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A GROWING ALGEBRA CONTAINING THE POLYNOMIAL RING

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1. Abstract

There are various papers on finding all the derivations of a non-associative algebra and an anti-symmetrized algebra (see [2], [3], [4], [5], [6], [10], [13], [15], [16]). We find all the derivations of the growing algebra $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ with the set of all right annihilators $T_3 = \{id, \partial_1, \partial_2, \partial_3\}$ in the paper. The dimension of $Der_{non}(WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]})$ of the algebra $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ is one and every derivation of the algebra $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ is outer. We show that there is a class $\mathfrak{P}$ of purely outer algebras in this work.

2. Preliminaries

Let $\mathbb{N}$ be the set of all non-negative integers and $\mathbb{Z}$ be the set of all integers. Let $\mathbb{N}^+$ be the set of all positive integers. Let $\mathbb{F}$ be a field of characteristic zero and $\mathbb{F}^\bullet$ the set of all non-zero elements in $\mathbb{F}$. Throughout the paper, we will assume that e is not the element of the field $\mathbb{F}$. For $n, t \in \mathbb{N}$, throughout the paper, m denotes a non-negative integer such that $m \leq n + t$. For fixed integers, $i_1, \cdots, i_m$ and for given irreducible polynomials $f_1, \cdots, f_m \in \mathbb{F}[x_1, \cdots, x_{n+t}]$, define $[f_1^{i_1}, \cdots, f_m^{i_m}]$ as the set $Poly_m = P_m = \{f_1^{i_1} \cdots f_m^{i_m}, f_1^{i_1} \cdots f_{m-1}^{i_{m-1}}, \cdots, f_2^{i_2} \cdots f_m^{i_m}, \cdots, f_1^{i_1}, \cdots, f_m^{i_m}\}$. For any subset P of P_m, define the $\mathbb{F}$-algebra $\mathbb{F}[e^{\pm[P]}, n, t] := \mathbb{F}[e^{\pm[P]}, x_1^{\pm 1}, \cdots, x_n^{\pm 1}, x_{n+1}, \cdots, x_{n+t}]$, which is

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spanned by

$$\mathbf{B} = \{e^{a_1 f_1} \cdots e^{a_r f_r} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} | f_1, \cdots, f_r \in P, a_1, \cdots, a_r \in \mathbb{Z}, \\ j_1, \cdots, j_n \in \mathbb{Z}, j_{n+1}, \cdots, j_{n+t} \in \mathbb{N}\}$$

We then denote $\partial_{h_1}^{k_1} \cdots \partial_{h_r}^{k_r}$ by the composition of the partial derivatives $\partial_{h_1}, \cdots, \partial_{h_r}$ on $\mathbb{F}[e^{\pm[P]}, n, t]$ with appropriate exponents where $1 \leq h_1, \cdots, h_r \leq n+t$ and ∂_h^0, $1 \leq h \leq n+t$, denotes the identity map on $\mathbb{F}[e^{\pm[P]}, n, t]$. For any $\alpha_u \in P \subset P_m$, let $\mathfrak{A}_{\alpha_u}$ be an additive subgroup of $\mathbb{F}$ such that $\mathfrak{A}_{\alpha_u}$ contains $\mathbb{Z}$. Consider now the (free) $\mathbb{F}$-vector space $N(e^{\mathfrak{A}_P}, n, t)_k$ (resp. $N(e^{\mathfrak{A}_P}, n, t)_{k^+}$) whose basis is the set

$$\mathbf{B}_1 = \{e^{a_1 f_1} \cdots e^{a_r f_r} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_{h_1}^{k_1} \cdots \partial_{h_r}^{k_r} | a_1 \in \mathfrak{A}_{\alpha_1}, \cdots, a_r \in \mathfrak{A}_{\alpha_r}, \\ f_1, \cdots, f_r \in P, h_1, \cdots, h_r \leq n+t, k_1 + \cdots + k_r \leq k \in \mathbb{N} \\ (\text{ resp. } \mathbb{N}^+)\} \tag{1}$$

If we define the multiplication $*$ on $N(e^{\mathfrak{A}_P}, n, t)_k$ as follows:

$$f\partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r} * g\partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q} = f(\partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r}(g))\partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q} \tag{2}$$

for any $f\partial_{h_1}^{p_1} \cdots \partial_{h_r}^{p_r}, g\partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q} \in N(e^{\mathfrak{A}_P}, n, t)_k$, then we define the combinatorial non-associative algebra $WN(e^{\mathfrak{A}_P}, n, t)_k$ whose underlying vector space is $N(e^{\mathfrak{A}_P}, n, t)_k$ and whose multiplication is $*$ in (2) (see [1], [2], [5], [14] and [15]). The non-associative subalgebra $WN(e^{\mathfrak{A}_P}, n, t)_{<k>}$ of the algebra $WN(e^{\mathfrak{A}_P}, n, t)_k$ is generated by

$$\{f\partial_{h_1}^{k_1} \cdots \partial_{h_r}^{k_r} | f \in \mathbf{B}, 1 \leq h_1, \cdots, h_r \leq n+t, k_1 + \cdots + k_r = k \in \mathbb{N}^+\}. \tag{3}$$

