

**PRECISE ASYMPTOTICS IN LOGLOG LAW FOR  
 $\rho$ -MIXING RANDOM VARIABLES**

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**Abstract.** Let  $X_1, X_2, \dots$  be identically distributed  $\rho$ -mixing random variables with mean zeros and positive finite variances. In this paper, we prove

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \log n} P(|S_n| \geq \epsilon \sqrt{n \log \log n}) = 1,$$

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \log n} P(M_n \geq \epsilon \sqrt{n \log \log n}) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

where  $S_n = X_1 + \dots + X_n$ ,  $M_n = \max_{1 \leq k \leq n} |S_k|$  and  $\sigma^2 = EX_1^2 + 2 \sum_{i=2}^{\infty} E(X_1, X_i) = 1$ .

**1. Introduction**

Let  $X, X_1, X_2, \dots$  be i.i.d. random variables with common distribution function  $F$ , mean zeros and positive finite variances, and set  $S_n = X_1 + \dots + X_n$ ,  $n \geq 1$ . Baum and Katz(1965) proved that for  $p < 2$  and  $r \geq p$ ,

$$(1.1) \quad \sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P(|S_n| \geq \epsilon n^{\frac{1}{p}}) < \infty, \quad \epsilon > 0$$

if and only if  $E|X|^r < \infty$  and, when  $r \geq 1$ ,  $EX = 0$ . For  $r = 2$  and  $p = 1$ , the result reduces to the theorem of Hsu and Robbins(1947, sufficiency) and Erdős(1949, necessity). For  $r = p = 1$ , the famous theorem of Spitzer(1956) was rediscovered. In view of the fact that the sums tend to infinity as  $\epsilon \downarrow 0$ , it is of interest to find the rate, that is, one would be interested in finding appropriate normalizations in terms of functions

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of  $\epsilon$  that yield non-trivial limits. Toward this end, Heyde(1975) proved that

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) = EX^2,$$

whenever  $EX = 0$  and  $EX^2 < \infty$ . By replacing  $n^{\frac{1}{p}}$  in (1.1) by  $\sqrt{n \log n}$ , corresponding results have been given in Gut and Spătaru(2000, Theorem 3).

At this point many papers try to establish similar results related to the law of the iterated logarithm. Sums analogous to those of (1.1) have been considered by Davis(1968). The following result holds: for the sufficiency, see Davis(1968, Theorem 4); for the necessity, see Gut(1980, Theorem 6.2): Suppose that  $EX = 0$  and that  $EX^2 = \sigma^2 < \infty$ . Then

$$(1.2) \quad \sum_{n=3}^{\infty} \frac{1}{n} P(|S_n| \geq \epsilon \sqrt{n \log \log n}) < \infty, \quad \epsilon > \sigma \sqrt{2}.$$

Conversely, if the sum is finite for some  $\epsilon$ , then  $EX = 0$  and  $EX^2 < \infty$ .

Recently, Gut and Spătaru(2000) proved the precise asymptotics in the law of the iterated logarithm for i.i.d. random variables as follows: Suppose that  $EX = 0$  and that  $EX^2 = \sigma^2 < \infty$ . Then

$$(1.3) \quad \lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \log n} P(|S_n| \geq \epsilon \sqrt{n \log \log n}) = \sigma^2.$$

In this paper, we consider the precise asymptotics in the law of the iterated logarithm for identically distributed  $\rho$ -mixing random variables.

Suppose that  $\{X_n, n \geq 1\}$  is a sequence of random variables on a probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ , set  $\mathcal{F}_n^- = \sigma(X_i : 1 \leq i \leq n)$ ,  $\mathcal{F}_n^+ = \sigma(X_i : i \geq n)$ ,

$$(1.4) \quad \rho(n) = \sup_{k \geq 1} \sup_{X \in L_2(\mathcal{F}_k^-)} \sup_{Y \in L_2(\mathcal{F}_{k+n}^+)} \frac{|EXY - EXEY|}{\sqrt{Var(X)Var(Y)}},$$

the sequence  $\{X_n, n \geq 1\}$  is said to be  $\rho$ -mixing if  $\rho(n) \rightarrow 0$  as  $n \rightarrow \infty$ . This definition was introduced by Kolmogorov and Rozanov(1960), and the limiting behaviors of  $\rho$ -mixing sequences have received more and more attention recently.(See Ibragimov(1975), Peligrad(1987), Bradley (1988), Lin and Lu(1996), Huang et al.(2005) and Zhao(2008)).

