Honam Mathematical J. **32** (2010), No. 3, pp. 453–465

ON STARCOMPACTNESS VERSUS COUNTABLE PRACOMPACTNESS

Junhui Kim and Myung Hyun Cho*

Abstract. In this paper, we consider countable version of star covering properties to get interesting results about the relationship between starcompactness and countable pracompactness. We also construct examples related to countable pracompactness and H-closedness.

1. Introduction

Throughout this paper, all spaces are assumed as T_1-topological spaces. Although the study of star covering properties of a topological space could be dated back to the seventies or even earlier [1, 2, 10, 11, 15, 22], a systematic investigation of them was done first by van Douwen, G. Reed, A. Roscoe and I. Tree [9] in 1991. Since then, this area has generated substantial interest of many topologists [6, 7, 8, 13, 14, 16, 20, 21, 23]. The most influential study of this topic is Matveev's survey article [18].

For a cover $\mathcal{U}$ of a space X and a subset $A \subset X$, we write $St(A, \mathcal{U}) = St^1(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $St^{n+1}(A, \mathcal{U}) = St(St^n(A, \mathcal{U}), \mathcal{U})$; for $A = \{x\}$ we write $St(x, \mathcal{U})$ instead of $St(\{x\}, \mathcal{U})$; it is also convenient to agree that $St^0(A, \mathcal{U}) = A$.

Definition 1. [9, 18] Let $n \in \mathbb{N}$.

(1) A space X is called *n-starcompact* if, for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $\mathrm{St}^n(F, \mathcal{U}) = X$.

(2) A space X is called *$n\frac{1}{2}$-starcompact* if, for every open cover $\mathcal{U}$ of X, there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$.

Received July 12, 2010. Accepted August 28, 2010.
2000 AMS subject classification: 54D20, 54B05, 54B10, 54C05.
Keywords and phrases: starcompact, $1\frac{1}{2}$-starcompact, 2-starcompact, countably pracompact.
* Corresponding author.
This paper was supported by Wonkwang University in 2008.

(3) A space X is *ω-starcompact* provided that for every open cover $\mathcal{U}$ of X there exist $k \in \mathbb{N}$ and a finite subset F of X such that $\mathrm{St}^k(F, \mathcal{U}) = X$.

It is clear that $\frac{1}{2}$-starcompactness coincides with compactness. For convenience, 1-starcompact spaces are simply called *starcompact*.

We notice that n-starcompact spaces in our terminology are called *n-pseudocompact* in [4] and *strongly n-starcompact* in [9], while $n\frac{1}{2}$-starcompact spaces are named *n-starcompact* in [9].

Let $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{n + \frac{1}{2} : n \in \mathbb{N}\}$. It can be easily seen from Definition 1 that an n-starcompact space is $(n + \frac{1}{2})$-starcompact for every $n \in \widetilde{\mathbb{N}}$.

Recall that a Tychonoff space is *pseudocompact* provided every continuous real-valued function is bounded. For Tychonoff spaces pseudocompactness is equivalent to (DFCC), the *discrete finite chain condition*, stating that all discrete sequences of nonempty open sets are finite. For regular spaces, (DFCC) is equivalent to 3-starcompactness [17]. Or even better: for regular spaces, (DFCC) is equivalent to $2\frac{1}{2}$-starcompactness [9]. We also have a well-known theorem.

Theorem 1.1.
(1) Countable compactness is equivalent to starcompactness in the class of Hausdorff spaces [11]*;*

(2) If $n \in \widetilde{\mathbb{N}}$ and $n \geq 3$, then every n-starcompact regular space is $2\frac{1}{2}$-starcompact [18]*;*

(3) $2\frac{1}{2}$-starcompactness is equivalent to pseudocompactness in the class of Tychonoff spaces [18]*.*

In accordance with Theorem 1.1, there are four types of n-starcompact ness.

Recall from [3] that a subspace Y of a space X is *relatively countably compact* in X if every infinite subset of Y has a cluster point in X. A space X is *countably pracompact* if it has a dense subspace Y such that Y is relatively countably compact in X. Clearly, every countably compact space is countably pracompact.

M.V. Matveev [17] obtained the following interesting result about the relationship between countable pracompactness and the above system of concepts: every countably pracompact space is 2-starcompact (see Theorem 2.5). He also constructed a locally compact 2-starcompact Hausdorff space that is not countably pracompact, and gave an example of a pseudocompact space which is not 2-starcompact.