The non-associative subalgebra $WN(e^{\mathfrak{A}_P}, n, t)_{[k]}$ of the algebra $WN(e_P^{\mathfrak{A}}, n, t)_k$ is generated by

$$\{f\partial_h^k | f \in \mathbf{B}, 1 \leq h \leq n+t\}. \tag{4}$$

For an algebra A and $l \in A$, an element $l_1 \in A$ is a right (resp. left) identity of l, if $l * l_1 = l$ (resp. $l_1 * l = l$) holds. The set of all right identities of $WN(e^{\mathfrak{A}_P}, n, t)_{[1]}$ is $\{\sum_{1 \leq u \leq n+t} x_u \partial_u + \sum_{1 \leq u \leq n+t} c_u \partial_u | c_u \in \mathbb{F}\}$. There is no left identity of $WN(e^{\mathfrak{A}_P}, n, t)_{k^+}$. The algebra $WN(e^{\mathfrak{A}_P}, n, t)_k$ has the left identity 1. If A is an associative $\mathbb{F}$-algebra, then the antisymmetrized algebra of A is a Lie algebra relative to the commutator $[x, y] := xy - yx$, (See [9]). For a general non-associative $\mathbb{F}$-algebra N we define in the same way its antisymmetrized algebra N^-. In case N^- is a Lie algebra we shall say that N is Lie admissible. For $S \subset N^-$, an element l is ad-diagonal with respect to S if for any $l_1 \in S$, $[l, l_1] = cl_1$ for $c \in \mathbb{F}$. The algebra $WN(e^{\mathfrak{A}_P}, n, t)_{[1]}$ is Lie admissible (see [1], [7], [16], and [18]).

Since the cardinality $|P|$ of P is 2^m, for all $\alpha \in P_m$, if $\mathfrak{A}_\alpha$ is $\mathbb{Z}$, then the algebra $WN(e^{\mathfrak{A}_{P_m}}, n, t)_k$ is $\mathbb{Z}^{2^m}$-graded as follows:

$$WN(e^{\mathfrak{A}_{P_m}}, n, t)_k = \bigoplus_{(a_1, \cdots, a_{m^2})} N_{(a_1, \cdots, a_{m^2})} \tag{5}$$

where $N_{(a_1, \cdots, a_{2^m})}$ is the vector subspace of $WN(e^{\mathfrak{A}_{P_m}}, n, t)_k$ spanned by

$$\{e^{a_1 f_1} \cdots e^{a_r f_r} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} | j_1, \cdots, j_n \in \mathbb{Z}, j_{n+1}, \cdots, j_{n+t} \in \mathbb{N}\}.$$

This implies that $WN(e^{\mathfrak{A}_P}, n, t)_k$ and $WN(e^{\mathfrak{A}_P}, n, t)_{k^+}$ are appropriate graded algebras as (5) (see [11]). Thus throughout the paper, the $(0, \cdots, 0)$-homogeneous component N_0 of $WN(e^{\mathfrak{A}_P}, n, t)_k$ is the subalgebra $WN(0, n, t)_k$ of $WN(e^{\mathfrak{A}_P}, n, t)_k$. For any standard basis element $e^{a_1 f_1} \cdots e^{a_r f_r}\ x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_{t_1}^{k_1} \cdots \partial_{t_r}^{k_r}$ of $WN(e^{\mathfrak{A}_{P_m}}, n, t)_k$, define the homogeneous degree as follows:

$$hd(e^{a_1 f_1} \cdots e^{a_r f_r} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_{t_1}^{k_1} \cdots \partial_{t_r}^{k_r}) = \sum_{u=1}^{n+t} |j_u|$$

where $|j_u|$ is the absolute value of j_u for $1 \leq u \leq n+t$. For any element $l \in WN(e^{\mathfrak{A}_P}, n, t)_k$, define $hd(l)$ as the highest homogeneous degree of each monomial of l. Note that the set of all right annihilators of $WN(e^{\mathfrak{A}_P}, n, t)_k$ is the subalgebra T_{n+t} of $WN(e^{\mathfrak{A}_P}, n, t)_k$ which is spanned by $\{\partial_{t_1}^{k_1} \cdots \partial_{t_r}^{k_r} | 1 \leq t_1, \cdots, t_r \leq n+t, k_1 + \cdots + k_r \leq k \in \mathbb{N}\}$. For a given algebra A, $Out(A)$ (resp. $Inn(A)$) is the set of all the outer (resp. inner) derivations of A and $Der(A)$ is the set of all the derivations of A. An algebra A is purely outer, if every derivation of A is outer i.e., $Der(A) = Out(A)$.

3. Derivations of the non-associative algebra $WN(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}$

For this section, the set of all right annihilators T_3 of $WN(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}$ is spanned by $\{id, \partial_1, \partial_2, \partial_3\}$. The algebra $WN(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}$ contains the polynomial ring and it is simple (see [4]).