## 2. Preliminaries

Now we assume that  $\{X_n, n \geq 1\}$  is a sequence of identically distributed random variables with mean zeros and finite variances. Without loss of generality we assume that  $\sigma^2 = EX_1^2 + 2 \sum_{j=2}^\infty Cov(X_1, X_j) = 1$ . Put  $b(\epsilon) = \exp\{\exp(M/\epsilon^2)\}$ , where  $M > 1$ .

**Lemma 2.1(Ibragimov(1975))** Let  $\{X_n, n \geq 1\}$  be a sequence of strictly stationary  $\rho$ -mixing random variables with  $EX_1 = 0$  and  $EX_1^2 < \infty$ . Assume that  $\sigma^2 = EX_1^2 + 2 \sum_{j=2}^\infty Cov(X_1, X_j) < \infty$  and  $\sum_{n=1}^\infty \rho(2^n) < \infty$ . Then

$$\frac{S_n}{\sigma\sqrt{n}} \rightarrow^{\mathcal{D}} N(0, 1) \text{ as } n \rightarrow \infty,$$

where  $S_n = X_1 + \dots + X_n$  and  $\rightarrow^{\mathcal{D}}$  means convergence in distribution.

**Lemma 2.2(Shao(1995))** Let  $\{X_n, n \geq 1\}$  be a sequence of  $\rho$ -mixing random variables with  $EX_n = 0$ . Then, for any  $q \geq 2$ , there exists  $K(q, \rho(\cdot))$  depending only on  $q$  and  $\rho(\cdot)$  such that

$$\begin{aligned} &P(\max_{1 \leq k \leq n} |S_k| \geq x) \\ &\leq \sum_{i=1}^n P(|X_i| \geq y) + Kx^{-q}n^{\frac{q}{2}} \exp(K \sum_{i=1}^{\lfloor \log n \rfloor} \rho(2^i)) \max_{1 \leq i \leq n} \|X_i I(|X_i| \leq y)\|_2^q \\ &\quad + Kx^{-q}n \exp(K \sum_{i=1}^{\lfloor \log n \rfloor} \rho^{2/q}(2^i)) \max_{1 \leq i \leq n} E|X_i|^q I(|X_i| \leq y) \end{aligned}$$

for any  $x > 0$  and  $y > 0$  with  $2n \max_{1 \leq i \leq n} E|X_i| I(|X_i| \leq y) \leq x$ .

**Proposition 2.3** Let  $N$  be a standard normal random variable. Then

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n \geq 3} \frac{1}{n \log n} P(|N| \geq \epsilon \sqrt{\log \log n}) = 1.$$

**Proof** By the fact that  $\int_0^\infty 2yP(|N| \geq y)dy = 1$ , we get

$$\begin{aligned}
& \lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n \geq 3} \frac{1}{n \log n} P(|N| \geq \epsilon \sqrt{\log \log n}) \\
&= \lim_{\epsilon \downarrow 0} \epsilon^2 \int_3^\infty \frac{1}{x \log x} P(|N| \geq \epsilon \sqrt{\log \log x}) dx \\
&\quad (\text{letting } y = \epsilon \sqrt{\log \log x}) \\
&= \lim_{\epsilon \downarrow 0} \int_{\epsilon \sqrt{\log \log 3}}^\infty 2y P(|N| \geq y) dy \\
&= 1.
\end{aligned}$$

□

**Proposition 2.4(Gut and Spătaru(2000))** Let  $b(\epsilon) = \exp\{\exp(M/\epsilon^2)\}$  and  $N$  be a standard normal random variable. Then, we have uniformly with respect to all sufficiently small  $\epsilon > 0$ ,

$$\lim_{M \rightarrow \infty} \epsilon^2 \sum_{n > b(\epsilon)} \frac{1}{n \log n} P(|N| \geq \epsilon \sqrt{\log \log n}) = 0.$$

