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EVALUATION $E(\exp(\int_0^t h(s)\tilde{d}x(s))$ ON ANALOGUE OF WIENER MEASURE SPACE

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Abstract. In this paper we evaluate the analogue of Wiener integral $\int_{C[0,t]} x(t_1)\cdots x(t_n)d\omega_\rho(x)$ where $0 = t_0 < t_1 < \cdots < t_n \le t$ and the Paley-Wiener-Zygmund integral $\int_{C[0,t]} \exp(\int_0^t h(s)\tilde{d}x(s))d\omega_\rho(x)$ where $h(s) = t - s$ and $(C[0,t], \omega_\rho)$ is the analogue of Wiener measure space.

1. Introduction and Preliminaries

In 2002, Kun Sik Ryu and Man Kyu Im [5] presented the definition and the theories of analogue of Wiener measure ω_ρ, which is a kind of generalization of concrete Wiener measure. In 2009, Kun Sik Ryu [9] introduced the generalized Fernique's Theorem for analogue of Wiener measure space. In this paper, in the first place, we evaluate the integral

$$\int_{C[0,t]} x(t_1)x(t_2)\cdots x(t_n)d\omega_\rho(x)$$

where $0 = t_0 < t_1 < \cdots < t_n \le t$. Also using our results, we evaluate the integral $\int_{C[0,t]} \exp(\int_0^t h(s)\tilde{d}x(s))d\omega_\rho(x)$ where $h(s) = t - s$ and $(C[0,t], \omega_\rho)$ is the analogue of Wiener measure space.

In this section we present some notations, definitions and Theorem from [5][8].

(A) Let $\mathbb{R}$ be the real number system. Let $\mathcal{B}(\mathbb{R})$ be the set of all Borel measurable subsets of $\mathbb{R}$ and let $\mathcal{B}(\mathbb{R}^n)$ be the set of all Borel measurable subsets of $\mathbb{R}^n$. Let m_L be the Lebesgue measure on $\mathbb{R}$.

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(B) For a positive real number t, let $C[0,t]$ be the space of all real valued continuous functions on a closed bounded interval $[0,t]$ with the supremum norm $\|\cdot\|_\infty$.

(C) Let $M(\mathbb{R})$ be the space of all finite complex valued countably additive measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. For $p\in\mathbb{R}$, let δ_p be the Dirac measure concentrated at p with total mass one.

(D) Let n be a non-negative integer. For $\overrightarrow{t}=(t_0,t_1,\cdots,t_n)$ with $0=t_0<t_1<\cdots<t_n\le t$, let $J_{\overrightarrow{t}}:C[0,t]\longrightarrow\mathbb{R}^{n+1}$ be a function with $J_{\overrightarrow{t}}(x)=(x(t_0),x(t_1),\cdots,x(t_n))$. For $B_j(j=0,1,2,\cdots,n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\overrightarrow{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all intervals. For a non-negative finite Borel measure ρ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, we let

$$\begin{aligned}&m_\rho\Big(J_{\overrightarrow{t}}^{-1}\Big(\prod_{j=0}^n B_j\Big)\Big)\\&=\int_{B_0}\Big[\int_{\prod_{j=1}^n B_j}W(n+1;\overrightarrow{t};u_0,u_1,\cdots,u_n)d\prod_{j=1}^n m_L(u_1,\cdots,u_n)\Big]d\rho(u_0),\end{aligned}$$

where

$$\begin{aligned}&W(n+1;\overrightarrow{t};u_0,u_1,\cdots,u_n)\\&=\Big(\prod_{j=1}^n\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\Big)\exp\{-\frac{1}{2}\sum_{j=1}^n\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\}.\end{aligned}$$

Then $\mathcal{B}(C[0,t])$, the set of all Borel subsets in $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive measure ω_ρ on $(C[0,t],\mathcal{B}(C[0,t]))$ such that $\omega_\rho(I)=m_\rho(I)$ for all I in $\mathcal{I}$. For ρ in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\rho=\sum_{j=1}^4\alpha_j\rho_j$, let $\omega_\rho=\sum_{j=1}^4\alpha_j\omega_{\rho_j}$. We say that ω_ρ is the complex-valued analogue of Wiener measure on $(C[0,t],\mathcal{B}(C[0,t]))$, associated with ρ. If ρ is a Dirac measure δ_0 at the origin in $\mathbb{R}$ then ω_ρ is the classical Wiener measure.

