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A GENERALIZATION OF THE ADAMS-BASHFORTH METHOD

Nahmwoo Hahm and Bum Il Hong†

Abstract. In this paper, we investigate a generalization of the Adams-Bashforth method by using the Taylor's series. In case of m-step method, the local truncation error can be expressed in terms of $m-1$ coefficients. With an appropriate choice of coefficients, the proposed method has produced much smaller error than the original Adams-Bashforth method. As an application of the generalized Adams-Bashforth method, the accuracy performance is demonstrated in the satellite orbit prediction problem. This implies that the generalized Adams-Bashforth method is applied to the orbit prediction of a low-altitude satellite. This numerical example shows that the prediction of the satellite trajectories is improved one order of magnitude.

1. Introduction

A multistep integration method is preferable to a single step method since the multistep integrator requires relatively small number of function evaluations. The Adams-Bashforth method is strongly stable in terms of round-off errors so that it produces relatively accurate approximation solutions. Therefore, the strong stability is the main reason for choosing the Adams-Bashforth method in many packages [1].

In this paper, the general multistep method associated with error control parameters will be formulated. Then, the Adams-Bashforth method becomes a special case of the general method. Although there may exist infinitely many strong stable multistep methods, only the method that

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† Corresponding author.
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can produce smaller local truncation error than the original Adams-Bashforth method will be considered.

For the comparison purposes, the two body problem of the Earth's satellite is integrated numerically. The two body problem is considered since it has an exact solution that can be used in the error quantification. The satellite for the numerical integration is a Geoscience Laser Altimeter System type low-altitude satellite [4].

2. Generalized Adams-Bashforth Method

This paper is concerned with approximating the solution $y(t)$ to the general first-order initial-value problem of the form

$$y' = f(t, y); \quad t_0 \le t \le t_N, \ \ y(t_0) = y_0. \tag{2.1}$$

Suppose that the mesh points $\{t_0, t_1, \ldots, t_N\}$ are selected with the uniform step size h, where

$$t_i = t_0 + ih \quad \text{for} \ \ i = 0, 1, \ldots, N \tag{2.2}$$

and the approximation $y_i \approx y(t_i)$ is specified for each $i = 0, 1, \ldots, m-1$.

The m-step Generalized Adams-Bashforth (GAB) method for solving the problem is represented by the difference equation

$$y_{i+1} = \sum_{k=0}^{m-1} a_k y_{i-k} + h \sum_{k=0}^{m-1} b_k f(t_{i-k}, y_{i-k}) \tag{2.3}$$

for $i = m-1, m, \ldots, N-1$, where $\{a_0, a_1, \ldots, a_{m-1}\}$ and $\{b_0, b_1, \ldots, b_{m-1}\}$ are constants to be determined. In practice, each a_k should be constrained to provide the method with the roundoff stability. To be a strongly stable method, the roots of the characteristic equation

$$\lambda^m - a_0\lambda^{m-1} - a_1\lambda^{m-2} - \cdots - a_{m-1} = 0 \tag{2.4}$$

must satisfy the following root conditions:

Criterion 1 $\lambda = 1$ is a simple root and is the only root of magnitude one.

Criterion 2 All roots except $\lambda = 1$ have absolute value less than 1.

In this section, it will be shown that each b_k can be expressed as a linear combination of $\{a_1, \ldots, a_{m-1}\}$. So it is sufficient to determine $\{a_1, \ldots, a_{m-1}\}$ only, since a_0 can be obtained by applying Criterion 1 to

(2.4). It is known that

$$1 = \sum_{k=0}^{m-1} a_k \tag{2.5}$$

or explicitly,

$$a_0 = 1 - \sum_{k=1}^{m-1} a_k. \tag{2.6}$$

The Taylor's series will be useful to derive the GAB method. The Taylor's series for y_{i+1} is given by

$$y_{i+1} = y_i + \sum_{j=1}^{\infty} D_{i,j} h^j \tag{2.7}$$

where $D_{i,j}$ is the j-th derivative of y at t_i divided by $j!$; that is,

$$D_{i,j} = \frac{1}{j!} \frac{d^j y}{dt^j} (t_i). \tag{2.8}$$

Since $t_{i-k} = t_i - kh$, the Taylor's series for y_{i-k} becomes

$$y_{i-k} = y_i + \sum_{j=1}^{\infty} (-k)^j D_{i,j} h^j. \tag{2.9}$$

By regarding f as a function of t, the relationship $f = y'$ results in the Taylor's series of the following form

$$f(t_{i-k}, y_{i-k}) = \sum_{j=1}^{\infty} j(-k)^{j-1} D_{i,j} h^{j-1}. \tag{2.10}$$