Theorem 1.2. [9] *If a regular space X contains a closed discrete subspace Y such that $|Y| = w(X) \geq \omega$, where $w(X)$ is the weight of X, then X is not $1\frac{1}{2}$-starcompact.*

In other words, $e(X) < w(X)$ for every regular $1\frac{1}{2}$-starcompact space X, where $e(X)$ is the *extent* of X, i.e., $e(X) = sup\{|D| : D \subset X$ is closed and discrete$\} + \omega$. In particular, a "very good" space $D^\tau \setminus \{p\}$ (where $D = \{0, 1\}$ is the discrete space, τ is an infinite cardinal, and p is an arbitrary point of D^τ) is 2-starcompact but not $1\frac{1}{2}$-starcompact. Indeed, $e(D^\tau \setminus \{p\}) = w(D^\tau \setminus \{p\}) = \tau$.

Since both inequalities $w(X) > |X|$ and $w(X) < |X|$ are possible, the next theorem is independent from the previous Theorem 1.2.

Theorem 1.3. [18] *If a regular space X contains a closed discrete subspace Y such that $|Y| = |X| \geq \omega$, then X is not $1\frac{1}{2}$-starcompact.*

In this paper, we consider countable version of star covering properties to get interesting results about the relationship between starcompactness and countable pracompactness. We prove the following theorems in Section 2:

(1) Every countably $2\frac{1}{2}$-starcompact space X is pseudocompact.

(2) In a regular space, countable $2\frac{1}{2}$-starcompactness is equivalent to $2\frac{1}{2}$-starcompactness.

(3) A countably pracompact metaLindelöf space is starcompact.

(4) Every feebly compact 1-cl-starLindelöf space X is 1-cl-starcompact (and hence 2-starcompact).

We also construct examples related to countable pracompactness and H-closedness. A Hausdorff space X is called *H-closed* if X is a closed subspace of every Hausdorff space in which it is contained. In Section 3, we give the following examples:

(1) There is a non-regular H-closed pracompact space which is not compact.

(2) Using the example in [9], there is a $2\frac{1}{2}$-starcompact Hausdorff space, but not countably 2-starcompact. Also, there is a 2-starcompact Hausdorff space, but not countably starcompact.

(3) We give a CH example of a Tychonoff space that is countably pracompact, but neither $1\frac{1}{2}$-starcompact(and so not starcompact) nor metaLindelöf.

(4) There is a countably $1\frac{1}{2}$-starcompact Tychonoff space which is not countably pracompact. (Moreover, the space is of first-countable.)

(5) There exists a T_2 Lindelöf $1\frac{1}{2}$-starcompact space which is not countably pracompact.

2. Countable version of star covering properties

Recall that X is an (a)-*space* provided that for every open cover $\mathcal{U}$ and every dense subspace $Y \subset X$ there is a closed in X and discrete $A \subset Y$ such that $St(A, \mathcal{U}) = X$. In the presence of countable compactness, A should be finite; this property is called *acc*. Restrict the definition of starcompactness, acc, to countable open covers $\mathcal{U}$. It turns out that for Hausdorff spaces countable versions of the property acc and starcompactness are equivalent to countable compactness. The countable version of property (a), which is a common generalization of property (a) and countable compactness, may be of some interest.

Definition 2. A space X is called *countably n-starcompact* if, for every countable open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $\mathrm{St}^n(F, \mathcal{U}) = X$.

A space X is called *countably $n\frac{1}{2}$-starcompact* if, for every countable open cover $\mathcal{U}$ of X, there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$.

For convenience, countably 1-starcompact spaces are simply called *countably starcompact.* Clearly, every n-starcompact (respectively, $n\frac{1}{2}$-starcompact) space is countably n-starcompact (respectively, countably $n\frac{1}{2}$-starcompact).

Theorem 2.1. *Every countably $2\frac{1}{2}$-starcompact space X is pseudocompact.*

Proof. Let $f : X \to \mathbb{R}$ be continuous and let $\mathcal{U}_n = \{x \in X : n-1 < f(x) < n+1\}$ for every $n \in \mathbb{Z}$. Then $\mathcal{U} = \{U_n : n \in \mathbb{Z}\}$ is a countable open cover of X. Since X is countably $2\frac{1}{2}$-starcompact, there is a finite subcollection $\mathcal{U}_0 = \{U_{n_1}, \cdots, U_{n_k}\}$ of $\mathcal{U}$ such that $St^2(\bigcup \mathcal{U}_0, \mathcal{U}) = X$. Put $M = max\{|n_1|, |n_2|, \cdots, |n_k|\} + 3$. Then $|f(x)| \le M$ for all $x \in X$. Therefore X is pseudocompact. □

Theorem 2.2. *Every regular countably $2\frac{1}{2}$-starcompact space is $2\frac{1}{2}$-starcompact.*