(E) For an odd natural number n, we let $n!!=1\cdot3\cdot5\cdots n$ and for an even natural number n, we let $n!!=2\cdot4\cdot6\cdots n$ and let $0!!=(-1)!!=1$.

By the change of variables formula, we have the following theorem.

Theorem 1.1. *(Theorem 3.1 in [5]) If $f:\mathbb{R}^{n+1}\longrightarrow\mathbb{C}$ is a Borel measurable function then the following equality holds.*

$$\int_{C[a,b]} f(x(t_0), x(t_1), \cdots, x(t_n)) d\omega_\rho(x)$$
$$= \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \cdots, u_n) W(n+1; \overrightarrow{t}; u_0, u_1, \cdots, u_n)$$
$$d\Big(\prod_{j=1}^{n} m_L \times \rho\Big)((u_1, \cdots, u_n), u_0)$$

where = means that if one side exists then both sides exist and two values are equal.

Lemma 1.2. *For $n \in \mathbb{N}$ and for $A > 0$,*

$$\frac{1}{\sqrt{2\pi A}} \int_{\mathbb{R}} u^n \exp\{-\frac{(u-v)^2}{2A}\} du = \sum_{p=0}^{[\frac{n}{2}]} \binom{n}{2p} v^{n-2p} A^p (2p-1)!!$$

where [ ] is the Gauss symbol.

Proof. Let $D_{n,A}(v) = \frac{1}{\sqrt{2\pi A}} \int_{\mathbb{R}} u^n \exp\{-\frac{(u-v)^2}{2A}\} du$ and let $x = u - v$. Then $D_{n,A}(v) = \sum_{k=0}^{n} \binom{n}{k} v^{n-k} \frac{1}{\sqrt{2\pi A}} \int_{\mathbb{R}} x^k \exp\{-\frac{x^2}{2A}\} dx$. $\int_{\mathbb{R}} x^k \exp\{-\frac{x^2}{2A}\} dx = 0$ if k is odd. If k is even, i.e. $k = 2p$ then, letting $\frac{x^2}{2A} = \omega$,

$$D_{n,A}(v) = \sum_{p=0}^{[\frac{n}{2}]} \binom{n}{2p} v^{n-2p} \frac{(2A)^p}{\sqrt{\pi}} \int_0^\infty \omega^{p-\frac{1}{2}} e^{-\omega} d\omega$$
$$= \sum_{p=0}^{[\frac{n}{2}]} \binom{n}{2p} v^{n-2p} \frac{(2A)^p}{\sqrt{\pi}} \Gamma(p + \frac{1}{2}) = \sum_{p=0}^{[\frac{n}{2}]} \binom{n}{2p} v^{n-2p} A^p (2p-1)!!.$$

□

Example 1.3. (1) Suppose that $f(u) = u^2$ is ρ-integrable. Then for $a = t_0 < t_1 < t_2 \leq b$,

$$\int_{C[a,b]} x(t_1)x(t_2) d\omega_\rho(x) = (t_1 - a)\rho(\mathbb{R}) + \int_{\mathbb{R}} u^2 d\rho(u).$$

If ρ has a normal distribution with mean α and variance σ^2, then

$$\int_{C[a,b]} x(t_1)x(t_2) d\omega_\rho(x) = (t_1 - a) + \alpha^2 + \sigma^2.$$

