

## GENERAL TYPES OF $(\alpha, \beta)$ -FUZZY IDEALS OF HEMIRINGS

Y. B. JUN, W. A. DUDEK, M. SHABIR AND M. S. KANG\*

### Abstract

W. A. Dudek, M. Shabir and M. Irfan Ali discussed the properties of  $(\alpha, \beta)$ -fuzzy ideals of hemirings in [9]. In this paper, we discuss the generalization of their results on  $(\alpha, \beta)$ -fuzzy ideals of hemirings. As a generalization of the notions of  $(\alpha, \in \vee q)$ -fuzzy left (right) ideals,  $(\alpha, \in \vee q)$ -fuzzy  $h$ -ideals and  $(\alpha, \in \vee q)$ -fuzzy  $k$ -ideals, the concepts of  $(\alpha, \in \vee q_m)$ -fuzzy left (right) ideals,  $(\alpha, \in \vee q_m)$ -fuzzy  $h$ -ideals and  $(\alpha, \in \vee q_m)$ -fuzzy  $k$ -ideals are defined, and their characterizations are considered. Using a left (right) ideal (resp.  $h$ -ideal,  $k$ -ideal), we construct an  $(\alpha, \in \vee q_m)$ -fuzzy left (right) ideal (resp.  $(\alpha, \in \vee q_m)$ -fuzzy  $h$ -ideal,  $(\alpha, \in \vee q_m)$ -fuzzy  $k$ -ideal). The implication-based fuzzy  $h$ -ideals ( $k$ -ideals) of a hemiring are considered.

### 1. Introduction

Hemirings (semirings with zero and commutative addition) which provide a common generalization of rings and distributive lattices arise naturally in such diverse areas of mathematics as combinatorics, functional analysis, graph theory, automata theory, formal language theory, mathematical modelling of quantum physics and parallel computation systems (see for example [3, 12, 13, 14]). Using the concept of soft sets which is introduced by Molodtsov, Feng, Jun and Zhao [10] initiated the study of soft semirings. Fuzzy semirings were first investigated in [2] and [1].

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Received July 26, 2010. Accepted August 26, 2010.

2000 Mathematics Subject Classification. 16Y60, 08A72.

Keywords :  $(\alpha, \in \vee q_m)$ -fuzzy left (right) ideal,  $(\alpha, \in \vee q_m)$ -fuzzy  $h$  ( $k$ )-ideal, fuzzifying left (right) ideal, fuzzifying  $h$  ( $k$ )-ideal,  $t$ -implication-based fuzzy left (right) ideal,  $t$ -implication-based fuzzy  $h$  ( $k$ )-ideal.

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Fuzzy  $k$ -ideals of semirings were studied by many authors, for example [11, 19]. Fuzzy  $h$ -ideals of hemirings were studied in [19, 23]. The idea of fuzzy point and its “belongingness” and “quasicoincidence” with a fuzzy set were given by Pu and Liu [20]. In [6], Bhakat and Das used this idea to define  $(\alpha, \beta)$ -fuzzy subgroups. In [4, 5, 6, 7, 8],  $(\alpha, \beta)$ -fuzzy substructures of algebraic structures are discussed. Jun [15] considered more general form of the notion of quasi-coincidence of a fuzzy point with a fuzzy set, and generalized results in the papers [16, 17]. He introduced the notions of  $(\in, q_k)$ -fuzzy subalgebras and  $(\in, \in \vee q_k)$ -fuzzy subalgebras in a BCK/BCI-algebra, and investigated several properties. He also discussed characterizations of  $(\in, \in \vee q_k)$ -fuzzy subalgebra in a BCK/BCI-algebra. Dudek et al. [9] restricted the study of such fuzzy substructures to different types of  $(\alpha, \beta)$ -fuzzy ideals, where  $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ . Jun [18] discussed more updated results than [9, Theorem 4.17] and [9, Corollary 4.18].

In this paper, we generalize the properties of  $(\alpha, \beta)$ -fuzzy ideals of hemirings, which were studied in [9] by W. A. Dudek, M. Shabir and M. Irfan Ali. We introduce the notions of  $(\alpha, \in \vee q_m)$ -fuzzy left (right) ideals,  $(\alpha, \in \vee q_m)$ -fuzzy  $h$ -ideals and  $(\alpha, \in \vee q_m)$ -fuzzy  $k$ -ideals which are a generalization of the notions of  $(\alpha, \in \vee q)$ -fuzzy left (right) ideals,  $(\alpha, \in \vee q)$ -fuzzy  $h$ -ideals and  $(\alpha, \in \vee q)$ -fuzzy  $k$ -ideals. We construct an  $(\alpha, \in \vee q_m)$ -fuzzy left (right) ideal (resp.  $(\alpha, \in \vee q_m)$ -fuzzy  $h$ -ideal,  $(\alpha, \in \vee q_m)$ -fuzzy  $k$ -ideal) by using a left (right) ideal (resp.  $h$ -ideal,  $k$ -ideal). We finally consider the implication-based fuzzy  $h$ -ideals ( $k$ -ideals) of a hemiring. The important achievement of the study with an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal ( $k$ -ideal) is that the notion of an  $(\in, \in \vee q)$ -fuzzy  $h$ -ideal ( $k$ -ideal) is a special case of an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal ( $k$ -ideal), and thus so many results in the papers [9] are corollaries of our results obtained in this paper.