**Lemma 2.5(Lin and Lu(1996))** Let  $\{X_n, n \geq 1\}$  be a strictly stationary sequence of  $\rho$ -mixing random variables with  $EX_1 = 0$ ,  $0 < EX_1^2 < \infty$  and  $\sum_{n=1}^\infty \rho(2^n) < \infty$ . If  $0 < \sigma^2 = EX_1^2 + 2 \sum_{j=2}^\infty Cov(X_1, X_j) < \infty$ , then  $W_n \Rightarrow W$ , where  $W_n(t) = (\sigma \sqrt{n})^{-1} S_{[nt]}$ ,  $0 \leq t \leq 1$  and  $\Rightarrow$  means weak convergence in  $D[0, 1]$  with the Skorohod topology. In particular,

$$\frac{M_n}{\sigma \sqrt{n}} \rightarrow^{\mathcal{D}} \sup_{0 \leq s \leq 1} |W(s)|.$$

**Proof** See Corollary 4.1.1 in Lin and Lu(1996). □

**Lemma 2.6(Billingsley(1968))** Let  $\{W(t), t \geq 0\}$  be a standard Wiener process and let  $N$  be a standard normal random variable. Then,

for any  $x > 0$ ,

$$\begin{aligned}
 P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq x\right) &= 1 - \sum_{k=-\infty}^{\infty} (-1)^k P((2k-1)x \leq N \leq (2k+1)x) \\
 &= 4 \sum_{k=-\infty}^{\infty} (-1)^k P(N \geq (2k+1)x) \\
 &= 2 \sum_{k=-\infty}^{\infty} (-1)^k P(|N| \geq (2k+1)x).
 \end{aligned}$$

### 3. Results

**Theorem 3.1** Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed  $\rho$ -mixing random variables with  $EX_1 = 0$  and  $0 < EX_1^2 < \infty$ . Assume that  $\sum_{n=1}^{\infty} \rho(2^n) < \infty$  and  $\sigma^2 = EX_1^2 + 2 \sum_{j=2}^{\infty} Cov(X_1, X_j) = 1$ . Then

$$(3.1) \quad \lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \log n} P(|S_n| \geq \epsilon \sqrt{n \log \log n}) = 1.$$

To prove Theorem 3.1 we need the following propositions.

**Proposition 3.2** Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed  $\rho$ -mixing random variables with  $EX_1 = 0$  and  $0 < EX_1^2 < \infty$ . Assume that  $\sum_{n=1}^{\infty} \rho(2^n) < \infty$  and  $\sigma^2 = EX_1^2 + 2 \sum_{j=2}^{\infty} Cov(X_1, X_j) = 1$ . Then uniformly with respect to all sufficiently small  $\epsilon > 0$ ,

$$(3.2) \quad \lim_{M \rightarrow \infty} \epsilon^2 \sum_{n > b(\epsilon)} \frac{1}{n \log n} P(|S_n| \geq \epsilon \sqrt{n \log \log n}) = 0.$$

**Proof** Applying Lemma 2.2 with  $x = \epsilon\sqrt{n \log \log n}$  and  $y = 2\epsilon\sqrt{n \log \log n}$ , we have

$$\begin{aligned}
 (3.3) \quad & \sum_{n>b(\epsilon)} \frac{1}{n \log n} P(|S_n| \geq \epsilon\sqrt{n \log \log n}) \\
 & \leq \sum_{n>b(\epsilon)} \frac{1}{n \log n} [nP(|X_1| \geq 2\epsilon\sqrt{n \log \log n}) \\
 & \quad + K(\epsilon\sqrt{n \log \log n})^{-q} n^{\frac{q}{2}} \exp(K \sum_{i=1}^{\lfloor \log n \rfloor} \rho(2^i)) \\
 & \quad (EX_1^2 I(|X_1| \leq 2\epsilon\sqrt{n \log \log n}))^{\frac{q}{2}} \\
 & \quad + K(\epsilon\sqrt{n \log \log n})^{-q} n \exp(K \sum_{i=1}^{\lfloor \log n \rfloor} \rho^{\frac{2}{q}}(2^i)) \\
 & \quad (EX_1^q I(|X_1| \leq 2\epsilon\sqrt{n \log \log n}))] \\
 & = I_1 + I_2 + I_3.
 \end{aligned}$$