Note 1. For any basis elements ∂_u, $x_1^{i_1} x_2^{i_2} x_3^{i_3}$, $x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_u$, $e^{p x_1 x_2 x_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$, $e^{p x_1 x_2 x_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} \partial_u$, $1 \leq u \leq 3$, of $WN(e^{\pm x_1 x_2 x_3}, 0, 3)_1$, and for any $c \in$

$\mathbb{F}$, $p \in \mathbb{Z}$, if we define an $\mathbb{F}$-linear map D_c from the algebra $WN(e^{\pm x_1x_2x_3}, 0, 3)_1$ to itself as follows:

$$
\begin{aligned}
&D_c(\partial_u) = 0,\\
&D_c(x_1^{i_1}x_2^{i_2}x_3^{i_3}) = 0,\\
&D_c(x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_u) = 0,\\
&D_c(e^{px_1x_2x_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}) = pce^{px_1x_2x_3}x_1^{i_1}x_2^{i_2}x_3^{i_3},\\
&D_c(e^{px_1x_2x_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_u) = pce^{px_1x_2x_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_u,
\end{aligned} \tag{6}
$$

then the map D_c can be linearly extended to a non-associative algebra derivation of
$WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ where $1 \leq u \leq 3$ (see [6], [7] and [10]). □

Lemma 3.1. *For any derivation D of $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ and for any basis elements ∂_u, $x_1^{i_1}x_2^{i_2}x_3^{i_3}$, $x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_u$, $1 \leq u \leq 3$, of $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$, we have that*

$$
\begin{aligned}
&D(\partial_u) = 0,\\
&D(x_1^ix_2^jx_3^k) = ic_{0,0,0,1}x_1^{i-1}x_2^jx_3^k + jd_{0,0,0,2}x_1^ix_2^{j-1}x_3^k + kr_{0,0,0,3}x_1^ix_2^jx_3^{k-1},\\
&D(x_1^ix_2^jx_3^k\partial_u) = ic_{0,0,0,1}x_1^{i-1}x_2^jx_3^k\partial_u + jd_{0,0,0,2}x_1^ix_2^{j-1}x_3^k\partial_u + kr_{0,0,0,3}x_1^i\\
&x_2^jx_3^{k-1}\partial_u
\end{aligned}
$$

hold with appropriate coefficients where $1 \leq u \leq 3$.

Proof. Let D be the derivation in the lemma. Since the algebra $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ is $\mathbb{Z}$-graded, $D(\partial_1)$ is the sum of terms in different homogeneous components of $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ in (5). Thus $D(\partial_1)$ can be written as follows:

$$
\begin{aligned}
D(\partial_1) &= \sum_{i,j,k\geq 0} \alpha_{i,j,k,0}e^{px_1x_2x_3}x_1^ix_2^jx_3^k + \sum_{i,j,k\geq 0} \alpha_{i,j,k,1}e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_1\\
&+ \sum_{i,j,k\geq 0} \alpha_{i,j,k,2}e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_2 + \sum_{i,j,k\geq 0} \alpha_{i,j,k,3}e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_3
\end{aligned}
$$

with appropriate coefficients. Since ∂_1 centralizes itself, we have that $D(\partial_1)$ is in the right annihilator of ∂_1, i.e.,

$$\begin{aligned}
\partial_1 * D(\partial_1) &= \sum_{i,j,k\geq 0} p\alpha_{i,j,k,0} e^{px_1x_2x_3} x_1^i x_2^{j+1} x_3^{k+1} \\
&+ \sum_{i\geq 1,j,k\geq 0} i\alpha_{i,j,k,0} e^{px_1x_2x_3} x_1^{i-1} x_2^j x_3^k \\
&+ \sum_{i,j,k\geq 0} p\alpha_{i,j,k,1} e^{px_1x_2x_3} x_1^i x_2^{j+1} x_3^{k+1}\partial_1 \\
&+ \sum_{i\geq 1,j,k\geq 1} i\alpha_{i,j,k,1} e^{px_1x_2x_3} x_1^{i-1} x_2^j x_3^k\partial_1 \\
&+ \sum_{i,j,k\geq 0} p\alpha_{i,j,k,2} e^{px_1x_2x_3} x_1^i x_2^{j+1} x_3^{k+1}\partial_2 \\
&+ \sum_{i\geq 1,j,k\geq 0} i\alpha_{i,j,k,2} e^{px_1x_2x_3} x_1^{i-1} x_2^j x_3^k\partial_2 \\
&+ \sum_{i,j,k\geq 0} p\alpha_{i,j,k,3} e^{px_1x_2x_3} x_1^i x_2^{j+1} x_3^{k+1}\partial_3 \\
&+ \sum_{i\geq 1,j,k\geq 0} i\alpha_{i,j,k,3} e^{px_1x_2x_3} x_1^{i-1} x_2^j x_3^k\partial_3 \\
&= 0 \qquad (7)
\end{aligned}$$