## 2. Preliminaries

A *semiring* is an algebraic system  $(R, +, \cdot)$  consisting of a non-empty set  $R$  together with two binary operations called addition ( $+$ ) and multiplication ( $\cdot$ ), here  $x \cdot y$  will be denoted by juxtaposition for all  $x, y \in R$ , such that  $(R, +)$  and  $(R, \cdot)$  are semigroups connected by the following distributive laws:  $a(b + c) = ab + ac$  and  $(b + c)a = ba + ca$  for all  $a, b, c \in R$ . An element  $0 \in R$  is called a *zero* of  $R$  if  $a + 0 = 0 + a = a$  for all  $a \in R$ . A semiring with zero and a commutative addition is called a *hemiring*. An element  $1 \in R$  is called the *identity* of  $R$  if  $1a = a1 = a$

for all  $a \in R$ . A semiring with a commutative multiplication is called a *commutative semiring*. A non-empty subset  $A$  of a semiring  $R$  is called a *subsemiring* of  $R$  if it is closed under the addition and multiplication. A non-empty subset  $I$  of a semiring  $R$  is said to be a *left* (resp. *right*) *ideal* of  $R$  if it is closed under the addition and  $RI \subseteq I$  (resp.  $IR \subseteq I$ ). A left ideal which is also a right ideal is called an *ideal*. A left (resp. right) ideal  $I$  of a hemiring  $R$  is called a *left* (resp. *right*) *k-ideal* of  $R$  if for any  $a, b \in I$  and  $x \in R$  whenever  $x + a = b$  then  $x \in I$ . A left (resp. right) ideal  $I$  of a hemiring  $R$  is called a *left* (resp. *right*) *h-ideal* of  $R$  if for any  $a, b \in I$  and all  $x, y \in R$  whenever  $x + a + y = b + y$  then  $x \in I$ . Every left (resp. right) *h-ideal* is a left (resp. right) *k-ideal* but the converse is not true in general. For a set  $R$ , let

$$\mathcal{F}(R) := \{\mu \mid \mu : R \rightarrow [0, 1] \text{ is a mapping}\}.$$

Elements of  $\mathcal{F}(R)$  are called *fuzzy subsets* of  $R$ .

A fuzzy subset  $\mu$  of a hemiring  $R$  is called a *fuzzy left* (resp. *right*) *ideal* of  $R$  if it satisfies:

$$(2.1) \quad \mu(x + y) \geq \min\{\mu(x), \mu(y)\},$$

$$(2.2) \quad \mu(yx) \geq \mu(x) \text{ (resp. } \mu(xy) \geq \mu(x))$$

for all  $x, y \in R$ .

A fuzzy subset  $\mu$  of a hemiring  $R$  is called a *fuzzy left* (resp. *right*) *h-ideal* of  $R$  if it is a fuzzy left (resp. right) ideal of  $R$  such that for all  $a, b, x, y \in R$ ,

$$(2.3) \quad x + a + y = b + y \implies \mu(x) \geq \min\{\mu(a), \mu(b)\}.$$

A fuzzy subset  $\mu$  of a hemiring  $R$  is called a *fuzzy left* (resp. *right*) *k-ideal* of  $R$  if it is a fuzzy left (resp. right) ideal of  $R$  such that for all  $a, b, x \in R$ ,

$$(2.4) \quad x + a = b \implies \mu(x) \geq \min\{\mu(a), \mu(b)\}.$$

For any  $\mu \in \mathcal{F}(R)$  and any  $t \in [0, 1]$ , the set

$$U(\mu; t) = \{x \in R \mid \mu(x) \geq t\}$$

is called a *level subset* of  $\mu$ . Given a point  $x \in R$ , consider a mapping

$$\mu : R \rightarrow [0, 1], y \mapsto \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Then  $\mu \in \mathcal{F}(R)$ , and it is said to be a *fuzzy point* with support  $x$  and value  $t$  and is denoted by  $[x; t]$ .

For any  $\mu \in \mathcal{F}(R)$ , we say that a fuzzy point  $[x; t]$  is

- (i) *contained* in  $\mu$ , denoted by  $[x; t] \in \mu$ , ([20]) if  $\mu(x) \geq t$ .

- (ii) *quasi-coincident* with  $\mu$ , denoted by  $[x; t] \text{q} \mu$ , ([20]) if  $\mu(x) + t > 1$ .  
 For a fuzzy point  $[x; t]$  and  $\mu \in \mathcal{F}(R)$ , we say that
  - (i)  $[x; t] \in \vee \text{q} \mu$  if  $[x; t] \in \mu$  or  $[x; t] \text{q} \mu$ .
  - (ii)  $[x; t] \bar{\alpha} \mu$  if  $[x; t] \alpha \mu$  does not hold for  $\alpha \in \{\in, \text{q}, \in \vee \text{q}, \in \wedge \text{q}\}$ .

