $(D(I)^G, \circ_t)$ is a groupoid.

Proposition 3.2. Let "$\circ_t$" be same as above, let $x_M, y_N \in$ IVFp(G) and let $A, B \in D(I)^G$. Then:

(a) $x_M \circ_t y_N = (xy)_{[M^L t N^L, M^U t N^U]}$.

(a) $A \circ_t B = \bigcup_{x_M \in A, y_N \in B} x_M \circ_t y_N$.

Proof. (a) Let $z \in G$. Then

$$(x_M \circ_t y_N)(z) = \begin{cases} [\bigvee_{z=x'y'} (x_{M^L}(x') \wedge y_{N^L}(y')), \bigvee_{z=x'y'} (x_{M^U}(x') \wedge y_{N^U}(y'))] & \text{if} \quad x'y' = z, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} [M^L t N^L, M^U t N^U] & \text{if} \quad z = xy \\ 0 & \text{otherwise.} \end{cases}$$

$$= (xy)_{[M^L t N^L, M^U t N^U]}$$

.

(b) Let $C = \bigcup_{x_M \in A, y_N \in B} x_M \circ_t y_N$, *i.e.*,

$C = [\bigvee_{x_{M^L} \in A^L, y_{N^L} \in B^L} (x_{M^L} \circ_t y_{N^L}), \bigvee_{x_{M^U} \in A^U, y_{N^U} \in B^U} (x_{M^U} \circ_t y_{N^U})]$.

For each $z \in G$, we may assume that $\exists u, v \in X$ such that $uv = z$, $x_M(u) \neq \mathbf{0}$ and $y_N(v) \neq \mathbf{0}$, i.e., ${x_M}^L(u) > 0, {x_M}^U < 1$ and ${y_N}^L(v) > 0, {y_M}^U(v) < 1$, whitout loss of generality. Then

$(A \circ_t B)^L(z) = \bigvee_{z=uv} [A^L(u) t B^L(v)]$

$\geq \bigvee_{z=uv} (\bigvee_{x_{M^L} \in A^L, y_{N^L} \in B^L} [x_{M^L}(u) t y_{N^L}(v)])$ [Since t is increasing]

$= (\bigcup_{x_{M^L} \in A^L, y_{N^L} \in B^L} x_{M^L} \circ_t y_{N^L})$

$= C^L(z)$.

Since $u_{A(u)} \in A$ and $v_{B(v)} \in B$,

$$\begin{aligned} C^L(z) &= \bigvee_{x_{M^L} \in A^L, y_{N^L} \in B^L} \Big(\bigvee_{z=uv} [x_{M^L}(u) t y_{N^L}(v)] \Big) \\ &= \bigvee_{z=uv} \Big(\bigvee_{x_{M^L} \in A^L, y_{N^L} \in B^L} [x_{M^L}(u) t y_{N^L}(v)] \Big) \\ &\geq \bigvee_{z=uv} [u_{A^L(u)}(u) t v_{B^L(v)}(v)] \\ &= \bigvee_{z=uv} [A^L(u) t B^L(v)] \\ &= (A \circ_t B)^L(z). \end{aligned}$$

Thus $(A \circ_t B)^L = C^L$. By the similar arguments, we have $(A \circ_t B)^U = C^U$.

Hence

$$A \circ_t B = \bigcup_{x_{M^L} \in A^L, y_{N^L} \in B^L} x_{M^L} \circ_t y_{N^L}. \qquad \blacksquare$$

Remark 3.2. Proposition 3.2 is the generalization of Proposition 3.2 in [4].

The following is the immediate result of Definition 3.1.

Proposition 3.3. Let $(G, \cdot)$ be a groupoid, and let "$\circ_t$" be same as above.

(a) if "$\cdot$" is associative[resp. commutative] in G, then so is "$\circ_t$" in $D(I)^G$.

(b) if "$\cdot$" is has an identity $e \in G$, then $e_{\mathbf{1}} \in \text{IVFp}(G)$ is an identity of "$\circ_t$" in $D(I)^G$, *i.e.*, $A \circ_t e_{\mathbf{1}} = A = e_{\mathbf{1}} \circ_t A$ for each $A \in D(I)^G$.