with appropriate coefficients. By (7), we have that $\alpha_{i,j,k,0}$, $\alpha_{i,j,k,1}$, $\alpha_{i,j,k,2}$, and $\alpha_{i,j,k,3}$, are zeros, $i,j,k \geq 0$. Thus $D(\partial_1)$ is zero. Similarly, we can prove that $D(\partial_2)$ and $D(\partial_3)$ are also zeros. By $D(\partial_u * x_1) = 0$, $1 \leq u \leq 3$, we can prove that $D(x_1) = b_{0,0,0,0} + b_{0,0,0,1}\partial_1 + b_{0,0,0,2}\partial_2 + b_{0,0,0,3}\partial_3$. Similarly, since ∂_u centralizes $x_1\partial_1$, we can also prove that

$$D(x_1\partial_1) = c_{0,0,0,0} + c_{0,0,0,1}\partial_1 + c_{0,0,0,2}\partial_2 + c_{0,0,0,3}\partial_3.$$

Since $x_1\partial_1$ is an idempotent, we can prove that $c_{0,0,0,2} = 0$, $c_{0,0,0,3} = 0$. This implies that $D(x_1\partial_1) = c_{0,0,0,0} + c_{0,0,0,1}\partial_1$. Since $D(\partial_1 * x_1^2\partial_1) = 2D(x_1\partial_1)$, we are also able to prove that

$$\begin{aligned}
D(x_1^2\partial_1) &= 2c_{0,0,0,0}x_1 + 2c_{0,0,0,1}x_1\partial_1 + \sum_{j,k} t_{0,j,k,0} x_2^j x_3^k \\
&+ \sum_{j,k} t_{0,j,k,1} x_2^j x_3^k \partial_1 + \sum_{j,k} t_{0,j,k,2} x_2^j x_3^k \partial_2 + \sum_{j,k} t_{0,j,k,3} x_2^j x_3^k \partial_3
\end{aligned}$$

where $t_{0,j,k,1}, t_{0,j,k,1}, t_{0,j,k,2}, t_{0,j,k,3} \in \mathbb{F}$ for all j and k. Since $D(x_1\partial_1 * x_1^2\partial_1) = 2D(x_1^2\partial_1)$, we have that $c_{0,0,0,0} = 0$, $t_{0,j,k,1} = t_{0,j,k,1} = t_{0,j,k,2} =$

$t_{0,j,k,3} = 0$. This implies that

$$D(x_1\partial_1) = c_{0,0,0,1}\partial_1,$$
$$D(x_1^2\partial_1) = 2c_{0,0,0,1}x\partial_1$$

hold. Since $D(x_1\partial_1 * x_1) = D(x_1)$, we also have that $D(x_1) = c_{0,0,0,1}$. By $D(\partial_1 * x_1^3\partial_1) = 3D(x_1^2\partial_1)$, we have that

$$\begin{aligned} D(x_1^3\partial_1) &= 3c_{0,0,0,1}x_1^2\partial_1 + \sum_{j,k} s_{0,j,k,0}x_2^jx_3^k + \sum_{j,k} s_{0,j,k,1}x_2^jx_3^k\partial_1 \\ &+ \sum_{j,k} s_{0,j,k,2}x_2^jx_3^k\partial_2 + \sum_{j,k} s_{0,j,k,3}x_2^jx_3^k\partial_3, \end{aligned}$$

where $s_{0,j,k,1}, s_{0,j,k,1}, s_{0,j,k,2}, s_{0,j,k,3} \in \mathbb{F}$ for all j and k. By $D(x_1\partial_1 * x_1^3\partial_1) = 3D(x_1^3\partial_1)$, we have that $D(x_1^3\partial_1) = 3c_{0,0,0,1}x_1^2\partial_1$. Since $D(x_1^2\partial_1 * x_1^{i-1}\partial_1) = (i-1)D(x_1^i\partial_1)$, by induction on i of $x_1^i\partial_1$, we are able to prove that

$$D(x_1^i\partial_1) = ic_{0,0,0,1}x_1^{i-1}\partial_1.$$

Similarly, we are also able to prove that

$$D(x_2^j\partial_2) = jd_{0,0,0,2}x_2^{j-1}\partial_2,$$
$$D(x_3^k\partial_3) = jr_{0,0,0,3}x_3^{j-1}\partial_3.$$