(2) Suppose that $f(u) = u^{n_1+n_2}$ is ρ-integrable and $n_k \in \mathbb{N}$. Then for $0 = t_0 < t_1 < t_2 \le t$,

$$\int_{C[0,t]} x^{n_1}(t_1)x^{n_2}(t_2)d\omega_\rho(x) = \sum_{p_2=0}^{[\frac{n_2}{2}]} \sum_{p_1=0}^{[\frac{n_1+n_2-2p_2}{2}]} \binom{n_2}{2p_2}$$
$$\binom{n_1+n_2-2p_2}{2p_1}(t_2-t_1)^{p_2}(t_1)^{p_1}(2p_2-1)!!(2p_1-1)!!$$
$$\int_{\mathbb{R}} u^{n_1+n_2-2p_1-2p_2} d\rho(u).$$

If ρ has a normal distribution with mean α and variance σ^2, then

$$\int_{\mathbb{R}} u^{n_1+n_2-2p_1-2p_2} d\rho(u)$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} u^{n_1+n_2-2p_1-2p_2} \exp\{\frac{(u-\alpha)^2}{2\sigma^2}\} dm_L(u)$$
$$= \sum_{p=0}^{[\frac{n_1+n_2-2p_1-2p_2}{2}]} \binom{n_1+n_2-2p_1-2p_2}{2p} \alpha^{n_1+n_2-2p_1-2p_2-2p}\sigma^{2p}$$
$$(2p-1)!!.$$

Thus

$$\int_{C[0,t]} x^{n_1}(t_1)x^{n_2}(t_2)d\omega_\rho(x) = \sum_{p_2=0}^{[\frac{n_2}{2}]} \sum_{p_1=0}^{[\frac{n_1+n_2-2p_2}{2}]} \sum_{p=0}^{[\frac{n_1+n_2-2p_1-2p_2}{2}]}$$
$$\binom{n_2}{2p_2}\binom{n_1+n_2-2p_2}{2p_1}\binom{n_1+n_2-2p_1-2p_2}{2p}$$
$$\prod_{j=1}^{2}(t_j-t_{j-1})^{p_j} \prod_{j=1}^{2}(2p_j-1)!!(2p-1)!!\alpha^{n_1+n_2-2p_1-2p_2-2p}\sigma^{2p}.$$

From Theorem 2.1, 2.3 in [9] and Lemma 2.9 in [7], we have the following Theorems.

Theorem 1.4. *$\int_{C[0,t]} \exp\{a \sup_{0\le s\le t} |x(s)-x(0)|\} d\omega_\rho(x)$ is finite for all positive real number a.*

Theorem 1.5. *If $\int_{\mathbb{R}} \exp(2a|u|)d\rho(u)$ is finite, then $\int_{C[0,t]} \exp\{a \sup_{0\le s\le t} |x(s)|\} d\omega_\rho(x)$ is finite for all positive real number a.*

2. Evaluation on analogue of Wiener Measure

In this section we evaluate the integral

$$\int_{C[0,t]} x(t_1)x(t_2)\cdots x(t_n)d\omega_\rho(x)$$

where $0 = t_0 < t_1 < \cdots < t_n \leq t$.

Theorem 2.1. *Suppose that $f(u) = u^n$ is ρ-integrable and Then for $0 = t_0 < t_1 < \cdots < t_n \leq t$,*

$$\int_{C[0,t]} x(t_1)x(t_2)\cdots x(t_n)d\omega_\rho(x) = \sum_{p_1=0}^{[\frac{1}{2}]} \sum_{p_2=0}^{[\frac{2-2p_1}{2}]} \cdots \sum_{p_n=0}^{[\frac{n-2(p_1+\cdots+p_{n-1})}{2}]}$$
$$\frac{\prod_{k=2}^n (k-2\sum_{i=1}^{k-1} p_i)\prod_{k=1}^n (t_k-t_{k-1})^{p_{n+1-k}}}{(n-2\sum_{i=1}^n p_i)!\prod_{k=2,p_k>0}^n (2p_k)!!} \int_{\mathbb{R}} u_0^{n-2(p_1+p_2+\cdots+p_n)} d\rho(u_0).$$