Definition 3.4. Let $(G, \cdot)$ be a groupoid and let $\widetilde{0} \neq A \in D(I)^G$. Then A is called an *interval-valued fuzzy subgroupoid* (in short, *t-IVGP*) in G if $A \circ_t A \subset A$, *i.e.*, $A^L \circ_t A^L \subset A^L$ and $A^U \circ_t A^U \subset A^U$.

It is clear that $\mathbf{0}$ and $\mathbf{1}$ are both *t-IVGPs* in G.

The followings are the immediate results of Definitions 3.1 and 3.4.

Proposition 3.5. Let $(G, \cdot)$ be a groupoid and let $\widetilde{0} \neq A \in D(I)^G$. Then the followings are equivalent:

(a) A is a t-IVGP in G.

(b) For any $x_M, y_N \in A$, $x_M \circ_t y_N \in A$, $i.e.$, $(A, \circ_t)$ is a groupoid.

(c) For any $x, y \in G$, $A^L(xy) \geq A^L(x)tA^L(y)$ and $A^U(xy) \geq A^U(x)tA^U(y)$.

Remark 3.5. Proposition 3.5 is the generalization of Proposition 3.5 in [4].

Proposition 3.6. Let A be a t-IVGP in a groupoid $(G, \cdot)$.

(a) If "$\cdot$" is associative in G, then so is "$\circ_t$ " in A, $i.e.$, for any $x_L, y_M, z_N \in A$,

$$x_L \circ_t (y_M \circ_t z_N) = (x_L \circ_t y_M) \circ_t z_N.$$

(b) If "$\cdot$" is commutative in G, then so is "$\circ_t$" in A, $i.e.$, for any $x_L, y_M \in A$,

$$x_L \circ_t y_M = y_M \circ_t x_L.$$

(c) If "$\cdot$" has an identity $e \in G$, then

$$e_1 \circ_t x_L = x_L = x_L \circ_t e_1, \ \forall x_L \in A.$$

Remark 3.6. Proposition 3.6 is the generalization of Proposition 3.6 in [4].

From Proposition 3.5, we can define a t-IVGP in G as follows.

Definition 3.4′. An interval-valued fuzzy set A in G is called a *t-interval-valued fuzzy subgroupoid*(in short, t-IVGP) in G if

$$A^L(xy) \geq A^L(x)tA^L(y) \text{ and } A^U(xy) \geq A^U(x)tA^U(y), \forall x, y \in G.$$

The following is the immediate result of Definition 3.4′.

Proposition 3.7. Let T be a subset of a groupoid $(G, \cdot)$. Then $A = [\chi_T, \chi_T]$ is a t-IVGP in G if and only if T is a subgroupoid of G, where χ_T is the charecteristic function of T.

Remark 3.7. Proposition3.7 is the generalization of Proposition 3.7 in [4].

Definition 3.8[**7**]. A t- norm t is said to be *continuous* if $t : I \times I \to I$ is continuous with respect to the usual topologies.

It is clear that t_0, t_1 and t_2 are all continuous t-norms.

Proposition 3.9. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be any family of t-IVGPs in a groupoid $(G, \cdot)$. If t is continuous, then $\bigcap_{\alpha\in\Gamma} A_\alpha$ is a t-IVGP in G.

Proof. Let $A = \bigcap_{\alpha\in\Gamma} A_\alpha$ and let $x, y \in G$. Then

$$A^L(xy) = \bigwedge_{\alpha\in\Gamma} A^L_\alpha(xy)$$

$$\geq \bigwedge_{\alpha\in\Gamma} [A^L_\alpha(x) t A^L_\alpha(y)].\ [\text{ Since } A_\alpha \text{ is a } t\text{-IVGP in } G]$$

