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SUPERSTABILITY OF FUNCTIONAL INEQUALITIES ASSOCIATED WITH GENERAL EXPONENTIAL FUNCTIONS

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Abstract. we prove the superstability of a functional inequality associated with general exponential functions as follows;

$$\left| f(x+y) - a^{x^2y+xy^2} g(x)f(y) \right| \leq H_p(x,y).$$

It is a generalization of the superstability theorem for the exponential functional equation proved by Baker.

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk in the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [22]). One of those was the question concerning the stability of homomorphisms :

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In the next year, Hyers [5] answered the Ulam's question for the case of the additive mapping on the Banach spaces G_1, G_2. Thereafter, the result of Hyers has been generalized by Rassias [15]. Since then, the stability problems

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of various functional equations have been investigated by many authors (see [1,3, 4, 6-13, 16-21]).

In particular, Baker et al. in [2] introduced the stability of the exponential functional equation in the following form : if f satisfies the inequality $|f(x+y) - f(x)f(y)| \le \varepsilon$, then either f is bounded or $f(x+y) = f(x)f(y)$. This is frequently referred to as *Superstability*.

In this paper, we will investigate the solution and the superstability of the exponential type functional equation

$$f(x+y) = a^{x^2y+xy^2} g(x)f(y), \tag{1}$$

which is a generalization of the superstability of the exponential functional equation given by Baker et al.[2]. Note that $f(x) = a^{\frac{x^3}{3}}$ is a solution of the equation (1).

2. Superstability of a functional inequality

Baker et al.[2] proved the superstability of Cauchy's exponential equation

$$f(x+y) = f(x)f(y).$$

That is, if the Cauchy difference $f(x+y) - f(x)(y)$ of a real-valued function f defined on a real vector space is bounded for all x, y, then f is either bounded or exponential. Their result was generalized by Baker [1] : let S be a semi-group and let f be a complex-valued function defined on S such that

$$|f(xy) - f(x)f(y)| < \delta$$

for all x, $y \in S$, then f is either bounded or multiplicative. The following our result is a generalization of Baker's theorem.

Theorem 1. *Let $\delta > 0$ and $a \ge 1$ be given. Let $f, g : R \to R$ be functions with $g(m) \ge max\{2, 4\delta/|f(m)|\}$ for some integer $m \ge 1$ such that*

$$|f(x+y) - a^{x^2y+xy^2} g(x)f(y)| < \delta \tag{2}$$

for all $x, y \in R$. *Then*

$$g(x+y) = a^{x^2y+xy^2} g(x)g(y)$$

for all $x, y \in R$.

Proof. If we replace x and y by m in (2), simultaneously, we get

$$|f(2m) - a^{2m^3} g(m)f(m)| < \delta.$$

By induction, we get that for all $m \geq 2$

$$\begin{aligned}
&|f(nm) - a^{m^3(1+2+\cdots+(n-1)+1^2+2^2+\cdots+(n-1)^2)} g(m)^{n-1} f(m)| \\
&\leq \delta + a^{m^3((n-1)+(n-1)^2}|g(m)|\delta + a^{m^3((n-2)+(n-2)^2+(n-1)+(n-1)^2)}|g(m)|^2\delta \\
&\quad + \cdots + a^{m^3(1+2+\cdots+(n-1)+1^2+2^2+\cdots+(n-1)^2)}|g(m)|^{n-2}\delta.
\end{aligned} \tag{3}$$

In fact, if the inequality (3) holds, we have

$$\begin{aligned}
&|f((n+1)m) - a^{m^3(1+2+\cdots+n+1^2+2^2+\cdots+n^2)} g(m)^n f(m)| \\
&\leq |f((n+1)m) - a^{m^2(nm)+m(nm)^2} g(m)f(nm)| \\
&+ |f(nm) - a^{m^3(1+2+\cdots+(n-1)+1^2+2^2+\cdots+(n-1)^2)} g(m)^{n-1} f(m)| a^{m^3(n+n^2)}|g(m)| \\
&\leq \delta + a^{m^3(n+n^2)}|g(m)|\delta + a^{m^3((n-1)+(n-1)^2+n+n^2)}|g(m)|^2\delta \\
&\quad + \cdots + a^{m^3(1+2+\cdots+n+1^2+2^2+\cdots+n^2)}|g(m)|^{n-1}\delta
\end{aligned}$$