**3. Generalizations of  $(\alpha, \beta)$ -fuzzy ideals**

In what follows, let  $R$  denote a hemiring and  $m$  an arbitrary element of  $[0, 1)$  unless otherwise specified. For a fuzzy point  $[x; t]$  and a fuzzy subset  $\mu$  of  $R$ , we say that

- (i)  $[x; t] \text{q}_m \mu$  if  $\mu(x) + t + m > 1$ .
- (ii)  $[x; t] \in \vee \text{q}_m \mu$  if  $[x; t] \in \mu$  or  $[x; t] \text{q}_m \mu$ .
- (iii)  $[x; t] \in \wedge \text{q}_m \mu$  if  $[x; t] \in \mu$  and  $[x; t] \text{q}_m \mu$ .
- (iv)  $[x; t] \bar{\alpha}_m \mu$  if  $[x; t] \alpha_m \mu$  does not hold for  $\alpha \in \{\text{q}_m, \in \vee \text{q}_m, \in \wedge \text{q}_m\}$ .

**DEFINITION 3.1.** Let  $\alpha \in \{\in, \text{q}, \in \vee \text{q}\}$ . A fuzzy subset  $\mu$  of  $R$  is called an  $(\alpha, \in \vee \text{q}_m)$ -fuzzy left (resp. right) ideal of  $R$  if for any  $x, y \in R$  and  $t, r \in (0, 1]$ ,

$$(3.1) \quad [x; t] \alpha \mu, [y; r] \alpha \mu \Rightarrow [x + y; \min\{t, r\}] \in \vee \text{q}_m \mu,$$

$$(3.2) \quad [x; t] \alpha \mu \Rightarrow [yx; t] \in \vee \text{q}_m \mu \text{ (resp. } [xy; t] \in \vee \text{q}_m \mu).$$

An  $(\alpha, \in \vee \text{q}_m)$ -fuzzy left and right ideal is called an  $(\alpha, \in \vee \text{q}_m)$ -fuzzy ideal. An  $(\alpha, \in \vee \text{q}_m)$ -fuzzy left (resp. right) ideal of  $R$  with  $m = 0$  is called an  $(\alpha, \in \vee \text{q})$ -fuzzy left (resp. right) ideal of  $R$  (see [9]).

**DEFINITION 3.2.** Let  $\alpha \in \{\in, \text{q}, \in \vee \text{q}\}$ . A fuzzy subset  $\mu$  of  $R$  is called an  $(\alpha, \in \vee \text{q}_m)$ -fuzzy  $k$ -ideal of  $R$  if it is an  $(\alpha, \in \vee \text{q}_m)$ -fuzzy ideal of  $R$  satisfying the following condition:

$$(3.3) \quad x + a = b, [a; t] \alpha \mu, [b; r] \alpha \mu \implies [x; \min\{t, r\}] \in \vee \text{q}_m \mu$$

for all  $a, b, x \in R$  and  $t, r \in (0, 1]$ .

An  $(\alpha, \in \vee \text{q}_m)$ -fuzzy  $k$ -ideal of  $R$  with  $m = 0$  is called an  $(\alpha, \in \vee \text{q})$ -fuzzy  $k$ -ideal of  $R$  (see [9]).

**DEFINITION 3.3.** Let  $\alpha \in \{\in, \text{q}, \in \vee \text{q}\}$ . A fuzzy subset  $\mu$  of  $R$  is called an  $(\alpha, \in \vee \text{q}_m)$ -fuzzy  $h$ -ideal of  $R$  if it is an  $(\alpha, \in \vee \text{q}_m)$ -fuzzy ideal of  $R$  satisfying the following condition:

$$(3.4) \quad a + y = b + y, [a; t] \alpha \mu, [b; r] \alpha \mu \implies [x; \min\{t, r\}] \in \vee \text{q}_m \mu$$

for all  $a, b, x, y \in R$  and  $t, r \in (0, 1]$ .

An  $(\alpha, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$  with  $m = 0$  is called an  $(\alpha, \in \vee q)$ -fuzzy  $h$ -ideal of  $R$  (see [9]).

**THEOREM 3.4.** *If  $I$  is a left (resp. right) ideal of  $R$ , then a fuzzy subset  $\mu$  of  $R$  such that  $\mu(x) \geq \frac{1-m}{2}$  for  $x \in I$  and  $\mu(x) = 0$  otherwise is an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$ .*

*Proof.* Let  $x, y \in R$  and  $t, r \in (0, 1]$  be such that  $[x; t] \in \mu$  and  $[y; r] \in \mu$ . Then  $\mu(x) \geq t > 0$  and  $\mu(y) \geq r > 0$ . Thus  $\mu(x) \geq \frac{1-m}{2}$  and  $\mu(y) \geq \frac{1-m}{2}$ , i.e.,  $x, y \in I$ . Hence  $x + y \in I$ , which implies that  $\mu(x + y) \geq \frac{1-m}{2}$ . If  $\min\{t, r\} \leq \frac{1-m}{2}$ , then  $\mu(x + y) \geq \frac{1-m}{2} \geq \min\{t, r\}$ , i.e.,  $[x + y; \min\{t, r\}] \in \mu$ . If  $\min\{t, r\} > \frac{1-m}{2}$ , then

$$\mu(x + y) + \min\{t, r\} + m > \frac{1-m}{2} + \frac{1-m}{2} + m = 1,$$

i.e.,  $[x + y; \min\{t, r\}] q_m \mu$ . Therefore  $[x + y; \min\{t, r\}] \in \vee q_m \mu$ . Now, let  $x \in R$  and  $t \in (0, 1]$  be such that  $[x; t] \in \mu$ . Then  $\mu(x) \geq t$ , and so  $\mu(x) \geq \frac{1-m}{2}$ , i.e.,  $x \in I$ . Since  $I$  is a left ideal of  $R$ , we obtain  $yx \in I$  for all  $y \in R$ . Hence  $\mu(yx) \geq \frac{1-m}{2}$ . If  $t \leq \frac{1-m}{2}$ , then  $\mu(yx) \geq \frac{1-m}{2} \geq t$ , i.e.,  $[yx; t] \in \mu$ . If  $t > \frac{1-m}{2}$ , then

$$\mu(yx) + t + m > \frac{1-m}{2} + \frac{1-m}{2} + m = 1$$

and thus  $[yx; t] q_m \mu$ . Therefore  $[yx; t] \in \vee q_m \mu$ . Similarly,  $[xy; t] \in \vee q_m \mu$ . Therefore  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$ .  $\square$