Since t is continous, t is continuous at $(\bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(x), \bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(y))$. Let $\epsilon > 0$. Then there exists a $\delta > 0$ such that if $r_1 \geq \bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(x) + \delta$ and $r_2 \geq \bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(y) + \delta$, then $r_1 t r_2 \geq (\bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(x)) t (\bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(y)) + \epsilon$. Let us choose $\alpha_0 \in \Gamma$ such that ${A_{\alpha_0}}^L(x) \geq \bigwedge_{\alpha\in\Gamma} {A_{\alpha_0}}^L(x) + \delta$ and ${A_{\alpha_0}}^L(y) \geq \bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(y) + \delta$. Then

$${A_{\alpha_0}}^L(x) t {A_{\alpha_0}}^L(y) \geq (\bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(x)) t (\bigvee_{\alpha\in\Gamma} {A_\alpha}^L(y)) + \epsilon.$$

Thus

$$\bigwedge_{\alpha\in\Gamma} [{A_{\alpha_0}}^L(x) t {A_{\alpha_0}}^L(y)] \geq (\bigwedge_{\alpha\in\Gamma} {A_\alpha}^L(x)) t (\bigvee_{\alpha\in\Gamma} {A_\alpha}^L(y)).$$

So

$$\bigwedge_{\alpha\in\Gamma} [A^L_\alpha(x) \wedge A^L_\alpha(y)] \geq (\bigcap_{\alpha\in\Gamma} A^L_\alpha)(x) t (\bigcap_{\alpha\in\Gamma} A^L_\alpha)(y) = A^L(x) t A^L(y).$$

Similarly, we can see that $A^U(xy) \geq A^U(x) t A^U(y)$. Hence $\bigcap_{\alpha\in\Gamma} A_\alpha$ is a t-IVGP in G. ■

Remark 3.9. Since $t_0 =$ " $\wedge$ " is continuous, Proposition 3.9 is the generalization of Proposition 3.8 in [4].

4. t-interval-valued fuzzy subgroups

Definition 4.1. Let be a group and let $A \in D(I)^G$. Then A is called an *interval-valued fuzzy subgroup* under a t-norm t (in short, t-IVG) in G if it satisfies the conditions : For any $x, y \in G$,

(i) $A^L(xy) \geq A^L(x)tA^L(y)$ and $A^U(xy) \leq A^U(x)tA^U(y)$,

(ii) $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$.

Proposition 4.2. Let A be a t-IVG in a group G. Then $A(x^{-1}) = A(x)$ for each $x \in G$.

Proof. Let $x \in G$. Then

$$A^L(x) = A^L((x^{-1})^{-1}) \geq A^L(x^{-1}) \geq A^L(x)$$

and

$$A^U(x) = A^U((x^{-1})^{-1}) \geq A^U(x^{-1}) \geq A^U(x).$$

Thus $A^L(x^{-1}) = A^L(x)$ and $A^U(x^{-1}) = A^U(x)$. So $A(x^{-1} = A(x)$ for each $x \in X$. ■

Proposition 4.3. If A is a t-IVG in a group G, then $H = \{x \in G : A(x) = \mathbf{1}\}$ is a subgroup of G.

Proof. Let $x, y \in H$. Then

$$A^L(xy^{-1}) \geq A^L(x)tA^L(y^{-1}) = A^L(x)tA^L(y) = 1t1 = 1$$

and

$$A^U(xy^{-1}) \geq A^U(x)tA^U(y^{-1}) = A^U(x)tA^U(y) = 1t1 = 1.$$

Thus $A^L(xy^{-1}) = 1$ and $A^U(xy^{-1}) = 1$. So $xy^{-1} \in H$. Hence H is a subgroup of X. ■

Proposition 4.4. If A is a t-IVG in a group G and if there is a sequence x_n in X such that $\lim_{n\to\infty} A^L(x_n)tA^L(x_n) = 1$ and $\lim_{n\to\infty} A^U(x_n)tA^U(x_n) = 1$, then $A(e) = \mathbf{1}$, where e is the identity in G.