Proof. Suppose X is a regular countably $2\frac{1}{2}$-starcompact space which is not $2\frac{1}{2}$-starcompact. Then X is not DFCC. So there is a discrete collection $\{U_n : n \in \omega\}$ of nonempty open subsets of X. For each $n \in \omega$, choose open subsets $V_{n,1}, V_{n,2}, V_{n,3}$ of X such that

$$U_n \supset \overline{V_{n,1}} \supset V_{n,1} \supset \overline{V_{n,2}} \supset V_{n,2} \supset \overline{V_{n,3}} \supset V_{n,3}.$$

Denote $\mathcal{O} = X \setminus \bigcup\{\overline{V_{n,1}} : n \in \omega\}$, $\mathcal{O}_{n,1} = U_n \setminus \overline{V_{n,2}}$, $\mathcal{O}_{n,2} = V_{n,1} \setminus \overline{V_{n,3}}$, $\mathcal{O}_{n,3} = V_{n,2}$ for each $n \in \omega$. Then $\mathcal{U} = \{\mathcal{O}\} \cup \{\mathcal{O}_{n,i} : n \in \omega, i = 1, 2, 3\}$ is a countable open cover of X. But for any finite subcollection $\mathcal{U}_0 \subset \mathcal{U}$, $St^2(\bigcup \mathcal{U}_0, \mathcal{U}) \neq X$. This contradicts the assumption that X is countably $2\frac{1}{2}$-starcompact. □

Corollary 2.3. *In a regular space, countable $2\frac{1}{2}$-starcompactness is equivalent to $2\frac{1}{2}$-starcompactness.*

A space X is *metaLindelöf* if every open cover of X has a point-countable open refinement.

Theorem 2.4. *A countably pracompact metaLindelöf space is starcompact.*

Proof. Let X be a countably pracompact metaLindelöf space and D be a dense subset of X which is countably compact in X. Suppose X is not starcompact. Then, by metaLindelöfness, there exists a point-countable open cover $\mathcal{U}$ of X destroying starcompactness. Let $x_0 \in D$. Then $\mathcal{V}_{x_0} = \{V \in \mathcal{U} : x_0 \in V\}$ is countable and so we may say that $\mathcal{V}_{x_0} = \{V_{0,i} \in \mathcal{U} : i \in \omega\}$. Because X is not starcompact, there exists $x_1 \in D \setminus V_{0,0}$. Take $\mathcal{V}_{x_1} = \{V_{1,i} : i \in \omega\}$ as before. Choose $x_2 \in D \setminus \bigcup_{i,j<2} V_{i,j}$, and, inductively, we choose $x_n \in D \setminus \bigcup_{i,j<n} V_{i,j}$ and $\mathcal{V}_{x_n} = \{V_{n,i} : i \in \omega\}$. Then $\{x_n : n \in \omega\}$ has a cluster point $x \in X$ since D is countably compact in X. Choose $U \in \mathcal{U}$ such that $x \in U$. Then there exists $n \in \omega$ such that $x_n \in U$. Thus $U = V_{n,i}$ for some $i \in \omega$. Take $M = max\{n, i : i \in \omega\}$. Then if $k > M$, then $x_k \in V_{n,i}$. This is a contradiction.

□

Recall that a space X has the *discrete finite chain condition* (abbreviated DFCC) provided every discrete family of nonempty open sets is finite; X is *feebly compact* provided every locally finite family of nonempty open sets is finite. It is clear that

$$\begin{array}{ccccc} & & \text{countably compact} & & \\ & & \Downarrow & & \\ \text{pseudocompact} & \Leftarrow & \text{feebly compact} & \Rightarrow & \text{DFCC} \end{array}$$

The first arrow can be reversed for Tychonoff spaces [10], the second one can be reversed for regular spaces [5].

Definition 3. [6, 18] For $[X]^{\leq\kappa} = \{A \subset X : |A| \leq \kappa\}$, we define
$st_n\text{-}l(X) = \aleph_0 \cdot \min\{\kappa :$ for every open cover $\mathcal{U}$ of X there exists an $A \in [X]^{\leq\kappa}$ such that $\mathrm{St}^n(A, \mathcal{U}) = X\}$;
$st_{n\frac{1}{2}}\text{-}l(X) = \aleph_0 \cdot \min\{\kappa :$ for every open cover $\mathcal{U}$ of X there exists a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $\mathrm{St}^n(\bigcup\mathcal{V}, \mathcal{U}) = X\}$;
$a\text{-}st\text{-}l(X) = \aleph_0 \cdot \min\{\kappa :$ for every open cover $\mathcal{U}$ of X and every dense $D \subseteq X$, there is an $A \in [D]^{\leq\kappa}$ such that $\mathrm{St}(A, \mathcal{U}) = X\}$.