Proof. Let $g(u_0, u_1, \cdots, u_n) = u_1 \cdots u_n$ and $J_{\vec{t}} : C[0,t] \longrightarrow \mathbb{R}^{n+1}$ be a function with $J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n))$. Then $g(x(t_0), x(t_1), \cdots, x(t_n)) = g \circ J_{\vec{t}}(x) = x(t_1)x(t_2)\cdots x(t_n)$ is Borel measurable. By Theorem 1.1 and Lemma 1.2,

$$\int_{C[0,t]} x(t_1)x(t_2)\cdots x(t_n)d\omega_\rho(x) = \frac{1}{\sqrt{(2\pi)^n \prod_{i=1}^n (t_i - t_{i-1})}} \times$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} u_1\cdots u_n \exp\{-\sum_{i=1}^n \frac{(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\} dm_L(u_1,\cdots,u_n)d\rho(u_0)$$
$$= \frac{1}{\sqrt{(2\pi)^{n-1}\prod_{i=1}^{n-1}(t_i-t_{i-1})}} \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} u_1\cdots u_{n-1} \sum_{p_1=0}^{[\frac{1}{2}]} \binom{1}{2p_1}$$
$$\times (t_n - t_{n-1})^{p_1}(2p_1-1)!! \exp\{-\frac{1}{2}\sum_{i=1}^{n-1}\frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\} u_{n-1}^{1-2p_1}$$
$$\times dm_L(u_1,\cdots,u_{n-1})d\rho(u_0)$$

$$= \sum_{p_1=0}^{[\frac{1}{2}]} \sum_{p_2=0}^{[\frac{2-2p_1}{2}]} \binom{1}{2p_1}\binom{2-2p_1}{2p_2}(t_n - t_{n-1})^{p_1}(t_{n-1}-t_{n-2})^{p_2}$$

$$\times(2p_1-1)!!(2p_2-1)!!\frac{1}{\sqrt{(2\pi)^{n-2}\prod_{i=1}^{n-2}(t_i-t_{i-1})}}$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}^{n-2}} u_1\cdots u_{n-2}^{3-2p_1-2p_2}\exp\{-\frac{1}{2}\sum_{i=1}^{n-2}\frac{(u_i-u_{i-1})^2}{t_i-t_{i-1}}\}$$

$$\times dm_L(u_1,\cdots,u_{n-2})d\rho(u_0)$$

$$= \cdots = \sum_{p_1=0}^{[\frac{1}{2}]}\sum_{p_2=0}^{[\frac{2-2p_1}{2}]}\cdots\sum_{p_n=0}^{[\frac{n-2(p_1+\cdots+p_{n-1})}{2}]}\prod_{k=1}^{n}\binom{k-2\sum_{i=1}^{k-1}p_i}{2p_k}$$

$$\times\prod_{k=1}^{n}(t_k-t_{k-1})^{p_{n+1-k}}\prod_{k=0,p_k>0}^{n}(2p_k-1)!!\int_{\mathbb{R}}u_0^{n-2(p_1+\cdots+p_n)}d\rho(u_0)$$

$$= \sum_{p_1=0}^{[\frac{1}{2}]}\sum_{p_2=0}^{[\frac{2-2p_1}{2}]}\cdots\sum_{p_n=0}^{[\frac{n-2(p_1+\cdots+p_{n-1})}{2}]}$$

$$\frac{\prod_{k=2}^{n}(k-2\sum_{i=1}^{k-1}p_i)\prod_{k=1}^{n}(t_k-t_{k-1})^{p_{n+1-k}}}{(n-2\sum_{i=1}^{n}p_i)!\prod_{k=2,p_k>0}^{n}(2p_k)!!}$$

$$\int_{\mathbb{R}}u_0^{n-2(p_1+p_2+\cdots+p_n)}d\rho(u_0).$$

□

Remark 2.2. (1) If

$$F(u_0) = \frac{1}{\sqrt{(2\pi)^n\prod_{i=1}^{n}(t_i-t_{i-1})}}\int_{\mathbb{R}^n}u_1\cdots u_n\exp\{-\sum_{i=1}^{n}\frac{(u_i-u_{i-1})^2}{2(t_i-t_{i-1})}\}$$

$$\times dm_L(u_1,\cdots,u_n),$$

then F is a polynomial of degree n. If n is even then F is an even polynomial, i.e. for any odd natural number k, the coefficient of u_0^k in F is zero. If n is odd then F is an odd polynomial, i.e for any even natural number k, the coefficient of u_0^k in F is zero.