**THEOREM 3.5.** *If  $I$  is a left (resp. right) ideal of  $R$ , then a fuzzy subset  $\mu$  of  $R$  such that  $\mu(x) \geq \frac{1-m}{2}$  for  $x \in I$  and  $\mu(x) = 0$  otherwise is a  $(q, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$ .*

*Proof.* Let  $x, y \in R$  and  $t, r \in (0, 1]$  be such that  $[x; t] q \mu$  and  $[y; r] q \mu$ . Then  $x, y \in I$ ,  $\mu(x) + t > 1$  and  $\mu(y) + r > 1$ . Since  $x + y \in I$ ,  $\mu(x + y) \geq \frac{1-m}{2}$ . If  $\min\{t, r\} \leq \frac{1-m}{2}$ , then

$$\mu(x + y) \geq \frac{1-m}{2} \geq \min\{t, r\}.$$

Hence  $[x + y; \min\{t, r\}] \in \mu$ . If  $\min\{t, r\} > \frac{1-m}{2}$ , then

$$\mu(x + y) + \min\{t, r\} + m > \frac{1-m}{2} + \frac{1-m}{2} + m = 1$$

and so  $[x + y; \min\{t, r\}] q_m \mu$ . Thus  $[x + y; \min\{t, r\}] \in \vee q_m \mu$ . Now let  $x \in R$  and  $t \in (0, 1]$  be such that  $[x; t] q \mu$ . Then  $\mu(x) + t > 1$ , and so  $x \in I$ . Since  $I$  is a left ideal of  $R$ ,  $yx \in I$  for all  $y \in R$ . Hence  $\mu(yx) \geq \frac{1-m}{2}$ . If  $t \leq \frac{1-m}{2}$ , then  $\mu(yx) \geq \frac{1-m}{2} \geq t$ , i.e.,  $[yx; t] \in \mu$ . If  $t > \frac{1-m}{2}$ , then

$$\mu(yx) + t + m > \frac{1-m}{2} + \frac{1-m}{2} + m = 1$$

which implies that  $[yx; t] \text{q}_m \mu$ . Therefore  $[yx; t] \in \vee \text{q}_m \mu$ . Similarly, we get  $[xy; t] \in \vee \text{q}_m \mu$ . Consequently,  $\mu$  is a  $(\text{q}, \in \vee \text{q}_m)$ -fuzzy left (resp. right) ideal of  $R$ .  $\square$

**THEOREM 3.6.** *If  $I$  is a left (resp. right) ideal of  $R$ , then a fuzzy subset  $\mu$  of  $R$  such that  $\mu(x) \geq \frac{1-m}{2}$  for  $x \in I$  and  $\mu(x) = 0$  otherwise is an  $(\in \vee \text{q}, \in \vee \text{q}_m)$ -fuzzy left (resp. right) ideal of  $R$ .*

*Proof.* Assume that  $[x; t] \in \vee \text{q} \mu$  and  $[y; r] \in \vee \text{q} \mu$  for all  $x, y \in R$  and  $t, r \in (0, 1]$ . Then we have the following four cases:

- (i)  $[x; t] \in \mu$  and  $[y; r] \in \mu$ ,
- (ii)  $[x; t] \in \mu$  and  $[y; r] \text{q} \mu$ ,
- (iii)  $[x; t] \text{q} \mu$  and  $[y; r] \in \mu$ ,
- (iv)  $[x; t] \text{q} \mu$  and  $[y; r] \text{q} \mu$ .

Cases (i) and (iv) imply that  $[x + y; \min\{t, r\}] \in \vee \text{q}_m \mu$  by the proof of Theorems 3.4 and 3.5. Second case implies that  $\mu(x) \geq t$  and  $\mu(y) + r > 1$ . Then  $x, y \in I$ , and so  $x + y \in I$ . Thus  $\mu(x + y) \geq \frac{1-m}{2}$ . If  $\min\{t, r\} \leq \frac{1-m}{2}$ , then

$$\mu(x + y) \geq \frac{1-m}{2} \geq \min\{t, r\}$$

and so  $[x + y; \min\{t, r\}] \in \mu$ . If  $\min\{t, r\} > \frac{1-m}{2}$ , then

$$\mu(x + y) + \min\{t, r\} + m > \frac{1-m}{2} + \frac{1-m}{2} + m = 1$$

which implies that  $[x+y; \min\{t, r\}] \text{q}_m \mu$ . Thus  $[x+y; \min\{t, r\}] \in \vee \text{q}_m \mu$ . Similarly, the third case induces the desired result. Next, let  $x \in R$  and  $t \in (0, 1]$  be such that  $[x; t] \in \vee \text{q} \mu$ . Then  $[x; t] \in \mu$  or  $[x; t] \text{q} \mu$ . Using the same process with the proof of Theorems 3.4 and 3.5, we conclude that  $[yx; t] \in \vee \text{q}_m \mu$  and  $[xy; t] \in \vee \text{q}_m \mu$  for all  $y \in R$ . Consequently,  $\mu$  is an  $(\in \vee \text{q}, \in \vee \text{q}_m)$ -fuzzy left (resp. right) ideal of  $R$ .  $\square$

If we take  $m = 0$  in Theorems 3.4, 3.5 and 3.6, then we have the following corollary.