Since  $k > b(\epsilon)$  if and only if  $k < M^{-1}\epsilon^2 k \log \log k$ , it follows that

$$\begin{aligned}
 I_1 & = \sum_{n>b(\epsilon)} \frac{1}{\log n} P(|X_1| \geq 2\epsilon\sqrt{n \log \log n}) \\
 & = \sum_{n>b(\epsilon)} \frac{1}{\log n} \sum_{k \geq n} P(\epsilon\sqrt{k \log \log k} \leq \frac{|X_1|}{2} \\
 & \quad < \epsilon\sqrt{(k+1) \log \log(k+1)}) \\
 & = \sum_{k>b(\epsilon)} [ \sum_{b(\epsilon) < n \leq k} \frac{1}{\log n} ] P(\epsilon\sqrt{k \log \log k} \leq \frac{|X_1|}{2} \\
 & \quad < \epsilon\sqrt{(k+1) \log \log(k+1)}) \\
 & \leq \sum_{k>b(\epsilon)} M^{-1}\epsilon^2 k \log \log k P(\epsilon\sqrt{k \log \log k} \leq \frac{|X_1|}{2} \\
 & \quad < \epsilon\sqrt{(k+1) \log \log(k+1)}) \\
 & \leq \sum_{k>b(\epsilon)} \exp(-M/\epsilon^2)(k - b(\epsilon)) \\
 & \quad \times P(4\epsilon^2 k \log \log k \leq X_1^2 < 4\epsilon^2(k+1) \log \log(k+1)) \\
 & \leq (4M)^{-1} EX_1^2
 \end{aligned}$$

, which yields

$$(3.4) \quad \lim_{M \rightarrow \infty} \epsilon^2 I_1 = 0.$$

For  $I_2$ , we estimate, for  $q > 2$

$$\begin{aligned} I_2 &= \sum_{n > b(\epsilon)} \frac{1}{n \log n} K(\epsilon^2 n \log \log n)^{-\frac{q}{2}} n^{\frac{q}{2}} \exp\left(K \sum_{i=1}^{[\log n]} \rho(2^i)\right) \\ &\quad \times (EX_1^2 I(|X_1| \leq 2\epsilon \sqrt{n \log \log n}))^{\frac{q}{2}} \\ &\leq C \sum_{n > b(\epsilon)} \frac{1}{n \log n} (\epsilon^2 n \log \log n)^{-\frac{q}{2}} n^{\frac{q}{2}} \\ &= C \epsilon^{-q} \int_{b(\epsilon)}^{\infty} \frac{dx}{(x \log x)(\log \log x)^{\frac{q}{2}}} dx \\ &\quad \text{letting } y = \log \log x \\ &= C \epsilon^{-q} \int_{\frac{M}{\epsilon^2}}^{\infty} y^{-\frac{q}{2}} dy \\ &= \frac{C}{\epsilon^2 M^{\frac{q}{2}-1}}, \end{aligned}$$

which yields

$$(3.5) \quad \lim_{M \rightarrow \infty} \epsilon^2 I_2 = \lim_{M \rightarrow \infty} \frac{C}{M^{\frac{q}{2}-1}} = 0.$$