Substituting (2.7), (2.9) and (2.10) into (2.3) and equating coefficients of like powers of h yield the system of equations,

$$1 = \sum_{k=0}^{m-1} (jb_k - ka_k)(-k)^{j-1} \quad \text{for } j = 1, 2, \ldots, m. \tag{2.11}$$

The local truncation error $\tau_{i+1}(h)$ at this step is

$$\begin{aligned} \tau_{i+1}(h) &= \big(y_{i+1} - \sum_{k=0}^{m-1} a_k y_{i-k}\big)/h - \sum_{k=0}^{m-1} b_k f(t_{i-k}, y_{i-k}) \\ &= \left\{ 1 - \sum_{k=0}^{m-1} \big[(m+1)b_k - ka_k\big](-k)^m \right\} D_{i,m+1} h^m. \end{aligned} \tag{2.12}$$

Note that the original Adams-Bashforth (AB) method is obtained by taking $a_1 = \cdots = a_{m-1} = 0$. Consider the equation when $j = 1$ in (2.11). It is given by

$$1 = \sum_{k=0}^{m-1} (b_k - k a_k) \tag{2.13}$$

which implies that the sum of all b_k equals 1 in the AB method. Hence,

$$\text{AB method:} \quad 1 = \sum_{k=0}^{m-1} b_k. \tag{2.14}$$

To derive the GAB method, it is convenient to use vectors and matrices. Let $\tilde{\mathbf{a}}$, $\mathbf{b}$ and $\mathbf{1}$ be m-dimensional column vectors:

$$\tilde{\mathbf{a}} = \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{bmatrix} \text{ and } \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{2.15}$$

Also, let $\mathbf{A}$ and $\mathbf{B}$ be $m \times m$ matrices defined by

(2.16)

$$\mathbf{A} = \begin{bmatrix} & \overset{(k+1)\text{-th}}{\vdots} & \\ \cdots & -(-k)^j & \cdots \\ & \vdots & \end{bmatrix} j\text{-th} \text{ and } \mathbf{B} = \begin{bmatrix} & \overset{(k+1)\text{-th}}{\vdots} & \\ \cdots & j(-k)^{j-1} & \cdots \\ & \vdots & \end{bmatrix} j\text{-th}$$

for $j = 1, 2, \ldots, m$ and $k = 0, 1, \ldots, m-1$.

In fact, a_0 disappears in (2.11) because its coefficient is actually zero. Thus the system of equations in (2.11) can be written in a vector-matrix form,

$$\mathbf{1} = \mathbf{B}\mathbf{b} - \mathbf{A}\tilde{\mathbf{a}}. \tag{2.17}$$

Since the first column of $\mathbf{A}$ is a zero vector, it will be replaced by $\mathbf{1}$ to obtain a new matrix $\tilde{\mathbf{A}}$. The following expression illustrates how to construct $\tilde{\mathbf{A}}$ explicitly:

$$\tilde{\mathbf{A}} = \left[\mathbf{1} \middle| \mathbf{A}_{j,k+1}\right] \quad \text{for } j = 1, 2, \ldots, m; \; k = 1, \ldots, m-1. \tag{2.18}$$

Here, the vertical line is used to separate the column replacement from the other original columns of $\mathbf{A}$. It is preferable to use $\tilde{\mathbf{A}}$ rather than $\mathbf{A}$, since

$$\tilde{\mathbf{A}}\tilde{\mathbf{a}} = \mathbf{1} + \mathbf{A}\tilde{\mathbf{a}}. \tag{2.19}$$

