Honam Mathematical J. **32** (2010), No. 3, pp. 515–523

THE ROLE OF $T(X)$ IN THE IDEAL THEORY OF Q-ALGEBRAS

Sun Shin Ahn and Seung Eel Kang

Abstract. In this paper, we introduced a special set, called T-part, in a Q-algebra X. We also show that the T-part of X is a subalgebra of X. By using T-part, we provide an equivalent condition that every ideal is a T-ideal.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4,5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2,3] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([8]) introduced the notion of d-algebras, i.e., (I) $x*x = 0$; (VII) $0*x = 0$; (VI) $x*y = 0$ and $y*x = 0$ imply $x = y$, which is another useful generalization of BCK-algebras, and investigated several relations between d-algebras and BCK-algebras, and then investigated other relations between d-algebras and oriented digraphs. On the while, Y. B. Jun, E. H. Roh and H. S. Kim ([6]) introduced a new notion, called a BH-algebra, i.e., (I) $x*x = 0$; (II) $x*0 = x$; (VI) $x*y = 0$ and $y*x = 0$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras, and showed that there is a maximal ideal in bounded BH-algebras. J. Neggers, S. S. Ahn and H. S. Kim ([7]) introduced a new notion, called an Q-algebra, which is a generalization of $BCH/BCI/BCK$-algebras, and generalized some theorems discussed in BCI-algebras.

In this paper, we introduced a special set, called T-part, in a Q-algebra X. We also show that the T-part of X is a subalgebra of X.

Received June 14, 2010. Accepted July 14, 2010.
2000 Mathematics Subject Classification. 06F35, 03G25.
Key words and phrases. atom, T-part, T-ideal, QS-algebra, Q-algebra.

By using T-part, we provide an equivalent condition that every ideal is a T-ideal.

2. Preliminaries

A Q-*algebra* ([7]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms:

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

For brevity we also call X a Q-*algebra*. In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$.

In a Q-algebra X the following property holds:

(IV) $(x * (x * y)) * y = 0$, for any $x, y \in X$.

A BCK-algebra is a Q-algebra X satisfying the additional axioms:

(V) $((x * y) * (x * z)) * (z * y) = 0$,
(VI) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(VII) $0 * x = 0$,

for all $x, y, z \in X$.

Definition 2.1([7]). Let $(X; *, 0)$ be a Q-algebra and $\emptyset \neq I \subset X$. I is called a *subalgebra* of X if

(S) $x * y \in I$ whenever $x \in I$ and $y \in I$.

I is called an *ideal* of X if it satisfies:

(Q_0) $0 \in I$,
(Q_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

A Q-algebra X is called a QS-*algebra* ([1]) if it satisfies the following identity:

$$(x * y) * (x * z) = z * y, \qquad \text{for any } x, y, z \in X.$$

Example 2.2([1]). Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{nz | z \in \mathbb{Z}\}$, where $n \in \mathbb{Z}$. Then $(\mathbb{Z}; -, 0)$ and $(n\mathbb{Z}; -, 0)$ are both Q-algebras and QS-algebras, where "$-$" is the usual subtraction of integers. Also, $(\mathbb{R}; -, 0)$ and $(\mathbb{C}; -, 0)$ are Q-algebras and QS-algebras where $\mathbb{R}$ is the set of all real numbers, $\mathbb{C}$ is the set of all complex numbers.

Example 2.3. (1) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then X is a Q-algebra, but not a QS/BCI-algebra, since $(2*0)*(2*1) = 2 \neq 1 = 1 * 0$.
(2) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then X is both a Q-algebra and QS-algebra.
(3) Let $X = \{0, 1, 2\}$ be a set with the table as follows:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Then X is both a Q-algebra and BCI-algebra , but not a QS-algebra, since $(0 * 1) * (0 * 2) = 0 \neq 1 = 2 * 1$.

Lemma 2.4. *Every Q-algebra X satisfies the following property:*

$$0 * (x * y) = (0 * x) * (0 * y) \text{ for any } x, y \in X.$$

Proof. For any $x, y \in X$, we have

$$\begin{aligned} 0 * (x * y) =&((0 * y) * (0 * y)) * (x * y) \\ =&((0 * y) * (x * y)) * (0 * y) \\ =&(((x * y) * x) * (x * y)) * (0 * y) \\ =&(((x * y) * (x * y)) * x) * (0 * y) \\ =&(0 * x) * (0 * y). \end{aligned}$$

This competes the proof. □

3. T-parts and T-ideals

In the following, let X denote a Q-algebra unless otherwise specified.