A space X is said to be *n-starLindelöf* (*$n\frac{1}{2}$-starLindelöf*, *absolutely starLindelöf*) if $st_n\text{-}l(X) = \aleph_0$ (respectively, $st_{n\frac{1}{2}}\text{-}l(X) = \aleph_0$, $a\text{-}st\text{-}l(X) = \aleph_0$). We write $st\text{-}l(X)$ instead of $st_1\text{-}l(X)$, i.e., 1-starLindelöfness is the same as starLindelöfness.

Definition 4. A space X is called *n-cl-starLindelöf* if for every open cover $\mathcal{U}$ of X there exists a countable subset $F \subset X$ such that $\overline{St^n(F, \mathcal{U})} = X$.

A space X is called *$n\frac{1}{2}$-cl-starLindelöf* if for every open cover $\mathcal{U}$ of X there exists a countable subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\overline{St^n(\bigcup\mathcal{V}, \mathcal{U})} = X$.

Theorem 2.5. [18] *Every countably pracompact space is 1-cl-starcom pact (and hence 2-starcompact).*

Proposition 2.6. [18] *A pseudocompact starLindelöf space is 1-cl-starcompact (and hence 2-starcompact).*

Corollary 2.7. *Every separable pseudocompact space is 2-starcompact.*

Proof. We first claim that if X is separable, then X is starLindelöf. Let $\mathcal{U}$ be an open cover of X and D be a countable dense subset of X. We want to show that $St(D, \mathcal{U}) = X$. Let $x \in X$. Then there exists $U \in \mathcal{U}$ such that $x \in U$. Since D is dense in X, $U \cap D \neq \emptyset$.. Thus $x \in U \subset St(D, \mathcal{U})$. This proves $St(D, \mathcal{U}) = X$. Therefore X is starLindelöf, proving our claim. So by Proposition 2.6, X is 2-starcompact. □

It is easy to show (without assumption of any axiom of separation) that
(1) every n-starLindelöf space is n-cl-starLindelöf,
(2) every n-cl-starLindelöf space is $(n+1)$-starLindelöf.

If we strengthen psedocompactness by feebly compactness, but weaken n-star-Lindelöfness to n-cl-starLindelöfness in Proposition 2.6, then we get the following theorem.

Theorem 2.8. *Every feebly compact 1-cl-starLindelöf space X is 1-cl-starcompact (and hence 2-starcompact).*

Proof. Suppose that X is not 1-cl-starcompact. Then there exists an open cover $\mathcal{U}$ such that for all finite $F \subset X, \overline{St(F, \mathcal{U})} \neq X$. Since X is 1-cl-starLindelöf, there exists a countable subset $A = \{x_n : n \in \omega\} \subset X$ such that $\overline{St(A, \mathcal{U})} = X$. Choose $U_0 \in \mathcal{U}$ such that $U_0 \not\subset \overline{St(x_0, \mathcal{U})}$. Denote $V_0 = U_0 \setminus \overline{St(x_0, \mathcal{U})}$ and choose $y_0 \in V_0$. Choose $U_1 \in \mathcal{U}$ such that $U_1 \not\subset \overline{St(\{x_0, x_1, y_0\}, \mathcal{U})}$. Denote $V_1 = U_1 \setminus \overline{St(\{x_0, x_1, y_0\}, \mathcal{U})}$ and choose $y_1 \in V_1$. Inductively , we can construct $\{V_n : n \in \omega\}$ such that $V_n = U_n \setminus \overline{St(\{x_1, \cdots, x_n, y_0, \cdots, y_{n-1}\}, \mathcal{U})}$ for each $n \in \omega$. Then $\{V_n : n \in \omega\}$ is a pairwise disjoint countably infinite collection. We also note that that$\{V_n : n \in \omega\}$ is locally finite.(If $x \in X$, then there exists $x_n \in A$ such that $x \in St(x_n, \mathcal{U})$. If $k \geq n$, then $V_k \cap St(x_n, \mathcal{U}) = \emptyset$.) Since X is feebly compact, $\{V_n : n \in \omega\}$ must be finite and so we have a contradiction. □

Corollary 2.9. *Every regular starcompact, 1-cl-starLindelöf space is 1-cl-starcompact.*

3. Related Examples

Note that X is countably pracompact iff there is a dense subspace D of X such that every countable open cover $\mathcal{U}$ of X has a finite subcover of D. So, similarly, we can define *pracompactness* with respect to covering property.

Definition 5. *A space X is called pracompact if there is a dense subspace D of X such that every open cover $\mathcal{U}$ of X has a finite subcover of D.*

The following is an example of a countably pracompact space which is not countably compact.