Consequently, $\mathbf{b}$ can be expressed in a simple form when (2.19) is applied to (2.17); that is,

$$\mathbf{b} = \tilde{\mathbf{C}}\tilde{\mathbf{a}} \tag{2.20}$$

where $\tilde{\mathbf{C}}$ is a coefficient matrix computed by

$$\tilde{\mathbf{C}} = \mathbf{B}^{-1}\tilde{\mathbf{A}}. \tag{2.21}$$

Numerical values of $\tilde{\mathbf{C}}$ are provided in Appendix A. Equation (2.20) implies that b_k for each $k = 0, 1, \ldots, m-1$ is a linear combination of $\{a_1, \ldots, a_{m-1}\}$.

Let $\mathbf{b}_0$ and $\mathbf{C}$ be a column vector and a matrix, respectively;

$$\begin{aligned} \mathbf{b}_0 &= \mathbf{B}^{-1}\mathbf{1}, \\ \mathbf{C} &= \mathbf{B}^{-1}\mathbf{A}. \end{aligned} \tag{2.22}$$

Analogous to $\tilde{\mathbf{A}}$, the values of $\tilde{\mathbf{C}}$ agree with those in $\mathbf{C}$ except in the first column. The first column of $\tilde{\mathbf{C}}$ is $\mathbf{b}_0$ so that $\tilde{\mathbf{C}}$ can be constructed as

$$\tilde{\mathbf{C}} = \left[\mathbf{b}_0 \middle| \mathbf{C}_{j,k+1}\right] \quad \text{for } j = 1, 2, \ldots, m; \; k = 1, \ldots, m-1. \tag{2.23}$$

Now, it is clear that $\mathbf{b}$ can be expressed in an additional form

$$\mathbf{b} = \mathbf{b}_0 + \mathbf{C}\tilde{\mathbf{a}} \tag{2.24}$$

and it is obvious that $\mathbf{b} = \mathbf{b}_0$ in the AB method; namely,

$$\text{AB method:} \quad \tilde{\mathbf{a}}^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \text{ and } \mathbf{b} = \mathbf{b}_0. \tag{2.25}$$

As shown in (2.14), $\mathbf{1}^T\mathbf{b}_0 = 1$. It is interesting to compare (2.13) to (2.24). In doing so, one can show that

$$\mathbf{1}^T\mathbf{C} = \begin{bmatrix} 0 & 1 & \cdots & m-1 \end{bmatrix}. \tag{2.26}$$

3. Error Analysis

The main purpose of this paper is to find numerical values of $\tilde{\mathbf{a}}$ such that the local truncation error $\tau_{i+1}(h)$ in (2.12) becomes smaller than that of the AB method. For convenience, define δ as the coefficient of $\tau_{i+1}(h)$:

$$\begin{aligned} \tau_{i+1}(h) &= \delta D_{i,m+1} h^m, \\ \delta &= 1 - \sum_{k=0}^{m-1} \left[(m+1)b_k - k a_k\right](-k)^m. \end{aligned} \tag{3.1}$$

Let $\mathbf{c}$ and $\mathbf{d}$ be m-dimensional column vectors defined by

$$(3.2)\qquad \mathbf{c} = \begin{bmatrix} \vdots \\ -(-k)^{m+1} \\ \vdots \end{bmatrix} (k+1)\text{-th and } \mathbf{d} = \begin{bmatrix} \vdots \\ -(m+1)(-k)^{m} \\ \vdots \end{bmatrix} (k+1)\text{-th}$$

for $k = 0, 1, \ldots, m-1$. Then, δ in (3.1) can be represented by these vectors,

$$(3.3)\qquad \delta = 1 + \mathbf{d}^T\mathbf{b} + \mathbf{c}^T\tilde{\mathbf{a}}.$$

In fact, δ can be written in many ways by applying (2.20) or (2.24) to (3.3).

$$(3.4)\qquad \begin{aligned} \delta &= 1 + (\mathbf{d}^T\tilde{\mathbf{C}} + \mathbf{c}^T)\tilde{\mathbf{a}} \\ &= 1 + \mathbf{d}^T\mathbf{b}_0 + (\mathbf{d}^T\mathbf{C} + \mathbf{c}^T)\tilde{\mathbf{a}}. \end{aligned}$$