**COROLLARY 3.7.** [9, Theorem 3.5] *If  $I$  is a left (resp. right) ideal of  $R$ , then a fuzzy subset  $\mu$  of  $R$  such that  $\mu(x) \geq 0.5$  for  $x \in I$  and  $\mu(x) = 0$  otherwise is an  $(\alpha, \in \vee \text{q})$ -fuzzy left (resp. right) ideal of  $R$  for  $\alpha \in \{\in, \text{q}, \in \vee \text{q}\}$ .*

The following example shows that there exists  $m \in [0, 1)$  such that the fuzzy subset  $\mu$  defined in Theorem 3.4 may not be an  $(\in, \in \wedge \text{q}_m)$ -fuzzy left (resp. right) ideal of  $R$ .

TABLE 1. Cayley tables for binary operations “+” and “.”

+	0	1	a	b	c	·	0	1	a	b	c
0	0	1	a	b	c	0	0	0	0	0	0
1	1	b	1	a	1	1	0	1	a	b	c
a	a	1	a	b	a	a	0	a	a	a	c
b	b	a	b	1	b	b	0	b	a	1	c
c	c	1	a	b	c	c	0	c	c	c	0

**EXAMPLE 3.8.** Consider a set  $R = \{0, 1, a, b, c\}$  with two binary operations defined by Table 1. Then  $(R, +, \cdot)$  is a hemiring and  $I := \{0, a, c\}$  is an ideal of  $R$  (see [9, Example 3.7]). Let  $\mu$  be a fuzzy subset of  $R$  defined by

$$\mu(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0.4 & \text{if } x \in \{a, c\}, \\ 0 & \text{if } x \in \{1, b\}. \end{cases}$$

Then  $\mu$  is an  $(\alpha, \in \vee q_{0.3})$ -fuzzy left (resp. right) ideal of  $R$  for  $\alpha \in \{\in, q, \in \vee q\}$  by Theorems 3.4, 3.5 and 3.6. But it is not an  $(\in, \in \wedge q_{0.3})$ -fuzzy left (resp. right) ideal of  $R$  since  $[a; 0.2] \in \mu$  and  $[c; 0.23] \in \mu$ , but

$$[a + c; \min\{0.2, 0.23\}] = [a; 0.2] \overline{\in \wedge q_{0.3} \mu}.$$

**THEOREM 3.9.** For any  $h$ -ideal  $I$  of  $R$ , let  $\mu$  be a fuzzy subset of  $R$  defined by  $\mu(x) \geq \frac{1-m}{2}$  for  $x \in I$  and  $\mu(x) = 0$  otherwise. Then  $\mu$  is an  $(\alpha, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$ , where  $\alpha \in \{\in, q, \in \vee q\}$ .

*Proof.* According to Theorems 3.4, 3.5 and 3.6,  $\mu$  is an  $(\alpha, \in \vee q_m)$ -fuzzy ideal of  $R$  for  $\alpha \in \{\in, q, \in \vee q\}$ . Assume that  $I$  is an  $h$ -ideal of  $R$  and let  $a, b, x, y \in R$  and  $t, r \in (0, 1]$  be such that  $x + a + y = b + y$ .

(1) If  $[a; t] \in \mu$  and  $[b; r] \in \mu$ , then  $\mu(a) \geq t$  and  $\mu(b) \geq r$ , and so  $a, b \in I$ . Since  $I$  is an  $h$ -ideal,  $x \in I$ . Hence  $\mu(x) \geq \frac{1-m}{2}$ . If  $\min\{t, r\} \leq \frac{1-m}{2}$ , then

$$\mu(x) \geq \frac{1-m}{2} \geq \min\{t, r\},$$

i.e.,  $[x; \min\{t, r\}] \in \mu$ . If  $\min\{t, r\} > \frac{1-m}{2}$ , then

$$\mu(x) + \min\{t, r\} + m > \frac{1-m}{2} + \frac{1-m}{2} + m = 1$$

and thus  $[x; \min\{t, r\}] q_m \mu$ . Therefore  $[x; \min\{t, r\}] \in \vee q_m \mu$ , and consequently  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$ .

(2) If  $[a; t] q \mu$  and  $[b; r] q \mu$ , then  $\mu(a) + t > 1$  and  $\mu(b) + r > 1$ , which imply that  $\mu(a) \geq \frac{1-m}{2}$  and  $\mu(b) \geq \frac{1-m}{2}$ . Hence  $a, b \in I$ , and so  $x \in I$

since  $I$  is an  $h$ -ideal. Therefore  $\mu(x) \geq \frac{1-m}{2}$ . By the similar method to the case (1), we conclude that  $[x; \min\{t, r\}] \in \vee q_m \mu$ . Hence  $\mu$  is a  $(q, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$ .