Example 3.1. ([19], see also [10] or [12]) **The Isbell-Mrówka Ψ-space**: *Let ω be a countable discrete space, and let $\mathcal{A}$ be a maximal almost disjoint (to be short, m.a.d.) family of infinite subsets of ω. Then the set $\Psi = \omega \cup \mathcal{A}$ is topologized as follows: the points of ω are isolated while a basic neighborhood of the point $A \in \mathcal{A}$ takes the form $O_F(A) = \{A\} \cup (A \setminus F)$, where F is an arbitrary finite set.*

(1) Equipped with this topology, Ψ is a Hausdorff, zero-dimensional (hence Tychonoff) space. It is countably pracompact (ω is countably compact and dense in Ψ) and hence pseudocompact. Note that ω is an open discrete subspace of Ψ, and $\mathcal{A}$ is a closed discrete subspace of Ψ. Henceforward, we do not consider the trivial case $|\mathcal{A}| < \omega$. Since $\mathcal{A}$ is infinite, closed and discrete in Ψ, Ψ is not countably compact. Moreover, it is not even $1\frac{1}{2}$-starcompact by Theorem 1.3.

(2) Ψ is a locally compact Moore space (and hence subparacomapct) which is not metaLindelöf. (Consider the open cover $\mathcal{U} = \{\{A\} \cup \omega : A \in \mathcal{A}\}$. Using the fact that $\mathcal{A}$ is uncountable it follows that $\mathcal{U}$ cannot have a point-countable open refinement.)

(3) Also, it is sometimes convenient to consider a modification of the Ψ space starting from an uncountable open discrete subspace ω_1 instead of ω. In this case, we denote the Isbell-Mrówka Ψ-space by $\Psi(\omega_1)$.

Definition 6. A space X is called *n-cl-starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite subset $F \subset X$ such that $\overline{St^n(F, \mathcal{U})} = X$.

A space X is called *$n\frac{1}{2}$-cl-starcompact* if for every open cover $\mathcal{U}$ of X there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\overline{St^n(\bigcup \mathcal{V}, \mathcal{U})} = X$.

Recall that a Hausdorff space X is called *H-closed* if X is a closed subspace of every Hausdorff space in which it is contained.

Theorem 3.2. [10] *A Hausdorff space X is H-closed if and only if for every open cover $\mathcal{U} = \{U_s : s \in S\}$ of X there exists a finite subfamily $\{U_{s_1}, U_{s_2}, \cdots, U_{s_k}\}$ such that $\overline{U_{s_1}} \cup \overline{U_{s_2}} \cup \cdots \cup \overline{U_{s_k}} = X$.*

Note that H-closedness is equivalent to $\frac{1}{2}$-cl-starcompactness. From the above Theorem, we can also deduce the following corollary which is already known from Alexandroff and Urysohn in 1971.

Corollary 3.3. [10] *A regular space is H-closed if and only if it is compact.*

However, not every H-closed space is compact. We give an example of a non-regular H-closed pracompact space which is not compact.

Example 3.4. *There is a H-closed pracompact T_2-space which is not compact.*

Let $I = [0, 1]$ with the usual topology. Then I is compact. Let $X = I$ with a topology τ generated by a base $\mathcal{B}$ with neighborhoods of any nonzero point being as in the usual topology, while neighborhoods of 0 will have the form $U \backslash A$, where U is a neighborhood of 0 in the usual topology and $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then (X, τ) is clearly a T_2-space which *is not regular. Since A has no cluster point in X, X is not countably compact and so it is not compact.*

Claim : X is H-closed and pracompact.

Let $D = I \cap (\mathbb{Q} \backslash A)$. Then D is dense in X. Let $\mathcal{U}$ be an open cover of X. Then for each $x \in X$ there exist $U_x \in \mathcal{U}$ and $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_x$.

Since I is compact, there exist $x_1, x_2, \cdots, x_n$ in I such that $\bigcup_{i=1}^{n} B_{x_i} = I$.

Note that there exists B_{x_k} such that $0 \in B_{x_k}$.

Thus $\bigcup_{i=1}^{n} U_{x_i} \subsetneq I$, but $\bigcup_{i=1}^{n} U_{x_i} \supset D$. Therefore $\bigcup_{i=1}^{n} \overline{U_{x_i}} \supset \overline{D} = X$, and so X is H-closed. Also, D is a dense subspace of X such that every (countable) open cover $\mathcal{U}$ of X has a finite subcover of D. Hence X is a pracompact space.