Definition 3.1. Let X be a Q-algebra. The set

$$T(X) := \{y \in X | y = (0 * x) * x \text{ for some } x \in X\}$$

is called the *T-part* of X.

Clearly, $0 \in T(X)$.

Theorem 3.2. *Let X be a Q-algebra. Then $T(X)$ is a subalgebra of X.*

Proof. Let $a, b \in T(X)$. Then $a = (0 * x) * x$ and $b = (0 * y) * y$ for some $x, y \in X$. Thus

$$\begin{aligned} a * b =& ((0 * x) * x) * ((0 * y) * y) \\ =& ((0 * ((0 * y) * y)) * x) * x) \\ =& [((0 * (0 * y)) * (0 * y)) * x] * x \\ =& [((0 * x) * (0 * y)) * (0 * y)] * x \\ =& [((0 * (x * y)) * (0 * y)] * x \\ =& [(0 * (0 * y)) * x] * (x * y) \\ =& [(0 * x) * (0 * y)] * (x * y) \\ =& [0 * (x * y)] * (x * y). \end{aligned}$$

Hence $a * b \in T(X)$, which completes the proof. □

In general, the T-part of a Q-algebra X may not be an ideal of X as shown in the following example.

Example 3.3. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the table as follows:

$*$	0	1	2	3	4
0	0	0	3	2	3
1	1	0	3	2	3
2	2	2	0	3	0
3	3	3	2	0	2
4	4	2	1	3	0

Then $(X; *, 0)$ is a Q-algebra. $T(X) = \{0, 2, 3\}$ is not an ideal of X, since $1 * 2 = 3 \in T(X)$, but $1 \notin T(X)$.

Definition 3.4. An element a is an *atom* of X if, for all $x \in X$, $x * a = 0$ implies $x = a$.

Obviously, 0 is an atom of X. The set of all atoms of X is denoted by $L(X)$.

Theorem 3.5. *Let X be a Q-algebra. Then for all x, y, z, u of X, the following conditions are equivalent:*

(i) *x is an atom;*
(ii) $x = z * (z * x)$;
(iii) $(z * u) * (z * x) = x * u$;
(iv) $x * (z * y) = y * (z * x)$;
(v) $0 * (z * x) = x * z$;
(vi) $0 * (0 * x) = x$;
(vii) $0 * (0 * (x * z)) = x * z$;
(viii) $z * (z * (x * u)) = x * u$.

Proof. (i)⇒(ii): Let x be an atom of X. Since $z * (z * x) \leq x$, we have $x = z * (z * x)$ for any $x, z \in X$.
(ii)⇒(iii): Using (ii), we get $(z * u) * (z * x) = (z * (z * x)) * u = x * u$ for any $x, z, u \in X$.
(iii)⇒(iv): Replacing u by $z * y$ in (iii), we obtain $x * (z * y) = (z * (z * y)) * (z * x) = (z * (z * x)) * (z * y) = y * (z * x)$. Hence (iv) holds.
(iv)⇒(v): Put $y := 0$ in (iv). Then $x * (z * 0) = 0 * (z * x)$. It follows from $z * 0 = z$ that $x * z = 0 * (z * x)$.
(v)⇒(vi): Set $z := 0$ in (v). Then $0 * (0 * x) = x * 0 = x$. Thus we have $0 * (0 * x) = x$.
(vi)⇒(vii): For any $x, z \in X$, we have

$$\begin{aligned} 0 * (0 * (x * z)) &= 0 * ((0 * x) * (0 * z)) \\ &= (0 * (0 * x)) * (0 * (0 * z)) \\ &= x * z. \end{aligned}$$

Hence $0 * (0 * (x * z)) = x * z$.
(vii)⇒(viii): For any $x, z, u \in X$, we get

$$\begin{aligned} x * u &= 0 * (0 * (x * u)) \\ &= 0 * ((z * z) * (x * u)) \\ &= 0 * ((z * (x * u)) * z) \\ &= [0 * (z * (x * u))] * (0 * z) \\ &= (0 * (0 * z)) * (z * (x * u)) \\ &= (0 * (0 * (z * 0))) * (z * (x * u)) \\ &= (z * 0) * (z * (x * u)) \\ &= z * (z * (x * u)). \end{aligned}$$

Thus (viii) holds.