Since ∂_u, $1 \leq u \leq 3$, is in the left annihilator of $x_1\partial_2$, we can prove that $D(x_1\partial_2) = \alpha_{0,0,0,0} + \alpha_{0,0,0,1}\partial_1 + \alpha_{0,0,0,2}\partial_2 + \alpha_{0,0,0,3}\partial_3$. By $D(x_1\partial_1 * x_1\partial_2) = D(x_1\partial_2)$, we can also prove that $\alpha_{0,0,0,0} = \alpha_{0,0,0,2} = \alpha_{0,0,0,3} = 0$, $\alpha_{0,0,0,1} = c_{0,0,0,1}$. This implies that $D(x_1\partial_2) = c_{0,0,0,1}\partial_2$. Since $D(x_1^2\partial_1 * x_1^{i-1}\partial_2) = (i-1)D(x_1^i\partial_2)$, by induction on i of $x_1^i\partial_2$, we can prove that

$$D(x_1^i\partial_2) = ic_{0,0,0,1}x_1^{i-1}\partial_2.$$

Similarly, we are able to prove that

$$D(x_1^i\partial_3) = ic_{0,0,0,1}x_1^{i-1}\partial_3,$$
$$D(x_2^j\partial_u) = jd_{0,0,0,2}x_2^{j-1}\partial_u,$$
$$D(x_3^k\partial_u) = kr_{0,0,0,3}x_3^{k-1}\partial_u$$

where $1 \leq u \leq 3$. By $D(x_1^i\partial_2 * x_2^{j+1}\partial_1) = (j+1)D(x_1^ix_2^j\partial_1)$, we have that

$$D(x_1^ix_2^j\partial_1) = ic_{0,0,0,1}x_1^{i-1}x_2^j\partial_1 + jd_{0,0,0,2}x_1^ix_2^{j-1}\partial_1.$$

Similarly, we are able to prove that

$$D(x_1^ix_2^j\partial_u) = ic_{0,0,0,1}x_1^{i-1}x_2^j\partial_2 + jd_{0,0,0,2}x_1^ix_2^{j-1}\partial_u$$

where $2 \leq u \leq 3$. Since $D(x_1^i x_2^j \partial_3 * x_3^{k+1}\partial_1) = (k+1)D(x_1^i x_2^j x_3^k \partial_1)$, we are also able to prove that

$$D(x_1^i x_2^j x_3^k \partial_1) = ic_{0,0,0,1}x_1^{i-1}x_2^j x_3^k \partial_1 + jd_{0,0,0,2}x_1^i x_2^{j-1}x_3^k \partial_1 + kr_{0,0,0,3}x_1^i x_2^j x_3^{k-1}\partial_1.$$

By $D(x_1^i x_2^j x_3^k \partial_1 * x_1) = D(x_1^i x_2^j x_3^k)$, we also have that

$$D(x_1^i x_2^j x_3^k) = ic_{0,0,0,1}x_1^{i-1}x_2^j x_3^k + jd_{0,0,0,2}x_1^i x_2^{j-1}x_3^k + kr_{0,0,0,3}x_1^i x_2^j x_3^{k-1}.$$

Similarly, we also have that

$$\begin{aligned} D(x_1^i x_2^j x_3^k \partial_u) &= ic_{0,0,0,1}x_1^{i-1}x_2^j x_3^k \partial_u + jd_{0,0,0,2}x_1^i x_2^{j-1}x_3^k \partial_u \\ &+ kr_{0,0,0,3}x_1^i x_2^j x_3^{k-1}\partial_u \end{aligned}$$

where $2 \leq u \leq 3$. So we have proven the lemma. □

Lemma 3.2. *For any derivation D of the algebra $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ and for basis elements $x_1^{i_1}x_2^{i_2}x_3^{i_3}$, $e^{x_1x_2x_3}$, $e^{-x_1x_2x_3}$, $x_1^{i_1}x_2^{i_2}x_3^{i_3}\partial_u$, $e^{x_1x_2x_3}\partial_u$, $e^{-x_1x_2x_3}\partial_u$, $1 \leq u \leq 3$, of $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$, we have that*

$$\begin{aligned} &D(x_1^i x_2^j x_3^k) = 0, \\ &D(x_1^i x_2^j x_3^k \partial_u) = 0, \\ &D(e^{x_1x_2x_3}) = ce^{x_1x_2x_3}, \\ &D(e^{x_1x_2x_3}\partial_u) = ce^{x_1x_2x_3}\partial_u, \\ &D(e^{-x_1x_2x_3}) = -ce^{-x_1x_2x_3}, \\ &D(e^{-x_1x_2x_3}\partial_u) = -ce^{-x_1x_2x_3}\partial_u \end{aligned}$$

hold where $c \in \mathbb{F}$.

Let D be the derivation in the lemma. Since the algebra $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ is $\mathbb{Z}$-graded, $D(e^{x_1x_2x_3}\partial_1)$ is the sum of terms in different homogeneous components of $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ in (5). Assume that

$$\begin{aligned} D(e^{x_1x_2x_3}\partial_1) &= \sum_{i,j,k\geq 0} a_{i,j,k,0}e^{px_1x_2x_3}x_1^i x_2^j x_3^k \\ &+ \sum_{i,j,k\geq 0} a_{i,j,k,1}e^{px_1x_2x_3}x_1^i x_2^j x_3^k \partial_1 \\ &+ \sum_{i,j,k\geq 0} a_{i,j,k,2}e^{px_1x_2x_3}x_1^i x_2^j x_3^k \partial_2 \\ &+ \sum_{i,j,k\geq 0} a_{i,j,k,3}e^{px_1x_2x_3}x_1^i x_2^j x_3^k \partial_3 \end{aligned}$$