The starLindelöf property is a natural generalization of starcompactness. Recall that a space X is *starLindelöf* if for every open cover $\mathcal{U}$ there is a countable subset $A \subset X$ such that $St(A, \mathcal{U}) = X$. This property generalizes not only starcompactness (i.e. in fact countable compactness) but also the Lindelöf property as well as separability. The implications from Lindelöfness and starcompactness can be, assuming T_1 or T_2, passed through countable extent:

Proposition 3.5. [6] *Every T_1-space of countable extent, i.e., $e(X) < \omega$, is starLindelöf.*

The converse is not true: a Ψ-space is separable, hence star-Lindelöf, but its extent can be as big as $\mathfrak{c}$ where $\mathfrak{c}$ is the continuum.

Note that a space X is countably compact and Lindelöf if and only if it compact.

It is clear that if X is a starcompact space, then it is countably starcompact and starLindelöf. But the converse is not true, e.g, Ψ-space. How about the following question?

Question 1. *Is every countably 2-starcompact, 2-star Lindelöf space 2-starcompact?*

The following example is from Example 2.3.8 in [9].

Example 3.6. [9] *Let $I = [0, 1]$ and express it as the union of pairwise disjoint sets $A_1, \cdots, A_{2n+1}$, each dense in I with $0, 1 \in A_{2n+1}$. Let $E_k = A_{k-1} \cup A_k \cup A_{k+1}$ for $k = 1, 3, 5, \cdots, 2n+1$ and $E_k = A_k$ for $k = 2, 4, 6, \cdots, 2n$, where $A_0 = A_1$ and $A_{2n+2} = A_{2n+1}$. Note that $E_i \cap E_j$ is dense in I if and only if $|i - j| \leq 1$ or i and j are consecutive* *odd numbers. Note also that for each $x \in I$ there is a unique index $k(x)$ such that $x \in A_{k(x)}$.*

Now let X denote the set I with the topology $\mathcal{T}_n(X)$ consisting of all subsets $G \subset I$ such that for every $x \in G$ there is an interval I_x satisfying $x \in I_x \cap E_{k(x)} \subset G$. This topology makes X a Hausdorff space that is $n\frac{1}{2}$-starcompact, but not n-starcompact.

Using the above example, we have the following two examples: There is a $2\frac{1}{2}$-starcompact Hausdorff space, but not countably 2-starcompact. Also, there is a 2-starcompact Hausdorff space, but not countably starcompact.

Example 3.7. *Let $X = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ with the topology $\mathcal{T}_2(X)$ as in Example 3.6. Then X is a $2\frac{1}{2}$-starcompact Hausdorff space, but not countably 2-starcompact.*

Example 3.8. *Let $X = A_1 \cup A_2 \cup A_3 \cup A_4$ as in Example 3.6 again where $0, 1 \in A_4$, $E_1 = A_1 \cup A_2$, $E_2 = A_2$, $E_3 = A_2 \cup A_3 \cup A_4$ and $E_4 = A_4$. Then X is a 2-starcompact Hausdorff space, but not countably $1\frac{1}{2}$-starcompact.*

We give a CH example of a Tychonoff space that is countably pracompact, but neither $1\frac{1}{2}$-starcompact(and so not starcompact) nor metaLindelöf.

Example 3.9. (**CH**) *Let C be a Cantor set and let $S = C \times C$ with the order topology given by the lexicographic order(i.e., $(a, b) < (c, d)$ if and only is $b < d$ or ($b = d$ and $a < c$)). Then S is T_2 compact, $dim(S) = 0$, and $\pi w(S) = w(S) = \mathfrak{c}$. Let $\{A_\alpha : \alpha \in \omega_1\}$ be a pairwise disjoint collection of clopen subsets of S. Let $Y = S \times \omega_1$ with the product topology. Topologize $X = Y \cup \omega_1$ as follows :*

(a) *Y is open in X,*

(b) $U_F(\alpha) = \begin{cases} (A_\alpha \times \{\alpha\} \setminus F) \cup \{\alpha\} & \text{if } \alpha \text{ is a non-limit ordinal,} \\ (A_\alpha \times \{\alpha + 1\} \setminus F) \cup \{\alpha\} & \text{if } \alpha \text{ is a limit ordinal,} \end{cases}$

where F is a proper clopen subset of X.

Then X is Tychonoff and countably pracompact since Y is countably compact. Note that X is not $1\frac{1}{2}$-starcompact by Theorem 1.2. Hence X is not starcompact. Also, X is not metaLindelöf because every pseudocompact metaLindelöf space is Lindelöf, but X is not Lindelöf.

There is a countably $1\frac{1}{2}$-starcompact Tychonoff space which is not countably pracompact. (Moreover, the space is of first-countable.)