To get more simple form of δ, define a coefficient δ_0 and a column vector $\mathbf{e}$ as follows.

$$(3.5)\qquad \begin{aligned} \delta_0 &= 1 + \mathbf{d}^T\mathbf{b}_0, \\ \mathbf{e}^T &= \mathbf{d}^T\mathbf{C} + \mathbf{c}^T. \end{aligned}$$

Obviously, (3.4) is equivalent to

$$(3.6)\qquad \delta = \delta_0 + \mathbf{e}^T\tilde{\mathbf{a}}.$$

The first entry of $\mathbf{e}$ is zero, so it is replaced by δ_0 to get a new column vector $\tilde{\mathbf{e}}$ which is given by

$$(3.7)\qquad \tilde{\mathbf{e}}^T = \left[\delta_0 \middle| \quad \mathbf{e}^T_{k+1}\right] \text{ for } k = 1, 2, \ldots, m-1.$$

Then, δ can be written in a compact form

$$(3.8)\qquad \delta = \tilde{\mathbf{e}}^T\tilde{\mathbf{a}}.$$

Numerical values of $\tilde{\mathbf{e}}$ are provided in Appendix A.

In the AB method, it is evident that $\mathbf{e}^T\tilde{\mathbf{a}} = 0$. Therefore the local truncation error is as follows,

$$(3.9)\qquad \text{AB method:} \quad \delta = \delta_0 \text{ and } \tau_{i+1}(h) = \delta_0 D_{i,m+1}h^m.$$

4. Numerical Example

The satellite motion of the two body model is selected for a numerical example. This problem has an exact solution that can be used for the accuracy assessment of the GAB method. The equation of a satellite orbit prediction problem is given by

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -(\mu/r^3)\mathbf{r} \end{bmatrix} \tag{4.1}$$

where $\mathbf{r}$ and $\mathbf{v}$ are position and velocity vectors, respectively; μ is a gravitational constant and r is the magnitude of $\mathbf{r}$.

The satellite in this investigation has a low altitude about 800 km. The initial condition of the orbit in SI unit is

$$\mathbf{r}_0 = \begin{bmatrix} 7082414.740 \\ 3.957 \\ -56.618 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_0 = \begin{bmatrix} -9.567 \\ -1039.545 \\ 7485.424 \end{bmatrix}. \tag{4.2}$$

For a practical reason, low-step methods are not popular in the orbit prediction problem. Thus, the 7-step method is applied to this example.

As shown in (A.6), all entries of $\tilde{\mathbf{e}}$ are non-positive except for the first element. Therefore, the error of the GAB method should be less than or equal to the error of the AB method in each case of

$$\tilde{\mathbf{a}}_1 = \begin{bmatrix} 1 \\ a_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_2 = \begin{bmatrix} 1 \\ 0 \\ a_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ a_4 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_5 \\ 0 \end{bmatrix}, \ \tilde{\mathbf{a}}_6 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ a_6 \end{bmatrix} \tag{4.3}$$

for $a_k \geq 0$; $k = 1, 2, \ldots, 6$. Figure 1 shows the results of these when the a_k is increased by 0.1 from 0 to 1. The error is represented by the positional root mean squares (rms). As the a_k approaches to 1, it becomes unstable because of the Criterion 2. Since the unstable method is not necessary, unstable cases are omitted in the figure.