Proof. Let $x \in G$. Then

$$A^L(e) = A^L(xx^{-1}) \geq A^L(x)tA^L(x^{-1}) = A^L(x)tA^L(x)$$

and

$$A^U(e) = A^U(xx^{-1}) \geq A^U(x)tA^U(x^{-1}) = A^U(x)tA^U(x).$$

Then, for each n,

$$A^L(e) \geq A^L(x_n)tA^L(x_n)$$

and

$$A^u(e) \geq A^u(x_n)tA^u(x_n).$$

On the other hand,

$$1 \geq A^L(e) \geq \lim_{n\to\infty} A^L(x_n)tA^L(x_n) = 1$$

and

$1 \geq A^U(e) \geq \lim_{n\to\infty} A^U(x_n)tA^U(x_n) = 1.$

Hence $A(e) = \mathbf{1}$. ■

Proposition 4.5. Let A be a t-IVG in a group G. If $A(xy^{-1}) = \mathbf{1}$, then $A(x) = A(y)$.

Proof. Let $x, y \in G$. Then

$A^L(x) = A^L((xy^{-1})y) \geq A^L(xy^{-1})tA^L(y) = 1tA^L(y)$
$= A^L(y) = A^L(y^{-1}) = A^L(x^{-1}(xy^{-1}))$
$\geq A^L(x^{-1})tA^L(xy^{-1}) = A^L(x)t1 = A^L(x)$

and

$A^U(x) = A^U((xy^{-1})y) \geq A^U(xy^{-1})tA^U(y) = 1tA^U(y)$
$= A^U(y) = A^U(y^{-1}) = A^U(x^{-1}(xy^{-1}))$
$\geq A^U(x^{-1})tA^U(xy^{-1}) = A^U(x)t1 = A^U(x).$

Hence $A(x) = A(y)$. ■

Proposition 4.6. Let G be a group and let $0 \neq A \in D(I)^G$ with $A(e) = \mathbf{1}$. Then A is a t-IVG in$G$ if and only if $A^L(xy^{-1}) \geq A^L(x)tA^L(y)$ and $A^U(xy^{-1}) \geq A^U(x)tA^U(y)$ for any $x, y \in G$.

Proof. (⇒): Suppose A is a t-IVG in G and let $x, y \in G$. Then, by Proposition 4.2, $A^L(xy^{-1}) \geq A^L(x)tA^L(y)$ and $A^U(xy^{-1}) \geq A^U(x)tA^U(y)$.

(⇐): Suppose the necessary conditions hold and let $x, y \in G$. Then

$A^L(x^{-1}) = A^L(ex^{-1}) \geq A^L(e)tA^L(x)$
$= 1tA^L(x) = A^L(x)$

and

$A^U(x^{-1}) = A^U(ex^{-1}) \geq A^U(e)tA^U(x)$
$= 1tA^U(x) = A^U(x).$

So $A^L(x^{-1}) \geq A^L(x)$ and $A^U(x^{-1}) \geq A^U(x)$ for each $x \in G$.

On the other hand,

$A^L(xy) = A^L(x(y^{-1})^{-1}) \geq A^L(x)tA^L(y^{-1})$
$\geq A^L(x)tA^L(y)$

and

$A^U(xy) = A^U(x(y^{-1})^{-1}) \geq A^U(x)tA^U(y^{-1})$
$\geq A^U(x)tA^U(y).$

Hence A is a t-IVG in G. ■

Proposition 4.7. Let G_p be the cyclic group of prime order p and let $A \in D(I)^{G_p}$ with $A(e) = \mathbf{1}$, where e is the identity in G_p. If $A(x) =$

$A(a) \leq A(e)$, for each $e \neq x \in G_p$ where $G_p = (a) = e = a^0, a^1, a^2, \cdots a^{p-1}$, then A is a t-IVG in G_p.

Proof. Let $x, y \in G_p$.