(2) If $\rho = \delta_q$ is a dirac measure at q with total mass one and $q \neq 0$, then

$$\int_{C[0,t]} x(t_1)x(t_2)\cdots x(t_n)d\omega_\rho(x) = \sum_{p_1=0}^{[\frac{1}{2}]}\sum_{p_2=0}^{[\frac{2-2p_1}{2}]}\cdots\sum_{p_n=0}^{[\frac{n-2(p_1+\cdots+p_{n-1})}{2}]} \frac{\prod_{k=2}^{n}(k-2\sum_{i=1}^{k-1}p_i)\prod_{k=1}^{n}(t_k-t_{k-1})^{p_{n+1-k}}}{(n-2\sum_{i=1}^{n}p_i)!\prod_{k=2,p_k>0}^{n}(2p_k)!!}q^{n-2(p_1+\cdot+p_n)}.$$

Lemma 2.3. *For $t>0$, let*

$$I_n(t) = \int_{u_i>0}\cdots\int_{u_1+\cdots+u_n\le t} u_1^{a_1-1}u_2^{a_2-1}\cdots u_n^{a_n-1}du_n du_{n-1}\cdots du_1.$$

Then

$$I_n(t) = t^{(a_1+\cdots+a_n)}\frac{\Gamma(a_n)\cdots\Gamma(a_1)}{\Gamma(a_1+\cdots+a_n+1)}.$$

Proof. Let $u_i = tx_i$. Then $I_n(t) = t^{(a_1+\cdots+a_n)}I_n(1)$. So

$$I_n(1) = \int_{u_i>0}\cdots\int_{u_1+\cdots+u_n\le 1} u_1^{a_1-1}u_2^{a_2-1}\cdots u_n^{a_n-1}du_n du_{n-1}\cdots du_1$$
$$= \int_{u_i>0,0<u_n<1}\cdots\int_{0<u_1+\cdots+u_{n-1}\le 1-u_n} u_1^{a_1-1}u_2^{a_2-1}\cdots u_n^{a_n-1}du_n du_{n-1}$$
$$\cdots du_1$$
$$= \int_0^1 u_n^{a_n-1}\Big(\int_{u_i>0}\cdots\int_{u_1+\cdots+u_{n-1}\le 1-u_n} u_1^{a_1-1}u_2^{a_2-1}\cdots u_{n-1}^{a_{n-1}-1}du_{n-1}$$
$$\cdots du_1\Big)du_n$$
$$= \int_0^1 u_n^{a_n-1}I_{n-1}(1-u_n)du_n$$
$$= \int_0^1 u_n^{a_n-1}(1-u_n)^{(a_1+\cdots+a_{n-1}+1)-1}I_{n-1}(1)du_n$$
$$= I_{n-1}(1)\frac{\Gamma(a_n)\Gamma(a_1+\cdots+a_{n-1}+1)}{\Gamma(a_1+\cdots+a_n+1)}$$

Since $I_1(1) = \frac{1}{a_1}$, $I_n(1) = \frac{\Gamma(a_n)\cdots\Gamma(a_1)}{\Gamma(a_1+\cdots+a_n+1)}$. Thus $I_n(t) = t^{(a_1+\cdots+a_n)}$ $\frac{\Gamma(a_n)\cdots\Gamma(a_1)}{\Gamma(a_1+\cdots+a_n+1)}$. □