Note that the last entry of $\tilde{\mathbf{e}}$ in (A.6) is zero. Theoretically, it implies that $\tilde{\mathbf{a}}_6$ has no effect on the GAB method. However, Figure 1 illustrates that this is not true in reality. In fact, $\tilde{\mathbf{a}}_6$ reduces the error significantly. Since the error has been accumulated by its nature, this contradictable behavior can be explained in a way that the approximate solution y_{i-6}

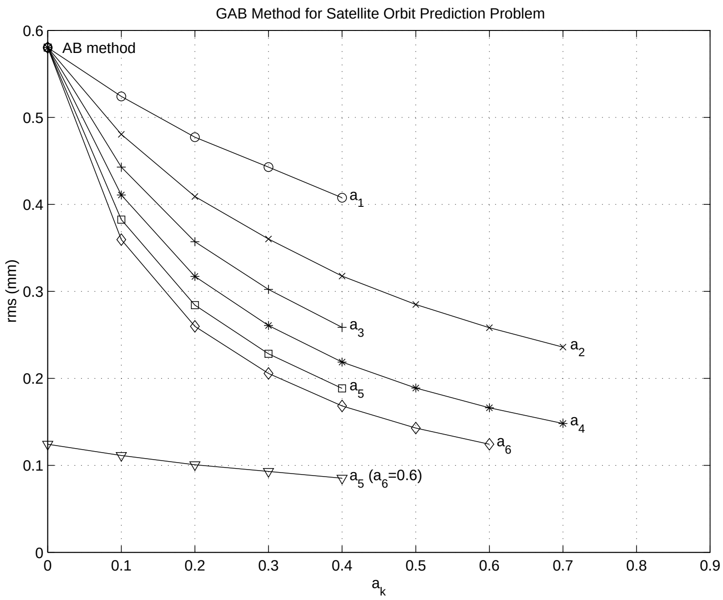

FIGURE 1. GAB method for Satellite Orbit Prediction Problem

contains less error than $y_{i-5}, y_{i-4}, \ldots, y_i$. For the same reason, it is shown that $\tilde{\mathbf{a}}_5$ reduces the error much than $\tilde{\mathbf{a}}_1$.

Among all cases of (4.3), the minimum error occurs when $a_6 = 0.6$ in $\tilde{\mathbf{a}}_6$. Thus, the following two cases are tried to find the better method: $\tilde{\mathbf{a}}_{4,6}^T = \begin{bmatrix}1 & 0 & 0 & 0 & a_4 & 0 & 0.6\end{bmatrix}$ and $\tilde{\mathbf{a}}_{5,6}^T = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & a_5 & 0.6\end{bmatrix}$.

Unexpectedly, $\tilde{\mathbf{a}}_{4,6}$ is very unstable whereas $\tilde{\mathbf{a}}_{5,6}$ provides the minimum error, even compared to (4.3). The method for $\tilde{\mathbf{a}}_{5,6}$ is shown as the lowest curve in the figure. Consequently, the best empirical values for $\tilde{\mathbf{a}}$ is

$$\text{Best 7-step GAB method (empirical):} \quad \tilde{\mathbf{a}}^T = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0.4 & 0.6\end{bmatrix}.$$

5. Conclusions

A general form of the multistep Adams-Bashforth method is derived by utilizing the Taylor's series. The coefficient matrix and the error vector of the generalized Adams-Bashforth method are formulated. Strongly stable multistep methods can be obtained by choosing appropriate values of parameters associated with the local truncation error. The formula for the local truncation error gives an idea how to choose such values, however, it might not meet with good results because of the accumulative error. It is known that those parameters should be non-negative. Only the method that can produce smaller error than the original Adams-Bashforth method is provided in the figure.

The two body problem of the Earth's satellite is integrated as a numerical example. A low-altitude satellite is taken for the numerical integration. The generalized 7-step method is applied to this satellite orbit prediction problem. Many cases are found to be better than the original Adams-Bashforth method. The best empirical method is obtained by selecting the parameter as $[\,1\ 0\ 0\ 0\ 0\ 0.4\ 0.6\,]$. This method shows that the prediction of the satellite trajectories is improved one order of magnitude in rms.

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Appendix A. Coefficient Matrix $\tilde{\mathbf{C}}$ and Error Vector $\tilde{\mathbf{e}}$

The coefficient matrix $\tilde{\mathbf{C}}$ and the error vector $\tilde{\mathbf{e}}$ of the GAB method will be provided in this section. As explained in Sections 2 and 3, the first column of $\tilde{\mathbf{C}}$ and the first element of $\tilde{\mathbf{e}}$ represent the AB method. It is interesting to note that all values of $\tilde{\mathbf{e}}$, except the first entry, are non-positive for each step method. This fact implies that the GAB method has a smaller error than the AB method when the values of $\{a_1, \ldots, a_{m-1}\}$ are positive.