Case(i) : Suppose $x \neq e$, $y \neq e$ and $xy^{-1} \neq e$. Then, by the hypothesis,

$$A^L(xy^{-1}) = A^L(x) = A^L(y)$$

and

$$A^U(xy^{-1}) = A^U(x) = A^U(y).$$

Thus

$$A^L(xy^{-1}) \geq A^L(x)tA^L(y)$$

and

$$A^U(xy^{-1}) \geq A^U(x)tA^U(y).$$

Case(ii) : Suppose $x \neq e$, $y \neq e$ and $xy^{-1} = e$. Then, by the hypothesis,

$$A^L(x) = A^L(y) \leq A^L(e) = A^L(xy^{-1})$$

and

$$A^U(x) = A^U(y) \leq A^U(e) = A^U(xy^{-1}).$$

Thus

$$A^L(xy^{-1}) \geq A^L(x)tA^L(y)$$

and

$$A^U(xy^{-1}) \geq A^U(x)tA^U(y).$$

Case(iii) : Suppose $x \neq e$, $y = e$ and $xy^{-1} \neq e$. Then, by the hypothesis,

$$A^L(x) = A^L(xy^{-1}) \leq A^L(e) = A^L(y) = 1$$

and

$$A^U(x) = A^U(xy^{-1}) \leq A^U(e) = A^U(y) = 1.$$

Thus

$$A^L(xy^{-1}) \geq A^L(x)t1 = A^L(x)tA^L(y)$$

and

$$A^U(xy^{-1}) \geq A^U(x)t1 = A^L(x)tA^U(y).$$

Case(iv) : Suppose $x = e$, $y \neq e$, $xy^{-1} \neq e$. Then it is the same as case (iii).

In all,

$$A^L(xy^{-1}) \geq A^L(x)tA^L(y)$$

and

$$A^U(xy^{-1}) \geq A^U(x)tA^U(y).$$

Hence A is a t-IVG in G_p. ■

Definition 4.8. Let A be a t-IVG in a group G. Then A is called a *t-interval-valued fuzzy normal subgroup* (in short, *t-IVNG*) in G if $A(xy) = A(yx)$ for any $x, y \in X$.

Proposition 4.9. Let A be a t-IVNG in a group G.
(a) For each $B \in D(I)^G$, $A \circ_t B = B \circ_t A$.
(b) If B is a t-IVG in G, then so is $B \circ_t A$.

Proof. (a) Let $z \in G$ with $z = xy$. Then
$(A \circ_t B)^L(z) = \bigvee_{xy=z} A^L(x)tB^L(y)$
$= \bigvee_{x=zy^{-1}} A^L(x)tB^L(y)$
$= \bigvee_{x=zy^{-1}} A^L(zy^{-1})tB^L(y)$
$= \bigvee_{x'=y^{-1}z} A^L(x')tB^L(y)$
(Since A is a t-IVNG in G)
$= \bigvee_{yx'=z} B^L(y)tA^L(x') = (B \circ_t A)^L$.
Similarly, $(A \circ_t B)^U(z) = (B \circ_t A)^U(z)$. So $A \circ_t B = B \circ_t A$. ■

(b) By Definition 3.4 and (a),
$(B \circ_t A) \circ_t (B \circ_t A) = B \circ_t (A \circ_t B) \circ_t A$
$= B \circ_t (B \circ_t A) \circ_t A$
$= (B \circ_t B) \circ_t (A \circ_t A) \subset B \circ_t A$.
Thus $B \circ_t A$ is a t-IVGP in G. Now let $x \in G$ with $x^{-1} = yz$.
Then
$(B \circ_t A)^L(x^{-1}) = \bigvee_{yz=x^{-1}} B^L(y)tA^L(z)$
$= \bigvee_{x=z^{-1}y^{-1}} B^L((y^{-1})^{-1})tA^L((z^{-1})^{-1})$
$\geq \bigvee_{x=z^{-1}y^{-1}} B^L(y^{-1})tA^L(z^{-1})$
$= \bigvee_{x=z^{-1}y^{-1}} A^L(z^{-1})tB^L(y^{-1})$
$= (A \circ_t B)^L(x) = (B \circ_t A)^L(x)$. (By (a)).
Similarly, we have $(B \circ_t A)^U(x^{-1}) \geq (B \circ_t A)^U(x)$.
Hence $B \circ_t A$ is a t-IVG in G. ■

5. t-interval-valued fuzzy rings and ideals

Definition 5.1. Let $(R, +, \cdot)$ be a ring, let t be a *t-interval-valued fuzzy subring* (in short, t-IVR) in R if it satisfies the following conditions:
(i) A is a t-IVG in R with respect to " + " (in the sense of Definition 4.1),

(ii) A is a t-IVGP in R with respect to " $\cdot$" (in the sense of Definition 3.4 or Definition 3.4′).