$$\begin{aligned}
I_3 &\leq C \sum_{n>b(\epsilon)} (n \log n)^{-1} (\epsilon^2 n \log \log n)^{-\frac{q}{2}} n EX_1^q I(|X_1| \\
&\leq 2\epsilon \sqrt{n \log \log n}) \\
&\leq C\epsilon^{-q} \sum_{n>b(\epsilon)} (\log n)^{-1} n^{-\frac{q}{2}} (\log \log n)^{-\frac{q}{2}} \\
&\quad \times \sum_{1 \leq j \leq n} EX_1^q I(2\epsilon \sqrt{j \log \log j} \leq |X_1| \\
&\leq 2\epsilon \sqrt{(j+1) \log \log(j+1)}) \\
&\leq C\epsilon^{-q} \sum_{j>b(\epsilon)} E|X_1|^q I(2\epsilon \sqrt{j \log \log j} \leq |X_1| \\
&\leq 2\epsilon \sqrt{(j+1) \log \log(j+1)}) \times \sum_{n=j} n^{-\frac{q}{2}} (\log n)^{-1} \\
&\leq C\epsilon^{-q} \sum_{j>b(\epsilon)} j^{-\frac{q}{2}+1} (\log j)^{-1} (\log \log j)^{-\frac{q}{2}} \\
&\quad \times EX_1^q I(2\epsilon \sqrt{j \log \log j} \leq |X_1| \leq 2\epsilon \sqrt{(j+1) \log \log(j+1)}) \\
&\leq C\epsilon^{-2} \sum_{j>b(\epsilon)} (\log j)^{-1} (\log \log j)^{-1} \\
&\quad \times EX_1^2 I(2\epsilon \sqrt{j \log \log j} \leq |X_1| \leq 2\epsilon \sqrt{(j+1) \log \log(j+1)}) \\
&\leq \frac{C}{M \exp(\frac{M}{\epsilon^2})} E|X_1|^2 I(|X_1| \geq 2\epsilon \sqrt{b(\epsilon) \log \log b(\epsilon)}).
\end{aligned}$$

Notice that, in particular,  $n > b(\epsilon)$ , it follows that

$$(3.6) \quad \epsilon^2 I_3 \leq C\epsilon^2 (M \exp(\frac{M}{\epsilon^2}))^{-1} E|X_1|^2 I(|X_1| \geq 2\epsilon \sqrt{b(\epsilon) \log \log b(\epsilon)}) \rightarrow 0$$

as  $\epsilon \searrow 0$  and  $M \rightarrow \infty$ . Combining (3.3)-(3.6) we obtain (3.2).  $\square$

**Proposition 3.3** Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed  $\rho$ -mixing random variables with  $EX_1 = 0$  and  $EX_1^2 < \infty$ . Assume that  $\sigma^2 = EX_1^2 + 2 \sum_{j=2}^{\infty} Cov(X_1, X_j) = 1$  and  $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ . Then

$$(3.7) \quad \lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n \leq b(\epsilon)} \frac{1}{n \log n} |P(|S_n| \geq \epsilon \sqrt{n \log \log n}) - P(|N| \geq \epsilon \sqrt{\log \log n})| = 0.$$



**Proof** Write  $\Delta_n = \sup_x |P(|S_n| \geq x\sqrt{n}) - P(|N| \geq x)|$ , note that  $P(|N| \geq x)$  is a continuous function for  $x \geq 0$ , and this combined with Lemma 2.2 yields, for any  $x \geq 0$   $\Delta_n = \sup_x |P(|S_n| \geq \sqrt{nx}) - P(|N| \geq x)| \rightarrow 0$  as  $n \rightarrow \infty$ . Then, applying Toeplitz's lemma(see, e.g. Stout(1995) p.120) we have

$$(3.8) \quad \frac{1}{\log \log m} \sum_{n=1}^m \frac{\Delta_n}{n \log n} \rightarrow 0 \text{ as } m \rightarrow \infty,$$

Hence, using (3.8) we obtain

$$\begin{aligned} & \epsilon^2 \sum_{n \leq b(\epsilon)} \frac{1}{n \log n} |P(|S_n| \geq \epsilon \sqrt{n \log \log n}) - P(|N| \geq \epsilon \sqrt{\log \log n})| \\ & \leq \epsilon^2 \sum_{n \leq [b(\epsilon)]} \frac{\Delta_n}{n \log n} \\ & = \epsilon^2 \log \log [b(\epsilon)] \times \frac{1}{\log \log [b(\epsilon)]} \sum_{n \leq b(\epsilon)} \frac{\Delta_n}{n \log n} \\ & \leq M \frac{1}{\log \log [b(\epsilon)]} \sum_{n \leq b(\epsilon)} \frac{\Delta_n}{n \log n} \rightarrow 0 \text{ as } \epsilon \downarrow 0. \end{aligned}$$