(3) If  $[a; t] \in \vee q \mu$  and  $[b; r] \in \vee q \mu$ , then we have the following four cases:

- (i)  $[a; t] \in \mu$  and  $[b; r] \in \mu$ ,
- (ii)  $[a; t] \in \mu$  and  $[b; r] q \mu$ ,
- (iii)  $[a; t] q \mu$  and  $[b; r] \in \mu$ ,
- (iv)  $[a; t] q \mu$  and  $[b; r] q \mu$ .

By the similar way to the method in (1) and (2), we have desired results for cases (i) and (iv). Case (ii) (resp. (iii)) implies that  $\mu(a) \geq t$  and  $\mu(b) + r > 1$  (resp.  $\mu(a) + t > 1$  and  $\mu(b) \geq r$ ) so that  $\mu(a) \geq \frac{1-m}{2}$  and  $\mu(b) \geq \frac{1-m}{2}$ . Thus  $a, b \in I$ , and so  $x \in I$  since  $I$  is an  $h$ -ideal. Therefore  $\mu(x) \geq \frac{1-m}{2}$ , which induces  $[x; \min\{t, r\}] \in \vee q_m \mu$ . Hence  $\mu$  is an  $(\in \vee q, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$ .  $\square$

**COROLLARY 3.10.** For any  $k$ -ideal  $I$  of  $R$ , let  $\mu$  be a fuzzy subset of  $R$  defined by  $\mu(x) \geq \frac{1-m}{2}$  for  $x \in I$  and  $\mu(x) = 0$  otherwise. Then  $\mu$  is an  $(\alpha, \in \vee q_m)$ -fuzzy  $k$ -ideal of  $R$ , where  $\alpha \in \{\in, q, \in \vee q\}$ .

*Proof.* It is straightforward by taking  $y = 0$  in Theorem 3.9.  $\square$

If we take  $m = 0$  in Theorem 3.9, then we obtain the following corollary.

**COROLLARY 3.11.** [9, Corollary 3.6] For any  $h$ -ideal  $I$  of  $R$ , let  $\mu$  be a fuzzy subset of  $R$  defined by  $\mu(x) \geq 0.5$  for  $x \in I$  and  $\mu(x) = 0$  otherwise. Then  $\mu$  is an  $(\alpha, \in \vee q)$ -fuzzy  $h$ -ideal of  $R$ , where  $\alpha \in \{\in, q, \in \vee q\}$ .

**THEOREM 3.12.** A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$  if and only if it satisfies:

- (1)  $\mu(x + y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ ,
- (2)  $\mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\}$  (resp.  $\mu(xy) \geq \min\{\mu(x), \frac{1-m}{2}\}$ ).

for all  $x, y \in R$ .

*Proof.* Let  $\mu$  be an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$ . Assume that (1) is not valid. Then there exist  $a, b \in R$  such that

$$\mu(a + b) < \min\{\mu(a), \mu(b), \frac{1-m}{2}\}.$$

If  $\min\{\mu(a), \mu(b)\} < \frac{1-m}{2}$ , then  $\mu(a + b) < t \leq \min\{\mu(a), \mu(b)\}$  for some  $t \in (0, \frac{1-m}{2})$ . It follows that  $[a; t] \in \mu$  and  $[b; t] \in \mu$ , but  $[a + b; t] \notin \mu$ . Moreover,  $\mu(a + b) + t < 2t < 1 - m$  and so  $[a + b; t] \overline{q}_m \mu$ .

Therefore  $[a + b; t] \overline{\in} \vee q_m \mu$ , a contradiction. If  $\min\{\mu(a), \mu(b)\} \geq \frac{1-m}{2}$ , then  $\mu(a) \geq \frac{1-m}{2}$ ,  $\mu(b) \geq \frac{1-m}{2}$  and  $\mu(a + b) < \frac{1-m}{2}$ . Hence  $[a; \frac{1-m}{2}] \in \mu$  and  $[b; \frac{1-m}{2}] \in \mu$ , but  $[a+b; \frac{1-m}{2}] \overline{\in} \mu$ . Also,  $\mu(a+b) + \frac{1-m}{2} < \frac{1-m}{2} + \frac{1-m}{2} = 1 - m$ , i.e.,  $[a + b; \frac{1-m}{2}] \overline{q}_m \mu$ . Hence  $[a + b; \frac{1-m}{2}] \overline{\in} \vee q_m \mu$ , which is also a contradiction. Consequently, (1) is valid. Now, let  $x, y \in R$  and  $\mu(x) < \frac{1-m}{2}$ . If  $\mu(yx) < \mu(x)$ , then there exists  $t \in (0, 1)$  such that  $\mu(yx) < t \leq \mu(x)$ . This implies  $[x; t] \in \mu$  and  $[yx; t] \overline{\in} \vee q_m \mu$ , which contradicts (3.2). Hence  $\mu(yx) \geq \mu(x) = \min\{\mu(x), \frac{1-m}{2}\}$ . Next, let  $\mu(x) \geq \frac{1-m}{2}$ . Then  $[x; \frac{1-m}{2}] \in \mu$ . If  $\mu(yx) < \frac{1-m}{2}$ , then

$$\mu(yx) + \frac{1-m}{2} < \frac{1-m}{2} + \frac{1-m}{2} = 1 - m$$

and so  $[yx; \frac{1-m}{2}] \overline{\in} \vee q_m \mu$ , which contradicts (3.2). Hence  $\mu(yx) \geq \frac{1-m}{2} = \min\{\mu(x), \frac{1-m}{2}\}$ . Similarly, we have the desired result for the right case. Therefore (2) holds.