with appropriate coefficients. We have that

$$\begin{aligned} D(\partial_1 * e^{x_1x_2x_3}\partial_1) &= D(e^{x_1x_2x_3}x_2x_3\partial_1) \\ &= \sum_{i,j,k\geq 0} pa_{i,j,k,0}e^{px_1x_2x_3}x_1^i x_2^{j+1}x_3^{k+1} \\ &+ \sum_{j,k\geq 0, i\geq 1} ia_{i,j,k,0}e^{px_1x_2x_3}x_1^{i-1}x_2^j x_3^k \\ &+ \sum_{i,j,k\geq 0} pa_{i,j,k,1}e^{px_1x_2x_3}x_1^i x_2^{j+1}x_3^{k+1}\partial_1 \\ &+ \sum_{j,k\geq 0, i\geq 1} ia_{i,j,k,1}e^{px_1x_2x_3}x_1^{i-1}x_2^j x_3^k\partial_1 \\ &+ \sum_{i,j,k\geq 0} pa_{i,j,k,2}e^{px_1x_2x_3}x_1^i x_2^{j+1}x_3^{k+1}\partial_2 \\ &+ \sum_{j,k\geq 0, i\geq 1} ia_{i,j,k,2}e^{px_1x_2x_3}x_1^{i-1}x_2^j x_3^k\partial_2 \\ &+ \sum_{i,j,k\geq 0} pa_{i,j,k,3}e^{px_1x_2x_3}x_1^i x_2^{j+1}x_3^{k+1}\partial_3 \\ &+ \sum_{j,k\geq 0, i\geq 1} ia_{i,j,k,3}e^{px_1x_2x_3}x_1^{i-1}x_2^j x_3^k\partial_3 \end{aligned} \tag{8}$$

and

$$\begin{aligned} D(e^{x_1x_2x_3}\partial_1 * x_1x_2x_3\partial_1) &= D(e^{x_1x_2x_3}x_2x_3\partial_1) \\ &= \sum_{i,j,k\geq 0} a_{i,j,k,0}e^{px_1x_2x_3}x_1^{i+1}x_2^{j+1}x_3^{k+1} \\ &+ \sum_{i,j,k\geq 0} a_{i,j,k,1}e^{px_1x_2x_3}x_1^i x_2^{j+1}x_3^{k+1}\partial_1 \\ &+ \sum_{i,j,k\geq 0} a_{i,j,k,2}e^{px_1x_2x_3}x_1^{i+1}x_2^j x_3^{k+1}\partial_1 \\ &+ \sum_{i,j,k\geq 0} a_{i,j,k,3}e^{px_1x_2x_3}x_1^{i+1}x_2^{j+1}x_3^k\partial_1 \\ &+ d_{0,0,0,2}e^{x_1x_2x_3}x_3\partial_1 + r_{0,0,0,3}e^{x_1x_2x_3}x_2\partial_1. \end{aligned} \tag{9}$$

By comparing (8) and (9), we have that $p = 1$, $a_{i,j,k,0} = a_{i,j,k,2} = a_{i,j,k,3} = 0$, $i, j, k \geq 0$, $a_{i,j,k,1} = 0$, $i \geq 1$, and $d_{0,0,0,2} = r_{0,0,0,3} = 0$. This implies that

$$D(e^{x_1x_2x_3}\partial_1) = \sum_{j,k\geq 0} a_{0,j,k,1}e^{x_1x_2x_3}x_2^jx_3^k\partial_1. \tag{10}$$

Since

$$\begin{aligned} D(\partial_2 * e^{x_1x_2x_3}\partial_1) &= D(e^{x_1x_2x_3}x_1x_3\partial_1) \\ &= \sum_{j,k\geq 0} a_{0,j,k,1}e^{x_1x_2x_3}x_1x_2^jx_3^{k+1}\partial_1 \\ &+ \sum_{j\geq 1,k\geq 0} ja_{0,j,k,1}e^{x_1x_2x_3}x_2^{j-1}x_3^k\partial_1 \end{aligned}$$

and

$$\begin{aligned} D(e^{x_1x_2x_3}\partial_1 * x_1^2x_3\partial_1) &= 2D(e^{x_1x_2x_3}x_1x_3\partial_1) \\ &= 2\sum_{j,k\geq 0} a_{0,j,k,1}e^{x_1x_2x_3}x_1x_2^jx_3^{k+1}\partial_1 \\ &+ 2c_{0,0,0,1}e^{x_1x_2x_3}x_3\partial_1, \end{aligned}$$

we have that $a_{0,j,k,1} = 0$, $j \geq 1$ and $c_{0,0,0,1} = 0$. This implies that

$$D(e^{x_1x_2x_3}\partial_1) = \sum_{k\geq 0} a_{0,0,k,1}e^{x_1x_2x_3}x_3^k\partial_1. \tag{11}$$