(viii)⇒(i): If $z*x = 0$, then by (viii), we have $x = x*0 = z*(z*(x*0)) = z*(z*x) = z*0 = z$. This shows that x is an atom. The proof is complete. □

Corollary 3.6. *Let X be a Q-algebra. If a is an atom of X, then for all x of X, $a*x$ is an atom. Hence $L(X)$ is a subalgebra of X. For every x of X, there is an atom a such that $a \leq x$, i.e., every Q-algebra is generated by atoms.*

Proposition 3.7. *Let X be a QS-algebra. Then $T(X)$ is an ideal of X.*

Proof. Let $x*y, y \in T(X)$ for any $x, y \in X$. It follows from Theorem 3.5 that $x = y*(y*x) = y*(0*(x*y))$. By Theorem 3.2, $0*(x*y) \in T(X)$. Since $y*x = 0*(x*y), y \in T(X)$, by Theorem 3.2 we have $x \in X$. Thus $T(X)$ is an ideal of X. □

Lemma 3.8. *Let X be a Q-algebra. A non-zero element $a \in X$ is an atom of X if $\{0, a\}$ is an ideal of X.*

Proof. Assume that $x \leq a$ for any $x \in X$. Then $x*a = 0 \in \{0, a\}$. Since $\{0, a\}$ is an ideal of X, we have $x \in \{0, a\}$. Hence $x = 0$ or $x = a$. Thus a is an atom of X. □

The converse of Lemma 3.8 is not true as seen in the following example.

Example 3.9. Let $X = \{0, 1, 2\}$ be a Q-algebra as in Example 2.3(2). Then an element 2 of X is an atom, but $\{0, 2\}$ is not an ideal since $1*2 = 2 \in \{0, 2\}$, but $1 \notin \{0, 2\}$.

Lemma 3.10. *If every non-zero element of a Q-algebra X is an atom, then any subalgebra of X is an ideal of X.*

Proof. Let S be a subalgebra of X and let $x, y*x \in S$ for any $y \in X$. If follows from Theorem 3.5 that $y = x*(x*y) = x*(0*(y*x))$. Since $0, y*x \in S$ and S is a subalgebra of X, we have $0*(y*x) \in S$. Hence $y = x*(0*(y*x)) \in S$. Thus any subalgebra of X is an ideal of X. □

From the above Lemmas we obtain the following theorem.

Theorem 3.11. *A Q-algebra contains only atoms if and only if every its subalgebra is an ideal.*

Proof. The necessity follows from Lemma 3.10. Conversely, assume that every its subalgebra is an ideal. Let $S := \{0, a\}$ be a subalgebra of X for any $0 \neq a \in X$. By assumption, S is an ideal of X. It follows from Lemma 3.8 that a is an atom of X. □

For any atom a of X, the set $V(a) := \{x \in X | a \leq x\}$ is called a *branch* of X.

Theorem 3.12. *Let X be a Q-algebra and suppose a and b are atoms of X. Then the following properties hold:*
(i) *For all $x \in V(a)$ and all $y \in V(b), x * y \in V(a * b)$,*
(ii) *For all x and $y \in V(a)$, $x*y \in B(X)$, where $B(X) := \{x \in X | 0 \leq x\}$,*
(iii) *If $a \neq b$, then for all $x \in V(a)$ and $y \in V(b)$, we have $x * y \notin B(X)$,*
(iv) *For all $x \in V(b)$, $a * x = a * b$,*
(v) *If $a \neq b$, then $V(a) \cap V(b) = \emptyset$.*

Proof. (i) For all $x \in V(a)$ and all $y \in V(b)$, by Theorem 3.5 we have

$$\begin{aligned}(a * b) * (x * y) =&(0 * (0 * (a * b))) * (x * y)\\ =&(0 * (x * y)) * (0 * (a * b))\\ =&[(0 * x) * (0 * y)] * (0 * (a * b))\\ =&[(0 * (0 * (a * b))) * x] * (0 * y)\\ =&[(a * b) * x] * (0 * y)\\ =&[(a * x) * b] * (0 * y)\\ =&(0 * b) * (0 * y)\\ =&0 * (b * y) = 0 * 0 = 0.\end{aligned}$$