Example 3.10. *Let κ be an uncountable cardinal. Then $|\kappa+\omega_1| = \kappa$. For each $\alpha < \kappa + \omega_1$, suppose X_α is a first-countable compact space and $|X_\alpha| \geq 2$. Let $X = \prod\{X_\alpha : \alpha < \kappa + \omega_1\}$. Define a function $f : [\kappa + \omega_1]^\omega \to \kappa + \omega_1$ by*

$$f(s) > \sup\{s \in \kappa + \omega_1 : s \text{ is countable}\}.$$

For each α, pick different points $a(\alpha)$ and $b(\alpha)$ in X_α and pick open sets $U(\alpha)$ and $V(\alpha)$ such that $a(\alpha) \in \overline{U(\alpha)}, b(\alpha) \in \overline{V(\alpha)}$ and $\overline{U(\alpha)} \cap \overline{V(\alpha)} = \emptyset$.

For each $s \in [\kappa + \omega_1]^\omega$, define

$$Y(s) = \{p \in X : p(\alpha) = a(\alpha) \text{ if } \alpha \notin s \cup \{f(s)\}, \text{ and } p(f(s)) = b(f(s))\}.$$

Notice that each $Y(s)$ is a compact subset of X. Let $Y = \bigcup\{Y(s) : s \in [\kappa + \omega_1]^\omega\}$. Then Y is Tychonoff and of first-countable.

Claim 1*: Y is countably $1\frac{1}{2}$-starcompact(and hence pseudocompact).*

Suppose $\mathcal{U} = \{U_n : n \in \omega\}$ is a countable basic open (in X) cover of Y. Then for each $n \in \omega$, $U_n = \prod\{G^n_\alpha : \alpha \in \kappa + \omega_1\}$, where if $\alpha \notin s_n$, then $G^n_\alpha = X_\alpha$ for some finite s_n. Let $s = \cup_{n\in\omega} s_n$. Then $f(s) > \sup s$. Thus $U_n \cap Y(s) \neq \emptyset$ for each $n \in \omega$. Hence $St(Y(s), \mathcal{U}) = Y$.

Claim 2*: Y is not countably pracompact.*

Let D be dense in Y. Pick $p_1 \in D$. Then $p_1 \in Y(s_1)$ for some s_1. Let $\alpha_1 = f(s_1)$. Note that $p_1(\alpha_1) = b(\alpha_1)$. Choose $p_2 \in D$ such that $p_2(\alpha_1) \in V(\alpha_1)$ and $p_2(\alpha_1 + 1) \in V(\alpha_1 + 1)$, because D is dense in X and $X_0 \times \cdots \times X_\alpha \times \cdots \times V(\alpha_1 + 1) \times \cdots$ is open in X) Also $p_2 \in Y(s_2)$ for some s_2. Then $f(s_2) > \alpha_1$. Denote $\alpha_2 = f(s_2)$.

Inductively, we can choose $p_n \in D$ such that $p_n(\alpha_i) \in V(\alpha_i)$ for $i < n$ and $p_n(\alpha_{n-1} + 1) \in V(\alpha_{n-1} + 1)$, and denote $\alpha_n = f(s_n)$. Therefore $p_n(\alpha_n) = b(\alpha_n)$. Suppose $p \in Y$ is a cluster point of $\{p_n : n \in \omega\}$. Then $p(\alpha_n) \in \overline{V(\alpha_n)}$ for all $n \in \omega$. But if $p \in Y(s)$, then $\alpha_n \in s$ for all n. Thus $f(s) > \sup \alpha_n$. So $p(f(s)) = b(f(s))$, but $p_n(f(s)) = a(f(s))$ for all n. This is a contradiction. Thus Y is not countably pracompact.

Theorem 3.11. [5] *A regular space X is DFCC if and only if every countable open cover of X has a finite collection whose union is dense in X.*

However, there exists a T_2 Lindelöf $1\frac{1}{2}$-starcompact space which is not countably pracompact.

Example 3.12. *Let $I = [0,1]$ have a usual topology $\mathcal{T}_u$. Define a topology on $X = I$ generated by a base $\mathcal{B} = \{U \backslash A : U \in \mathcal{T}_u$ and $|A| \leq \omega\}$. Then X is T_2 , but not regular.*

Claim: X is $1\frac{1}{2}$-starcompact.

Let $\mathcal{V}$ be an open cover of X. Denote $\mathcal{V} = \{V = U \backslash A : U \in \mathcal{T}_u, |A| \leq \omega\}$. Since I is compact, there exists a finite subcover $\{U_1, \cdots, U_n\}$ of $\mathcal{T}_u$. Then $\bigcup_{i=1}^n V_i \subsetneq X$, but $\bigcup_{i=1}^n \overline{V_i} = X$. Thus $St(\bigcup_{i=1}^n V_i, \mathcal{V}) = X$.

However, X is not countably pracompact because every countable subset of X is closed and discrete. We also note that X is Lindelöf because we can see it in the Claim above.

Using Example 3.6 again, we have the following example.