Proposition 5.2. Let R be a ring and let $\widetilde{0} \neq A \in D(I)^R$ such that $A(0) = \mathbf{1}$ where 0 is the zero element for " $+$ " in R. Then A is a t-IVR in R if and only if any $x, y \in G$.

$A^L(x)tA^L(y) \leq A^L(x-y) \wedge A^L(xy)$

and

$A^U(x)tA^U(y) \leq A^U(x-y) \wedge A^U(xy)$

Proof. A is a t-IVR in R

if and only if

$A^L(x-y) \geq A^L(x)tA^L(y),\ A^U(x-y) \geq A^U(x)tA^U(y)$

(by Proposition 4.6)

and

$A^L(xy) \geq A^L(x)tA^L(y),\ A^U(xy) \geq A^U(x)tA^U(y)$ for any $x, y \in R$

(by Definition 3.4′)

if and only if

$A^L(x)tA^L(y) \leq A^L(x-y) \wedge A^L(xy)$

and

$A^U(x)tA^U(y) \leq A^U(x-y) \wedge A^U(xy)$ for any $x, y \in R$. ■

Corollary 5.2 [4,Proposition 6.2]. Let R be a ring and let $\widetilde{0} \neq A \in D(I)^R$. Then A is an IVR in R if and only if $A^L(x) \wedge A^L(y) \leq A^L(x-y) \wedge A^L(xy)$ and $A^U(x) \wedge A^U(y) \leq A^U(x-y) \wedge A^U(xy)$ for any $x, y \in R$.

Definition 5.3. Let R be a ring and let $\widetilde{0} \neq A \in D(I)^X$ be a t-IVR in R. Then A is called a:

(1) *t-interval-valued fuzzy left ideal* (in short, *t-IVLI*) in X if $A^L(xy) \geq A^L(y)$ and $A^U(xy) \geq A^U(y)$ for any $x, y \in R$.

(2) *t- interval-valued fuzzy right ideal* (in short, *t-IVRI*) in R if $A^L(xy) \geq A^L(x)$ and $A^U(xy) \geq A^U(x)$ for any $x, y \in R$.

(3) *t-interval-valued fuzzy ideal* (in short, *t-IVI*) in R if it is both t-IVLI and t-IVRL in X.

Proposition 5.4. Let R be a ring and let $\widetilde{0} \neq A \in D(I)^R$ such that $A(0) = \mathbf{1}$. Then A is a t-IVI [resp.t-IVLI, t-IVRI] in R if and only if $A^L(x-y) \geq A^L(x)tA^L(y)$, $A^U(x-y) \geq A^U(x)tA^U(y)$ and $A^L(xy) \geq A^L(x)s_tA^L(y)[\geq A^L(y), \geq A^L(x)]$, $A^U(xy) \geq A^U(x)s_tA^U(y)[\geq A^U(y), \geq$

$A^U(x)]$ for any $x, y \in R$.

Proof. It is obvious from Proposition 4.2 and Definition 4.3.

Corollary 5.4 [4, Proposition 6.5]. Let R be a ring and let $\widetilde{0} \neq A \in D(I)^R$. Then A is an [resp. IVLI, IVRI] in R if and only if $A^L(x-y) \geq A^L(x) \wedge A^L(y)$, $A^U(x-y) \geq A^U(x) \wedge A^U(y)$ and $A^L(xy) \geq A^L(x) \wedge A^L(y)[\geq A^L(y), \geq A^L(x)], A^U(xy) \geq A^U(x) \wedge A^U(y)[\geq A^U(y), > A^U(x)]$ for any $x, y \in R$.