for all $n \geq 2$. By (3), we get

$$\begin{aligned}
&\left| \frac{f(nm)}{a^{m^3(1+2+\cdots+(n-1)+1^2+2^2+\cdots+(n-1)^2)} g(m)^{n-1} f(m)} - 1 \right| \\
&\leq \Big(\frac{1}{|g(m)|^{n-1}|f(m)|} + \frac{1}{|g(m)|^{n-2}|f(m)|} + \cdots + \frac{1}{|g(m)||f(m)|} \Big)\delta \\
&< \frac{1}{|g(m)||f(m)|}(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots)\delta = \frac{2\delta}{|g(m)||f(m)|} \leq \frac{1}{2}
\end{aligned}$$

for all positive integer n. Since $g(m)^n \to \infty$ as $n \to \infty$,

$$f(nm) \to \infty \quad as \quad n \to \infty.$$

Then for any $x, y \in R$ we have

$$\begin{aligned}
&f(nm)|g(x+y) - a^{x^2y+xy^2}g(x)g(y)| \\
&\leq \Big|a^{(x+y)^2nm+(x+y)(nm)^2}g(x+y)f(nm) \\
&\qquad - f(nm+x+y)\Big|\frac{1}{a^{(x+y)^2nm+(x+y)(nm)^2}} \\
&+ \Big|f(nm+x+y) \\
&\qquad - a^{x^2(y+nm)+x(y+nm)^2}g(x)f(y+nm)\Big|\frac{1}{a^{(x+y)^2nm+(x+y)(nm)^2}} \\
&+ \Big|f(y+nm) - a^{y^2nm+y(nm)^2}g(y)f(nm)\Big||g(x)| \cdot \frac{a^{x^2y+xy^2}}{a^{y^2nm+y(nm)^2}}
\end{aligned}$$

and so

$$\begin{aligned}
&|g(x+y) - a^{x^2y+xy^2}g(x)g(y)| \\
&\leq \frac{2\delta}{f(nm)a^{(x+y)^2nm+(x+y)(nm)^2}} + \frac{a^{x^2y+xy^2}\delta|g(x)|}{f(nm)a^{y^2nm+y(nm)^2}} \to 0
\end{aligned}$$

as $n \to \infty$. Thus it follows that

$$g(x+y) = a^{x^2y+xy^2}g(x)g(y).$$

for any $x, y \in R$. □

Corollary 2. *Let $\delta > 0$ and $a \geq 1$ be given. Let $f : R \to R$ be a function with $f(m) \geq max\{2, 2\sqrt{\delta}\}$ for some integer $m \geq 1$ such that*

$$|f(x+y) - a^{x^2y+xy^2}f(x)f(y)| < \delta$$

for all $x, y \in R$. Then

$$f(x+y) = a^{x^2y+xy^2}f(x)f(y)$$

for all $x, y \in R$.

Corollary 3. *Let $\delta > 0$ and $a \geq 1$ be given. Let $f : R \to R$ be a function such that*

$$|f(x+y) - a^{x^2y+xy^2} f(x)f(y)| < \delta$$

for all $x, y \in R$. Then either f is bounded or

$$f(x+y) = a^{x^2y+xy^2} f(x)f(y)$$

for all $x, y \in R$.

Suppose that $H_p : R \times R \to R$ be a monotonically increasing (in both variables) homogeneous mapping, for which $H_p(tx, ty) = t^p H_p(x, y)$ holds for some $p > 1$, and for all $t, x, y \in R$. For examples, let $H_p(x, y) = ax^p + by^p$ for $a, b, x, y \in R$. Then H_p is a monotonically increasing (in both variables) homogeneous mapping.

Theorem 4. *Let $a \geq 1$ be given. If $f, g : R \to R$ satisfy the functional inequality*

$$|f(x+y) - a^{x^2y+xy^2} g(x)f(y)| \ \leq H_p(x, y),$$

then either $g(x) = o(x^p)$ as $x \to \infty$ or $g(x+y) = a^{x^2y+xy^2} g(x)g(y)$ for every $x, y \in R$.

Proof. By the same method as the proof of Theorem 1, we have

$$\begin{aligned}
&|f(nx) - a^{x^3(1+2+\cdots+(n-1)+1^2+2^2+\cdots+(n-1)^2)} g(x)^{n-1} f(x)| \\
&\leq H_p((n-1)x, x) + H_p((n-2)x, x)|g(x)|a^{x^3((n-1)+(n-1)^2} \\
&+ H_p((n-3)x, x)|g(x)|^2 a^{x^3((n-2)+(n-2)^2+(n-1)+(n-1)^2)} \\
&+ \cdots + H_p(x, x)|g(x)|^{n-2} a^{x^3(1+2+\cdots+(n-1)+1^2+2^2+\cdots+(n-1)^2)}
\end{aligned}$$