Since

$$\begin{aligned} D(\partial_3 * e^{x_1x_2x_3}\partial_1) &= D(e^{x_1x_2x_3}x_1x_2\partial_1) \\ &= \sum_{k\geq 0} a_{0,0,k,1}e^{x_1x_2x_3}x_1x_2x_3^k\partial_1 \\ &+ \sum_{k\geq 1} ka_{0,0,k,1}e^{x_1x_2x_3}x_3^{k-1}\partial_1 \end{aligned}$$

and

$$\begin{aligned} D(e^{x_1x_2x_3}\partial_1 * x_1^2x_2\partial_1) &= 2D(e^{x_1x_2x_3}x_1x_2\partial_1) \\ &= 2\sum_{k\geq 0} a_{0,0,k,1}e^{x_1x_2x_3}x_1x_2x_3^k\partial_1, \end{aligned}$$

we have that $a_{0,0,k,1} = 0$, $k \geq 1$. This implies that

$$D(e^{x_1x_2x_3}\partial_1) = a_{0,0,0,1}e^{x_1x_2x_3}\partial_1 \tag{12}$$

and we also have

$$\begin{aligned}
&D(x_1^i\partial_u)=0,\\
&D(x_2^j\partial_u)=0,\\
&D(x_3^k\partial_u)=0,\\
&D(x_1^ix_2^jx_3^k)=0,\\
&D(x_1^ix_2^jx_3^k\partial_u)=0
\end{aligned}$$

where $2\le u\le 3$. By (12) and $D(e^{x_1x_2x_3}\partial_1 * x_1)=D(e^{x_1x_2x_3})$, we also have that $D(e^{x_1x_2x_3})=a_{0,0,0,1}e^{x_1x_2x_3}$. By $D(e^{x_1x_2x_3}\partial_1 * x_1\partial_2)=D(e^{x_1x_2x_3}\partial_2)$ and $D(e^{x_1x_2x_3}\partial_1 * x_1\partial_3)=D(e^{x_1x_2x_3}\partial_3)$, we can prove that

$$\begin{aligned}
D(e^{x_1x_2x_3}\partial_2)&=a_{0,0,0,1}e^{x_1x_2x_3}\partial_2,\\
D(e^{x_1x_2x_3}\partial_3)&=a_{0,0,0,1}e^{x_1x_2x_3}\partial_3.
\end{aligned}$$

Since $D(e^{x_1x_2x_3}\partial_1 * e^{-x_1x_2x_3}\partial_1)=0$, we can prove that

$$\begin{aligned}
D(e^{-x_1x_2x_3}\partial_1) &= -a_{0,0,0,1}e^{-x_1x_2x_3}\partial_1+\sum_{1\le u\le 3,j,k\ge 0}\beta_{0,j,k,0}x_2^jx_3^k\\
&+\sum_{j,k\ge 0}\beta_{0,j,k,1}x_2^jx_3^k\partial_1+\sum_{j,k\ge 0}\beta_{0,j,k,2}x_2^jx_3^k\partial_2\\
&+\sum_{j,k\ge 0}\beta_{0,j,k,3}x_2^jx_3^k\partial_3.
\end{aligned}$$

By $D(e^{-x_1x_2x_3}\partial_1 * x_1\partial_1)=D(e^{-x_1x_2x_3}\partial_1)$, we can also prove that

$$D(e^{-x_1x_2x_3}\partial_1)=-a_{0,0,0,1}e^{-x_1x_2x_3}\partial_1.$$

By $D(e^{-x_1x_2x_3}\partial_1 * x_1)=D(e^{-x_1x_2x_3})$, we also have that

$$D(e^{-x_1x_2x_3})=-a_{0,0,0,1}e^{-x_1x_2x_3}.$$

Similarly, we can prove that

$$\begin{aligned}
D(e^{-x_1x_2x_3}\partial_2)&=-a_{0,0,0,1}e^{-x_1x_2x_3}\partial_2,\\
D(e^{-x_1x_2x_3}\partial_3)&=-a_{0,0,0,1}.e^{-x_1x_2x_3}\partial_3.
\end{aligned}$$

So we have proven the lemma. □

Theorem 3.1. *For any derivation D of the algebra $WN(e^{\pm x_1x_2x_3},0,3)_{[1]}$ and for basis elements*

$e^{px_1x_2x_3}x_1^ix_2^jx_3^k$ *and* $e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u$, $1 \le u \le 3$, *of* $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$, *we have that*

$$D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k) = pce^{px_1x_2x_3}x_1^ix_2^jx_3^k,$$
$$D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u) = pce^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u$$

hold where $1 \le u \le 3$, $p \in \mathbb{Z}$, *and* $c \in \mathbb{F}$.