Lemma 2.4. *Let* $0 = t_0 < t_1 < \cdots < t_n \le t$. *Let* $\Delta_n = \{(t_1, \cdots, t_n) \in (0,t)^n | 0 = t_0 < t_1 < \cdots < t_n \le t\}$. *Then*

$$\int_{\Delta_n} \prod_{j=1}^{n} (t_j - t_{j-1})^{k_j} dt_n \cdots dt_1 = t^{k_1 + \cdot + k_n + n} \frac{\prod_{j=1}^{n} k_j!}{(n + k_1 + \cdots + k_n)!}.$$

Proof. Let $t_j - t_{j-1} = x_j$. Then $t_k = \sum_{j=1}^{k} x_j$ and the Jacobian of this transform is $J = 1$. By above lemma,

$$\begin{aligned}
&\int_{\Delta_n} \prod_{j=1}^{n} (t_j - t_{j-1})^{k_j} dt_n \cdots dt_1 \\
&= \int_{x_i > 0} \cdots \int_{x_1 + \cdots + x_n \le t} \prod_{j=1}^{n} x_j^{(k_j+1)-1} dx_n \cdots dx_1 \\
&= I_n(t) = t^{k_1 + \cdot + k_n + n} \frac{\prod_{j=1}^{n} k_j!}{(n + k_1 + \cdots + k_n)!}.
\end{aligned}$$

□

Lemma 2.5. *If* $\int_{\mathbb{R}} \exp(2t|u|) d\rho(u)$ *is finite, then* $\int_{C[0,t]} \exp\{-tx(0)\} \exp\{\int_0^t x(s) dm_L(s)\} d\omega_\rho(x)$ *is finite.*

Proof. By Theorem 1.5 and assumption, we have

$$\begin{aligned}
&\int_{C[0,t]} \exp\{-tx(0)\} \exp\{\int_0^t x(s) dm_L(s)\} d\omega_\rho(x) \\
&\le \int_{C[0,t]} \exp\{-tx(0)\} \exp\{\int_0^t \sup\{|x(s)| | 0 \le s \le t\} dm_L(s)\} d\omega_\rho(x) \\
&= \int_{C[0,t]} \exp\{-tx(0)\} \exp\{t \sup |x(s)| | 0 \le s \le t\} d\omega_\rho(x) \\
&\le \int_{C[0,t]} \exp\{t|x(0)|\} \exp\{t \sup\{|x(s)| | 0 \le s \le t\} d\omega_\rho(x) \\
&\le \Big(\int_{\mathbb{R}} \exp(2t|u| d\rho(u)\Big)^{\frac{1}{2}} \Big(\int_{C[0,t]} \exp\{2t \sup\{|x(s)| | 0 \le s \le t\} d\omega_\rho(x)\Big)^{\frac{1}{2}}.
\end{aligned}$$

□

Remark 2.6. If $\int_{\mathbb{R}} \exp(2t|u|) d\rho(u)$ is finite, then $\int_{\mathbb{R}} |u|^n d\rho(u)$ is finite for each n.

Now we evaluate the integral of $\int_{C[0,t]} \exp(\int_0^t h(s) \tilde{dx}(s)) d\omega_\rho(x)$ where $h(s) = t - s$.

Theorem 2.7. *Suppose that* $\int_{\mathbb{R}} \exp(2t|u|)d\rho(u)$ *is finite and let* $h(s) = t - s$ *on* $[0,t]$ *and let* $\Delta_n = \{(s_1, \cdots, s_n) \in (0,t)^n | 0 = s_0 < s_1 < \cdots < s_n \leq t\}$. *Then*

$$\int_{C[0,t]} \exp(\int_0^t h(s)\tilde{d}x(s))d\omega_\rho(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p_1=0}^{[\frac{1}{2}]}\sum_{p_2=0}^{[\frac{2-2p_1}{2}]}\cdots$$

$$\sum_{p_n=0}^{[\frac{n-2(p_1+\cdots+p_{n-1})}{2}]} \frac{(-t^m)}{m!} \frac{\prod_{k=2}^n (k - 2\sum_{i=1}^{k-1} p_i)}{(n - 2\sum_{i=1}^n p_i)! \prod_{k=2,p_k>0}^n (2p_k)!!} t^{(p_1+\cdots+p_n+n)}$$

$$\frac{\prod_{i=1}^n p_i!}{(n + p_1 + \cdots + p_n)!} \int_{\mathbb{R}} u_0^{m+n-2(p_1+p_2+\cdots+p_n)} d\rho(u_0).$$