Conversely, suppose that two conditions (1) and (2) are valid. Let  $x, y \in R$  and  $t, r \in (0, 1]$  be such that  $[x; t] \in \mu$  and  $[y; r] \in \mu$ . Then

$$\mu(x + y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{t, r, \frac{1-m}{2}\}.$$

Assume that  $t \leq \frac{1-m}{2}$  or  $r \leq \frac{1-m}{2}$ . Then  $\mu(x + y) \geq \min\{t, r\}$ , and so  $[x + y; \min\{t, r\}] \in \mu$ . Now, suppose that  $t > \frac{1-m}{2}$  and  $r > \frac{1-m}{2}$ . Then  $\mu(x + y) \geq \frac{1-m}{2}$ , and thus

$$\mu(x + y) + \min\{t, r\} > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m,$$

i.e.,  $[x + y; \min\{t, r\}] \overline{q}_m \mu$ . Hence  $[x + y; \min\{t, r\}] \in \vee q_m \mu$ . By (2), we have  $\mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\} \geq \min\{t, \frac{1-m}{2}\}$ . If  $t \leq \frac{1-m}{2}$ , then  $\mu(yx) \geq t$ , i.e.,  $[yx; t] \in \mu$ . If  $t > \frac{1-m}{2}$ , then  $\mu(yx) + t > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m$ , which implies that  $[yx; t] \overline{q}_m \mu$ . Therefore  $[yx; t] \in \vee q_m \mu$ . Similarly,  $[xy; t] \in \vee q_m \mu$ . Consequently,  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$ .  $\square$

If we take  $m = 0$  in Theorem 3.12, then we have the following corollary.

**COROLLARY 3.13.** [9, Lemmas 4.3, 4.4 and 4.5] *A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q)$ -fuzzy left (resp. right) ideal of  $R$  if and only if it satisfies:*

- (1)  $\mu(x + y) \geq \min\{\mu(x), \mu(y), 0.5\}$ ,
- (2)  $\mu(yx) \geq \min\{\mu(x), 0.5\}$  (resp.  $\mu(xy) \geq \min\{\mu(x), 0.5\}$ ).

for all  $x, y \in R$ .

**LEMMA 3.14.** *Let  $\mu$  be a fuzzy subset of  $R$  and let  $a, b, x, y \in R$  be such that  $x + a + y = b + y$ . Then for any  $t, r \in (0, 1]$  the following statements are equivalent:*

- (1)  $[a; t] \in \mu, [b; r] \in \mu \Rightarrow [x; \min\{t, r\}] \in \vee q_m \mu,$
- (2)  $\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}.$

*Proof.* (1)  $\Rightarrow$  (2) Let  $a, b, x, y \in R$  be such that  $x + a + y = b + y$ . If (2) is false, then

$$\mu(x) < t \leq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$$

for some  $t \in (0, \frac{1-m}{2}]$ . Hence  $[a; t] \in \mu$  and  $[b; t] \in \mu$ , but  $[x; t] \notin \mu$ . Also,  $\mu(x) + t \leq 2t \leq 1 - m$ , i.e.,  $[x; t] \overline{q}_m \mu$ . Therefore  $[x; t] \notin \vee q_m \mu$  which contradicts (1). Thus (2) is valid.

(2)  $\Rightarrow$  (1) Let  $a, b, x, y \in R$  and  $t, r \in (0, 1]$  be such that  $x + a + y = b + y, [a; t] \in \mu$  and  $[b; r] \in \mu$ . Then

$$\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} \geq \min\{t, r, \frac{1-m}{2}\}.$$

If  $\min\{t, r\} \leq \frac{1-m}{2}$ , then  $\mu(x) \geq \min\{t, r\}$  and so  $[x; \min\{t, r\}] \in \mu$ . If  $\min\{t, r\} > \frac{1-m}{2}$ , then  $\mu(x) \geq \frac{1-m}{2}$  and thus

$$\mu(x) + \min\{t, r\} > \frac{1-m}{2} + \frac{1-m}{2} = 1 - m,$$

i.e.,  $[x; \min\{t, r\}] q_m \mu$ . Therefore  $[x; \min\{t, r\}] \in \vee q_m \mu$ . □

**COROLLARY 3.15.** *Let  $\mu$  be a fuzzy subset of  $R$ . For any  $a, b, x, y \in R$  and  $t, r \in (0, 1]$ , the following are equivalent:*

- (1)  $x + a = b, [a; t] \in \mu, [b; r] \in \mu \Rightarrow [x; \min\{t, r\}] \in \vee q_m \mu,$
- (2)  $x + a = b \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}.$

*Proof.* Straightforward by taking  $y = 0$  in Lemma 3.14. □

If we take  $m = 0$  in Lemma 3.14, then we have the following corollary.