2-step method:

$$\text{(A.1)} \qquad \tilde{\mathbf{C}} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } \tilde{\mathbf{e}}^T/3! = \frac{1}{12}\begin{bmatrix} 5 & -1 \end{bmatrix}.$$

3-step method:

$$\text{(A.2)} \qquad \tilde{\mathbf{C}} = \frac{1}{12}\begin{bmatrix} 23 & 5 & 4 \\ -16 & 8 & 16 \\ 5 & -1 & 4 \end{bmatrix} \text{ and } \tilde{\mathbf{e}}^T/4! = \frac{1}{24}\begin{bmatrix} 9 & -1 & 0 \end{bmatrix}.$$

4-step method:

(A.3)
$$\tilde{\mathbf{C}} = \frac{1}{24}\begin{bmatrix} 55 & 9 & 8 & 9 \\ -59 & 19 & 32 & 27 \\ 37 & -5 & 8 & 27 \\ -9 & 1 & 0 & 9 \end{bmatrix},$$

$$\text{and} \quad \tilde{\mathbf{e}}^T/5! = \frac{1}{720}\begin{bmatrix} 251 & -19 & -8 & -27 \end{bmatrix}.$$

5-step method:

(A.4)
$$\tilde{\mathbf{C}} = \frac{1}{720}\begin{bmatrix} 1901 & 251 & 232 & 243 & 224 \\ -2774 & 646 & 992 & 918 & 1024 \\ 2616 & -264 & 192 & 648 & 384 \\ -1274 & 106 & 32 & 378 & 1024 \\ 251 & -19 & -8 & -27 & 224 \end{bmatrix},$$

$$\text{and} \quad \tilde{\mathbf{e}}^T/6! = \frac{1}{1440}\begin{bmatrix} 475 & -27 & -16 & -27 & 0 \end{bmatrix}.$$

6-step method:

(A.5)
$$\tilde{\mathbf{C}} = \frac{1}{1440}\begin{bmatrix} 4277 & 475 & 448 & 459 & 448 & 475 \\ -7923 & 1427 & 2064 & 1971 & 2048 & 1875 \\ 9982 & -798 & 224 & 1026 & 768 & 1250 \\ -7298 & 482 & 224 & 1026 & 2048 & 1250 \\ 2877 & -173 & -96 & -189 & 448 & 1875 \\ -475 & 27 & 16 & 27 & 0 & 475 \end{bmatrix},$$

$$\tilde{\mathbf{e}}^T/7! = \frac{1}{60480}\begin{bmatrix} 19087 & -863 & -592 & -783 & -512 & -1375 \end{bmatrix}.$$

7-step method:

(A.6)

$$\tilde{\mathbf{C}} = \frac{1}{60480} \begin{bmatrix} 198721 & 19087 & 18224 & 18495 & 18304 & 18575 & 17712 \\ -447288 & 65112 & 90240 & 87480 & 89088 & 87000 & 93312 \\ 705549 & -46461 & 528 & 31347 & 24576 & 31875 & 11664 \\ -688256 & 37504 & 21248 & 58752 & 96256 & 80000 & 117504 \\ 407139 & -20211 & -12912 & -19683 & 11136 & 58125 & 11664 \\ -134472 & 6312 & 4224 & 5832 & 3072 & 28200 & 93312 \\ 19087 & -863 & -592 & -783 & -512 & -1375 & 17712 \end{bmatrix},$$

$$\tilde{\mathbf{e}}^T/8! = \frac{1}{120960}\begin{bmatrix}36799 & -1375 & -1024 & -1215 & -1024 & -1375 & 0\end{bmatrix}.$$

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Department of Mathematics
University of Incheon
Incheon 406-772 Korea
E-mail: nhahm@inchon.ac.kr

Department of Mathematics
Kyung Hee University
Yongin 446-701
Korea
E-mail: bihong@khu.ac.kr