Proof. Let $h(s) = t - s$ on $[0,t]$. Then $h(t) = 0, h(0) = t$. And The Paley-Wiener-Zygmund integral $\int_0^t h(s)\tilde{d}x(s)$ equals to the Riemann-Stieltjes integral $\int_0^t h(s)dx(s)$ $\omega_\rho - a.e.x$. By the integration by part of Riemann-Stieltjes integral [1], $\int_0^t h(s)\tilde{d}x(s) = -tx(0) + \int_0^t x(s)dm_L(s)$ $\omega_\rho - a.e.x$. Let

$$B = \int_{C[0,t]} \exp(\int_0^t h(s)\tilde{d}x(s))d\omega_\rho(x).$$

Then $\exp(\int_0^t h(s)\hat{d}x(s))$ is integrable by lemma 2.5 and

$$\begin{aligned}
B &= \int_{C[0,t]} \exp(-tx(0)) \exp(\int_0^t x(s)dm_L(s))d\omega_\rho(x) \\
&= \int_{C[0,t]} \sum_{m=0}^{\infty} \frac{(-tx(0))^m}{m!} \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\int_0^t x(s)dm_L(s)\Big)^n d\omega_\rho(x) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-t)^m}{m!n!} \int_{C[0,t]} n! \int_{\Delta_n} (x(0))^m x(s_1)\cdots x(s_n) ds_1 \cdots ds_n d\omega_\rho(x) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-t)^m}{m!} \int_{\Delta_n} [\int_{C[0,t]} (x(0))^m x(s_1)\cdots x(s_n) d\omega_\rho(x)] ds_1 \cdots ds_n.
\end{aligned}$$

Also let $A = \int_{C[0,t]}(x(0))^m x(s_1)\cdots x(s_n)d\omega_\rho(x)$. Then by Theorem 2.1, we have

$$A = \sum_{p_1=0}^{[\frac{1}{2}]} \sum_{p_2=0}^{[\frac{2-2p_1}{2}]} \cdots \sum_{p_n=0}^{[\frac{n-2(p_1+\cdots+p_{n-1})}{2}]}$$
$$\frac{\prod_{k=2}^{n}(k-2\sum_{i=1}^{k-1}p_i)\prod_{k=1}^{n}(s_k-s_{k-1})^{p_{n+1-k}}}{(n-2\sum_{i=1}^{n}p_i)!\prod_{k=2,p_k>0}^{n}(2p_k)!!}$$
$$\int_{\mathbb{R}} u_0^{m+n-2(p_1+p_2+\cdots+p_n)}d\rho(u_0).$$

Thus by lemma 2.4

$$B = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p_1=0}^{[\frac{1}{2}]} \sum_{p_2=0}^{[\frac{2-2p_1}{2}]} \cdots \sum_{p_n=0}^{[\frac{n-2(p_1+\cdots+p_{n-1})}{2}]}$$
$$\frac{(-t^m)}{m!}\frac{\prod_{k=2}^{n}(k-2\sum_{i=1}^{k-1}p_i)}{(n-2\sum_{i=1}^{n}p_i)!\prod_{k=2,p_k>0}^{n}(2p_k)!!}t^{(p_1+\cdots+p_n+n)}$$
$$\frac{\prod_{i=1}^{n}p_i!}{(n+p_1+\cdots+p_n)!}\int_{\mathbb{R}} u_0^{m+n-2(p_1+p_2+\cdots+p_n)}d\rho(u_0).$$

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