Proposition 5.5. Let R be a skew field and let $\widetilde{0} \neq A \in D(I)^R$. Then A is a t-IVI in R if and only if

(1) $A(x) = A(e)$, for each $x \in R$-$\{0\}$,

(2) $A^L(0) = A^L(0)s_tA^L(x) \geq A^L(x)tA^L(e)$

and

$A^U(0) = A^U(0)s_tA^U(e) \geq A^U(x)tA^U(e)$ for each $x \in X$,

(3) $A^L(e) = A^L(e)s_tA^L(e)$ and $A^U(e) = A^U(e)s_tA^U(e)$.

Proof. ($\Rightarrow$): Suppose A is a t-IVI in R and let $0 \neq x \in R$. Then

$A^L(x) = A^L(xe) \geq A^L(x)s_tA^L(e)$ [By Proposition 5.4]
$= A^L(e)s_tA^L(x) \geq A^L(e)s_t0$
$= 1-(1-A^L(e))t(1-0)$
$= 1-(1-A^L(e)) = A^L(e) = A^L(x^{-1}x)$
$\geq A^L(x^{-1})s_tA^L(x) = A^L(x)s_tA^L(x^{-1})$
$\geq A^L(x)s_t0 = A^L(x)$.

So $A^L(x) = A^L(e)$. Similarly, we can see that $A^U(x) = A^U(e)$ for each $x \in R$-$\{0\}$. Hence, the condition (1) holds.

Let $x \in R$. Then

$A^L(0) = A^L(x0) \geq A^L(x)s_tA^L(0)$
$= A^L(0)s_tA^L(x) \geq A^L(0)s_t0 = A^L(0)$
$= A^L(0)s_tA^L(0) = A^L(e-e)s_tA^L(e-e)$
$\geq [A^L(e)tA^L(e)]s_t[A^L(e)tA^L(e)]$
$\geq [A^L(e)tA^L(e)]s_t0 = A^L(e)tA^L(e)$.

Thus $A^L(0) = A^L(0)s_tA^L(x) \geq A^L(x)tA^L(e)$. Similarly, we can see that $A^U(0) = A^U(0)s_tA^U(x) \geq A^U(x)tA^U(e)$. So the condition (2) holds.

Now let $0 \neq x \in R$. Then, by (1),

$A^L(e) = A^L(x) = A^L(xe) \geq A^L(x)s_tA^L(e)$
$= A^L(e)s_tA^L(e) \geq A^L(e)s_t0 = A^L(e)$.

Thus $A^L(e) = A^L(e)s_tA^L(e)$. Similarly, we can see that $A^U(e) = A^U(e)s_tA^U(e)$. So the condition (3) holds.

($\Leftarrow$) : Suppose the necessary condition hold and let $x \in R$. Since $A^L(0) = A^L(-0)$ and $A^U(0) = A^U(-0)$, let $x \neq 0$. Then, by (1),

$A^L(x) = A^L(e) = A^L(-x)$

and

$A^U(x) = A^U(e) = A^U(-x)$.

Thus

$A^L(-x) = A^L(x)$ and $A^U(-x) = A^U(x)$ for each $x \in X$, (6.1)

Let $x, y \in R$.

Case (i): Suppose $x + y \neq 0$ with $y \neq 0$. Then

$A^L(x + y) = A^L(x + y)t1$
$\geq A^L(x + y)tA^L(x)$
$= A^L(e)tA^L(x)$
$= A^L(y)tA^L(x)$ (by (1))
$= A^L(x)tA^L(y)$ (by (1))

Similarly, we can see that $A^U(x + y) = A^U(x)tA^U(y)$.

Case(ii) : Suppose $x + y = 0$ with $x = 0$. Then

$A^L(x + y) = A^L(0) = A^L(0)t1 \geq A^L(0)tA^L(y)$
$= A^L(x)tA^L(y)$

Also, we can see that $A^U(x + y) = A^U(x)tA^U(y)$.