Hence $x * y \in V(a * b)$. Thus (i) holds.
(ii) and (iii) are simple consequences of (i).
(iv) For all $x \in V(b)$, by Theorem 3.5 we have $(a * x) * (a * b) = (a * (a * b)) * x = b * x = 0$. Moreover, $a * b$ is an atom by Corollary 3.6. Therefore $a * x = a * b$. Therefore (iv) holds.
(v) Let $a \neq b$ and $V(a) \cap V(b) \neq \emptyset$. Then there exists $c \in V(a) \cap V(b)$. By (i), we have $0 = c * c \in V(a * b)$ and so $a * b = 0$, which is a contradiction. Thus (v) is true. □

Definition 3.13. A non-empty subset A of a Q-algebra X is called a *T-ideal* of X if it satisfies

(i) $0 \in A$,
(ii) $x * (y * z) \in A$ and $y \in A$ imply $x * z \in A$ for all $x, y, z \in X$.

Every T-ideal of a Q-algebra is an ideal, but not converse. In fact, consider the Q-algebra $X := \{0, 1, 2, 3, 4\}$ as in Example 3.3. The set $A := \{0, 1\}$ is an ideal of X but not a T-ideal of X, since $4 * (0 * 3) = 1 \in A$, but $4 * 3 = 3 \notin A$.

Example 3.14. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	1	3
2	2	2	0	3
3	3	3	3	0

It is easily checked that $(X; *, 0)$ is a Q-algebra. Then $\{0, 1, 2\}$ is a T-ideal of X.

Lemma 3.15. *Let A be a T-ideal of a Q-algebra X. Then $(0*x)*x \in A$ for all $x \in A$.*

Proof. Straightforward. □

Theorem 3.16. *Let A be an ideal of a QS-algebra X. Then A is a T-ideal of X if and only if $T(X) \subseteq A$.*

Proof. Necessity follows from Lemma 3.15. Conversely, suppose that $T(X) \subseteq A$. Let $x * (y * z) \in A$ and $y \in A$ for all $x, y, z \in X$. Since X is a QS-algebra, we have

$$\begin{aligned}[(x*z)*(x*(y*z))]*y =& ((y*z)*z)*y \\ =& ((y*z)*y)*z \\ =& ((y*y)*z)*z \\ =& (0*z)*z \in T(X) \subseteq A.\end{aligned}$$

Hence $x * z \in A$, since A is an ideal of a QS-algebra X. Thus A is a T-ideal of X, completing the proof. □

Corollary 3.17. *Let A and B be ideals of a QS-algebra. If $A \subseteq B$ and A is a T-ideal of X, then B is also a T-ideal of X.*

Proof. Straightforward. □

References

[1] S. S. Ahn and H. S. Kim, *On QS-algebras*, J. Chungcheong Math. Soc., **12**(1999), 33-41.
[2] Q. P. Hu and X. Li, *On BCH-algebras*, Math. Seminar Notes, **11**(1983), 313-320.
[3] Q. P. Hu and X. Li, *On proper BCH-algebras*, Math. Japon., **30**(1985), 659-661.
[4] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon., **23**(1978), 1-26.
[5] K. Iséki, *On BCI-algebras*, Mathe. Seminar Notes, **8**(1980), 125-130.

[6] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Sci. Mathematica , **1**, (1998), 347-354.
[7] J. Neggers, S. S. Ahn and H. S. Kim, *On Q-algebras*, Inter. J. Math. Math. Sci., **27**(2001), 749-757.
[8] J. Neggers and H. S. Kim, *On d-algebras*, Math. Slovaca, **49** (1999), 19-26.

Sun Shin Ahn
Department of Mathematics Education
Dongguk University
Seoul 100-715, Korea
E-mail address: sunshine@dongguk.edu

Seung Eel Kang
Department of Mathematics Education
Dongguk University
Seoul 100-715, Korea
E-mail address: seungeel@naver.com