Example 3.13. *There exists a T_2, first-countable, Lindelöf, $1\frac{1}{2}$-starcompact space which is not countably pracompact.*

Let $I = A_1 \cup A_2 \cup A_3$. $E_1 = A_1 \cup A_2, E_2 = A_2, E_3 = A_2 \cup A_3$ and $\mathcal{B}$ be a countable base for I. Denote $\mathcal{B}' = \{B \cap E_i : B \in \mathcal{B}, i = 1,2,3\}$. Then $\mathcal{B}'$ is a countable base for X. Thus X is second countable. The rest of a verification is immediate.

References

[1] A. Arhangel'skii, *The star method, new classes of spaces, and countable compactness*, *Soviet Math. Dokl.*, **21** (1980), No. 2, 550–554.

[2] A. V. Arhangel'skii, *Spaces of functions in the topology of pointwise convergence and compact spaces* , *Uspekhi Mat. Nauk*, **39** (1984), No. 5, 11-50. (in Russian)

[3] A. V. Arhangel'skii, *Compactness*, Contemporary Problems in Mathematics. Fundamental Directions. General Topology - 2 Moscow, WINITI Publ. (1989) 5-128 (in Russian); *English translation:* Encyclopedia of Mathematical Sciences. General Topology II, Springer 1996.

[4] A. Arhangel'skii, *General topology II*, Encyclopedia of Mathematical Sciences, **50**, Springer-Verlag, Berlin, 1996.

[5] R. W. Bagley, E. H. Connel and J. D. McKnight, *On properties characterizing pseudocompact spaces*, Proc. Amer. Math. Soc. **9** (1958) 500-506.

[6] M. Bonanzinga, *Star-Lindelöf and absolutely star-Lindelöf spaces*, *Questions Answers Gen. Topology*, **16** (1998), 79-104.

[7] J. Cao, J. Kim, T. Nogura and Y. Song, *Cardinal invariants related to star covering properties*, *Topology Proc.*, **26** (2001-2002), No. 1, 83-96.

[8] M.H.Cho and J. Kim, *Topological operations of iterated star-covering properties*, *Bull. Korean Math. Soc.*, **40** (2003), 727-731.

[9] E. van Douwen, G. Reed, A. Roscoe and I. Tree, Star covering properties, *Topology Appl.*, **39** (1991), 71-103.

[10] R. Engelking, *General Topology, Revised and completed edition*, *Heldermann Verlag, Berlin*, 1989.

[11] W. M. Fleishman, *A new extension of countable compactness*, *Fund. Math.*, **67** (1970), 1-9.

[12] L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton, 1960.

[13] G. R. Hiremath, *On star with Lindelöf center property*, *J. Indian Math. Soc.*, **59** (1993), 227-242.

[14] S. Ikenaga, *Topological concepts between Lindelöf and Pseudo-Lindelöf*, *Research Reports of Nara National College of Technology*, **26** (1990), 103-108.

[15] S. Ikenaga and T. Tani, *On a topological concept between countable compactness and pseudocompactness*, *Research Reports of Numazu Technical College*, **15** (1980), 139-142.

[16] J. Kim, *Iterated starcompact topological spaces*, *Appl. Gen. Topol.*, **5** (2004), 1-10.

[17] M. V. Matveev, *On properties similar to countable compactness and pseudocompactness*, Vestnik MGU, Ser. Mat., Mekh. (1984) No 2 24-27 (in Russian), *English translation:* Moscow Univ. Math. Bull.

[18] M. Matveev, *A survey on star covering properties*, *Topology Atlas*, Preprint No. 330, 1998.

[19] S. Mrówka, *Some set-teoretic constructions in topology*, Fund. Math **44** No.2, (1977) 83-92.

[20] M. E. Rudin, I. Stares and J. E. Vaughan, *From countable compactness to absolute countable compactness*,*Proc. Amer. Math. Soc.*, **125** (1997), 927-934.

[21] Y. Song, *A study of star-covering properties in topological spaces*, Ph.D Thesis, Shizuoka University, Japan, 2000.

[22] A. H. Stone, *Paracompactness and product spaces*, *Bull. Amer. Math. Soc.*, **54** (1948), 977-982.

[23] I. Tree, *Constructing regular 2-starcompact spaces that are not strongly 2-star-Lindelöf*, *Topology Appl.*, **47** (1992), 129-132.

Junhui Kim
Division of Mathematics & Informational Statistics
Wonkwang University
Iksan 570-749, Korea
E-mail: `junhikim@wonkwang.ac.kr`

Myung Hyun Cho
Department of Mathematics Education
Wonkwang University
Iksan 570-749, Korea
E-mail: `mhcho@wonkwang.ac.kr`