for all positive integer $n \geq 2$ and $x > 0$. Since $a \geq 1$, we get

$$\begin{aligned}
&\left|\frac{f(nx)}{g(x)^{n-1} f(x) a^{x^3(1+2+\cdots+(n-1)+1^2+2^2+\cdots+(n-1)^2)}} - 1\right| \leq \sum_{i=2}^{n-1} \frac{H_p(ix, x)}{g(x)^{i-1} f(x)} \\
&\leq \sum_{i=1}^{\infty} \frac{i^p H_p(x, x)}{|g(x)|^i |f(x)|} \leq \frac{H_p(x, x)}{|g(x)||f(x)|} \sum_{i=1}^{\infty} \frac{i^p}{|g(x)|^{i-1}}
\end{aligned}$$

for any positive integer $n \geq 2$ and $x > 0$. Assume that $g(x) \neq o(x^p)$ as $x \to \infty$, that is, there exist some $\alpha > 0$ and a sequence $\{x_k\}$ in R such that $x_k \to \infty$ as $k \to \infty$ and $g(x_k) \geq \alpha {x_k}^p > 1$ for sufficiently large k. Then $\frac{i^p}{g(x_k)^{i-1}} \leq \frac{i^p}{(\alpha x_k^p)^{i-1}}$ and $\sum_{i=1}^{\infty} \frac{i^p}{(\alpha x_k^p)^{i-1}}$ converges. We can then let the series $\sum_{i=1}^{\infty} \frac{i^p}{g(x_k)^{i-1}}$ converge to a value less than $\frac{g(x)f(x)}{2H_p(x,x)}$ by taking sufficiently large k. Thus for some sufficiently large k and any $n \geq 2$, we have

$$\left| \frac{f(nx_k)}{g(x_k)^{n-1} f(x) a^{x^3} \prod_{i=1}^{n-1} a^i \prod_{i=1}^{n-1} a^{i^2}} - 1 \right| < \frac{1}{2}.$$

By the convergence of $\sum_{i=1}^{\infty} \frac{i^p}{g(x_k)^{i-1}}$, we have $\frac{n^p H_p(x_k,x_k)}{g(x_k)^{n-1}} \to 0$ as $n \to \infty$. Then we have

$$\left| \frac{\frac{f(nx_k)}{n^p H_p(x_k,x_k)}}{\frac{g(x_k)^{n-1} f(x) a^{x^3} \prod_{i=1}^{n-1} a^i \prod_{i=1}^{n-1} a^{i^2}}{n^p H_p(x_k,x_k)}} - 1 \right| < \frac{1}{2}$$

and so $\frac{n^p H_p(x_k,x_k)}{f(nx_k)} \to 0$ as $n \to \infty$. If k is sufficiently large, we have

$$\begin{aligned}
& f(nx_k)|g(x+y) - a^{x^2y+xy^2} g(x)g(y)| \\
& \leq \Big| a^{(x+y)^2 nx_k + (x+y)(nx_k)^2} g(x+y) f(nx_k) \\
& \qquad - f(nx_k + x + y) \Big| \frac{1}{a^{(x+y)^2 nx_k + (x+y)(nx_k)^2}} \\
& + \Big| f(nx_k + x + y) \\
& \qquad - a^{x^2(y+nx_k) + x(y+nx_k)^2} g(x) f(y + nx_k) \Big| \frac{1}{a^{(x+y)^2 nx_k + (x+y)(nx_k)^2}} \\
& + \Big| f(y + nx_k) - a^{y^2 nx_k + y(nx_k)^2} g(y) f(nx_k) \Big| |g(x)| \cdot \frac{a^{x^2y+xy^2}}{a^{y^2 nx_k + y(nx_k)^2}} \\
& \leq C_1 H_p(x+y, nx_k) + C_2 H_p(x, y + nx_k) + C_3 H_p(y, nx_k) \\
& \leq (C_1 + C_2 + C_3) H_p(nx_k, nx_k) = (C_1 + C_2 + C_3) n^p H_p(x_k, x_k)
\end{aligned}$$

for some $C_1, C_2, C_3 > 0$ and sufficiently large n. Thus

$$|g(x+y) - a^{x^2y+xy^2} g(x)g(y)| \leq \frac{(C_1 + C_2 + C_3) n^p H_p(x_k, x_k)}{f(nx_k)} \to 0$$

as $n \to \infty$. $\square$

Corollary 5. *Let $a \geq 1$ be given. If $f : R \to R$ satisfies the functional inequality*

$$\left| f(x+y) - a^{x^2y+xy^2} f(x)f(y) \right| \leq H_p(x,y),$$

then either $f(x) = o(x^p)$ as $x \to \infty$ or $f(x+y) = a^{x^2y+xy^2} f(x)f(y)$ for every $x, y \in R$.

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