Proof. Let D be the derivation in the lemma. By $D(e^{x_1x_2x_3}\partial_1 * x_1^{i+1}\partial_u) = (i+1)D(e^{x_1x_2x_3}x_1^i\partial_u)$, we are able to prove that $D(e^{x_1x_2x_3}x_1^i\partial_u) = a_{0,0,0,1}e^{x_1x_2x_3}x_1^i\partial_u$ for $1 \le u \le 3$, with appropriate coefficients. By $D(e^{x_1x_2x_3}\partial_2 * e^{x_1x_2x_3}\partial_u) = D(e^{2x_1x_2x_3}x_1x_3\partial_u)$, we are also able to prove that

$$D(e^{2x_1x_2x_3}x_1x_3\partial_u) = 2a_{0,0,0,1}e^{2x_1x_2x_3}x_1x_3\partial_u.$$

Since $D(e^{x_1x_2x_3}x_1\partial_2 * e^{x_1x_2x_3}x_1^{i-2}\partial_u) = D(e^{2x_1x_2x_3}x_1^ix_3\partial_u)$, we also have that

$$D(e^{2x_1x_2x_3}x_1^ix_3\partial_u) = 2a_{0,0,0,1}e^{2x_1x_2x_3}x_1^ix_3\partial_u.$$

By $D(e^{x_1x_2x_3}x_1x_3\partial_2 * e^{x_1x_2x_3}x_1^{i-2}\partial_u) = D(e^{2x_1x_2x_3}x_1^ix_3^2\partial_u)$, we prove that

$$D(e^{2x_1x_2x_3}x_1^ix_3^2\partial_u) = 2a_{0,0,0,1}e^{2x_1x_2x_3}x_1^ix_3^2\partial_u,$$

and by $D(e^{2x_1x_2x_3}x_1^ix_3^2\partial_3 * x_3^{k-1}\partial_u) = (k-1)D(e^{2x_1x_2x_3}x_1^ix_3^k\partial_u)$, we also prove that

$$D(e^{2x_1x_2x_3}x_1^ix_3^k\partial_u) = 2a_{0,0,0,1}e^{2x_1x_2x_3}x_1^ix_3^k\partial_u.$$

By $D(e^{2x_1x_2x_3}x_1^ix_3^k\partial_3 * x_2^jx_3\partial_u) = D(e^{2x_1x_2x_3}x_1^ix_2^jx_3^k\partial_u)$, we have that

$$D(e^{2x_1x_2x_3}x_1^ix_2^jx_3^k\partial_u) = 2a_{0,0,0,1}e^{2x_1x_2x_3}x_1^ix_2^jx_3^k\partial_u,$$

and by $D(e^{2x_1x_2x_3}x_1^ix_2^{j-1}x_3^{k-1}\partial_1 * e^{x_1x_2x_3}\partial_u) = D(e^{3x_1x_2x_3}x_1^ix_2^jx_3^k\partial_u)$, we also have that

$$D(e^{3x_1x_2x_3}x_1^ix_2^jx_3^k\partial_u) = 3a_{0,0,0,1}e^{3x_1x_2x_3}x_1^ix_2^jx_3^k\partial_u.$$

By induction on $p \in \mathbb{Z}$ of $e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u$ and $D(e^{(p-1)x_1x_2x_3}x_1^ix_2^{j-1}x_3^{k-1}\partial_1 * e^{x_1x_2x_3}\partial_u) = D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u)$, we are able to prove that

$$D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u) = pa_{0,0,0,1}e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u.$$

By putting $c = a_{0,0,0,1}$, we have that

$$D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u) = pce^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u.$$

By $D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_1 * x_1) = D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k)$, we also have that

$$D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k) = pce^{px_1x_2x_3}x_1^ix_2^jx_3^k.$$

Therefore we have proven the lemma. □

Theorem 3.2. *For any* $D \in Der_{non}(WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]})$, *D is the linear sum of the derivations* D_c *as shown in Note 1 where* $c \in \mathbb{F}$. *The additive group* $D \in Der_{non}(WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]})$ *is isomorphic to the additive group* $\mathbb{F}$. *Every derivation of the algebra* $WN(e^{\pm x_1^r}, 0, 3)_{[1]}$ *is outer.*

Proof. The proofs of the theorem are straightforward by Lemma 3.2, Theorem 3.1, and the fact that the derivation of Note 1 cannot be inner. This completes the proof of the theorem. □

Corollary 3.1. *The dimension of* $Der_{non}(WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]})$ *of the algebra* $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ *is one. For any derivation D of* Der_{non} $(WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]})$, $D(N_0') = 0$ *holds where* N_0' *is the zero-homogeneous component of* $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ *in (5) (see* [8] *and* [9]*).*

Proof. The proofs of the corollary are straightforward by Lemma 3.2 and Note 1. □

Proposition 3.1. *If A is not a purely outer algebra, then algebra A and* $WN(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ *are not isomorphic.*

Proof. The proof of the proposition is straightforward by Theorem 3.2. □

Remarks. By Theorem 3.2, there is a class $\mathfrak{P}$ of purely outer algebras, i.e., for any $A \in \mathfrak{P}$ and for any $D \in Der(A)$, D is outer. □

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