Case(iii) : Suppose $x + y = 0$ with $0 \neq x = -y$. Then

$A^L(x + y) = A^L(0) \geq A^L(e)tA^L(e)$ (by (2))
$= A^L(x)tA^L(-y)$ (by (1))
$= A^L(x)tA^L(y)$. (by (6.1))

and

$A^U(x + y) = A^U(0) \geq A^U(e)tA^U(e)$ (by (2))
$= A^U(x)s_tA^U(-y)$ (by (1))
$= A^U(x)s_tA^U(y)$. (by (6.1))

In all, for any $x, y \in X, A^L(x + y) \geq A^L(x)tA^L(y)$ and $A^U(x + y) \geq A^U(x)tA^U(y)$. (6.2)

Now let $x, y \in R$.

Case(i) : Suppose $xy = 0$ with, say, $x = 0$.

Then, by (2),

$A^L(xy) = A^L(0) = A^L(0)s_tA^L(y) = A^L(x)tA^L(y)$

and

$A^U(xy) = A^U(0) = A^U(0)s_tA^U(y) = A^U(x)tA^U(y)$.

Case(ii) : Suppose $xy \neq 0$. Then, by (1) and (3),

$A^L(xy) = A^L(e) = A^L(e)s_tA^L(e) = A^L(x)s_tA^L(y)$

and

$A^U(xy) = A^U(e) = A^U(e)s_tA^U(e) = A^U(x)s_tA^U(y)$.

In all, for any $x, y \in R, A^L(xy) \geq A^L(x)s_tA^L(y)$ and $A^U(xy) \geq A^U(x)s_t$ $A^U(y)$. (6.3)

On the other hand, by (6.3),

$A^L(xy) \geq A^L(x)s_tA^L(y) \geq A^L(x)s_t0$
$= A^L(x) = A^L(x)t1 \geq A^L(x)tA^L(y)$

and

$A^U(xy) \leq A^U(x)tA^U(y) \leq A^U(x)t1$
$= A^U(x) = A^U(x)s_t0 \leq A^U(x)s_tA^U(y)$.

So $A^L(xy) \geq A^L(x)tA^L(y)$ and $A^U(xy) \leq A^U(x)s_tA^U(y)$ for any $x, y \in R$. (6.4) Hence, by (6.1), (6.2), (6.3) and (6.4), A is a t- IVI in R.

Corollary 5.5[4, Proposition 6.7]. Let R be a skew field and let $\widetilde{0} \neq A \in \text{IVS}(R)$. Then A is an IVI[resp.IVLI, IVRI] in R if and only if $A(x) = A(e) \leq A(0)$, ie., $A^L(x) = A^L(e) \leq A^L(0)$ and $A^U(x) = A^U(e) \leq A^U(0)$ for each $0 \neq x \in R$.

Proposition 5.6. Let R be a commutative ring with a unity e. If for any t-IVI A in $R, A(x) = A(e) \leq A(0)$, i.e., $A^L(x) = A^L(e) \leq A^L(0)$ and $A^U(x) = A^U(e) \leq A^U(0)$ for each $0 \neq x \in R$, then R is a field.

Proof. Let A be an ideal of R such that $A \neq R$. Then clearly $A = [\chi_A, \chi_A]$ is a t-IFI in R such that $A \neq \widetilde{1}$. Then there exists $y \in R$ such that $y \in A$. Thus $\chi_A(y) = 0$. By the hypothesis, $\chi_A(x) = \chi_A(e) \leq \chi_A(0)$ for each $0 \neq x \in R$. Thus $\chi_A(0) = 1$ ie., $A = \{0\}$. Hence X is a field. ■

Corollary 5.6 [4, Proposition 6.9]. Let R be a commutative ring with a unity e. If for any IVI A in R, $A(x) = A(e) \leq A(0)$, ie., $A^L(x) = A^L(e) \leq A^L(0)$ and $A^U(x) = A^U(e) \leq A^U(0)$ for each $0 \neq x \in R$. Then R is a field.

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