□

**Proof of Theorem 3.1** Theorem 3.1 now follows from Propositions 3.2 and 3.3 and the triangle inequality.

**Theorem 3.4** Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed  $\rho$ -mixing random variables with  $EX_1 = 0$  and  $0 < EX_1^2 < \infty$ . Assume that  $0 < \sigma^2 = EX_1^2 + 2 \sum_{j=2}^{\infty} Cov(X_1, X_j) = 1$  and  $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ . Then

$$(3.9) \quad \lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \log n} P(M_n \geq \epsilon \sqrt{n \log \log n}) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

**Proposition 3.5** Let  $\{W(t), t \geq 1\}$  be a standard Wiener process. Then,

$$(3.10) \quad \lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=3}^{\infty} \frac{1}{n \log n} P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon \sqrt{\log \log n}\right) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

**Proof** By Lemma 2.6, for any  $m \geq 1$  and  $x > 0$ ,

$$\begin{aligned} 2 \sum_{k=0}^{2m+1} (-1)^k P(|N| \geq (2k + 1)x) &\leq P(\sup_{0 \leq s \leq 1} |W(s)| \geq x) \\ &\leq 2 \sum_{k=0}^{2m} (-1)^k P(|N| \geq (2k + 1)x), \end{aligned}$$

which yields (3.10) together with Proposition 2.3. □

From Proposition 2.4 and Lemma 2.6 now we obtain the following result.

**Proposition 3.6** Let  $\{W(t), t \geq 1\}$  be a standard Wiener process. Then,

$$(3.11) \quad \lim_{M \rightarrow \infty} \epsilon^2 \sum_{n > b(\epsilon)} \frac{1}{n \log n} P(\sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon \sqrt{\log \log n}) = 0.$$

**Proof**

$$\begin{aligned} &\lim_{M \rightarrow \infty} \epsilon^2 \sum_{n > b(\epsilon)} \frac{1}{n \log n} P(\sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon \sqrt{\log \log n}) \\ &\leq C \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)^2} \lim_{M \rightarrow \infty} \epsilon^2 \sum_{n > b(\epsilon)} \frac{1}{n \log n} P(|N| \geq \epsilon \sqrt{\log \log n}) \\ &= 0 \text{ by Proposition 2.4.} \end{aligned}$$

□

**Proposition 3.7** Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed  $\rho$ -mixing random variables satisfying conditions of Theorem 3.4. Then, uniformly with respect to all sufficiently small  $\epsilon > 0$

$$(3.12) \quad \lim_{M \rightarrow \infty} \epsilon^2 \sum_{n > b(\epsilon)} \frac{1}{n \log n} P(M_n \geq \epsilon \sqrt{n \log \log n}) = 0.$$

**Proof** By the similar proof to that of Proposition 3.2, (3.12) follows. □

**Proposition 3.8** Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed  $\rho$ -mixing random variables satisfying conditions of Theorem 3.4. Then,

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n \leq b(\epsilon)} \frac{1}{n \log n} |P(|M_n| \geq \epsilon \sqrt{n \log \log n}) - P(\sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon \sqrt{\log \log n})|$$

$> \epsilon\sqrt{\log \log n} = 0$ .

**Proof** Denote  $\Delta_n = \sup_x |P(M_n \geq x\sqrt{n}) - P(\sup_{0 \leq s \leq 1} |W(s)| > x)|$ . We can easily get that  $\Delta_n \rightarrow 0$  as  $n \rightarrow \infty$  by Lemma 2.5.

Applying Toeplitz lemma [Stout(1995), p. 120], we have

$$\frac{1}{\log \log m} \sum_{n=1}^m \frac{\Delta_n}{n \log n} \rightarrow 0, \text{ as } m \rightarrow \infty.$$

Hence,

$$\begin{aligned} & \epsilon^2 \sum_{n \leq b(\epsilon)} \frac{\Delta_n}{n \log n} \\ = & \epsilon^2 \log \log [b(\epsilon)] \times \frac{1}{\log \log [b(\epsilon)]} \sum_{n \leq b(\epsilon)} \frac{\Delta_n}{n \log n} \\ \leq & M \frac{1}{\log \log [b(\epsilon)]} \sum_{n \leq b(\epsilon)} \frac{\Delta_n}{n \log n} \rightarrow 0 \text{ as } \epsilon \searrow 0. \end{aligned}$$

□

**Proof of Theorem 3.4** By Propositions 3.5-3.8 and triangle inequality the result (3.9) follows.

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