**COROLLARY 3.16.** [9, Lemmas 4.7] *Let  $\mu$  be a fuzzy subset of  $R$  and let  $a, b, x, y \in R$  be such that  $x + a + y = b + y$ . Then for any  $t, r \in (0, 1]$  the following statements are equivalent:*

- (1)  $[a; t] \in \mu, [b; r] \in \mu \Rightarrow [x; \min\{t, r\}] \in \vee q \mu,$
- (2)  $\mu(x) \geq \min\{\mu(a), \mu(b), 0.5\}.$

Combining Theorem 3.12 and Lemma 3.14, we have the following theorem.

**THEOREM 3.17.** *A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$  if and only if it satisfies conditions (1) and (2) in Theorem 3.12, and*

$$(3.5) \quad x + a + y = b + y \implies \mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$$

for all  $a, b, x, y \in R$ .

**COROLLARY 3.18.** *A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q_m)$ -fuzzy  $k$ -ideal of  $R$  if and only if it satisfies conditions (1) and (2) in Theorem 3.12, and Corollary 3.15(2) .*

If we take  $m = 0$  in Theorem 3.17, then we have the following corollary.

**COROLLARY 3.19.** [9] *A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q)$ -fuzzy  $h$ -ideal of  $R$  if and only if it satisfies conditions (1) and (2) in Corollary 3.13, and*

$$(3.6) \quad x + a + y = b + y \implies \mu(x) \geq \min\{\mu(a), \mu(b), 0.5\}$$

for all  $a, b, x, y \in R$ .

**THEOREM 3.20.** *Let  $\mu$  be a fuzzy subset of  $R$ . Then  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$  if and only if its nonempty level set  $U(\mu; t)$  is a left (resp. right) ideal of  $R$  for all  $t \in (0, \frac{1-m}{2}]$ .*

*Proof.* Assume that  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$ . Let  $t \in (0, \frac{1-m}{2}]$  and  $x, y \in U(\mu; t)$ . Then  $\mu(x) \geq t$  and  $\mu(y) \geq t$ . It follows from Theorem 3.12(1) that

$$\mu(x + y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{t, \frac{1-m}{2}\} = t$$

so that  $x + y \in U(\mu; t)$ . For any  $x \in U(\mu; t)$  and  $y \in R$ , we get

$$\mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\} \geq \min\{t, \frac{1-m}{2}\} = t$$

by Theorem 3.12(2). Thus  $yx \in U(\mu; t)$ . Similarly,  $xy \in U(\mu; t)$ . Therefore  $U(\mu; t)$  is a left (resp. right) ideal of  $R$  for all  $t \in (0, \frac{1-m}{2}]$ .

Conversely, suppose that  $U(\mu; t)$  is a nonempty left (resp. right) ideal of  $R$  for all  $t \in (0, \frac{1-m}{2}]$ . If there exist  $x_0, y_0 \in R$  such that

$$\mu(x_0 + y_0) < \min\{\mu(x_0), \mu(y_0), \frac{1-m}{2}\},$$

then  $\mu(x_0 + y_0) < t_0 \leq \min\{\mu(x_0), \mu(y_0), \frac{1-m}{2}\}$  for some  $t_0 \in (0, \frac{1-m}{2}]$ . Thus  $x_0, y_0 \in U(\mu; t_0)$  and  $x_0 + y_0 \notin U(\mu; t_0)$ , which is a contradiction.

Hence

$$\mu(x + y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$$

for all  $x, y \in R$ . If  $\mu(y_0x_0) < \min\{\mu(x_0), \frac{1-m}{2}\}$  for some  $x_0, y_0 \in R$ , then there exists  $t \in (0, \frac{1-m}{2}]$  such that  $\mu(y_0x_0) < t \leq \min\{\mu(x_0), \frac{1-m}{2}\}$ . It follows that  $x_0 \in U(\mu; t)$  and  $y_0x_0 \notin U(\mu; t)$ . This is a contradiction, and so

$$\mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\}$$

for all  $x, y \in R$ . Similarly,  $\mu(xy) \geq \min\{\mu(x), \frac{1-m}{2}\}$  for all  $x, y \in R$ . Using Theorem 3.12, we know that  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$ .  $\square$

**COROLLARY 3.21.** [9] *Let  $\mu$  be a fuzzy subset of  $R$ . Then  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy left (resp. right) ideal of  $R$  if and only if its nonempty level set  $U(\mu; t)$  is a left (resp. right) ideal of  $R$  for all  $t \in (0, 0.5]$ .*

Let  $\mu$  be an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$  and let  $t \in (0, \frac{1-m}{2}]$  be such that  $U(\mu; t) \neq \emptyset$ . For any  $x, y \in R$  and  $a, b \in U(\mu; t)$ , if  $x + a + y = b + y$  then  $\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} \geq \min\{t, \frac{1-m}{2}\} = t$  by (3.5), and so  $x \in U(\mu; t)$ . Now let  $\mu$  be a fuzzy subset of  $R$  such that its nonempty level set  $U(\mu; t)$  is an  $h$ -ideal of  $R$  for all  $t \in (0, \frac{1-m}{2}]$ . Then  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy ideal of  $R$  by Theorem 3.20. Let  $a, b, x, y \in R$  be such that  $x + a + y = b + y$ . Assume that

$$\mu(x) < \min\{\mu(a), \mu(b), \frac{1-m}{2}\} = t_x.$$

Then  $t_x \in (0, \frac{1-m}{2}]$  and  $a, b \in U(\mu; t_x)$ , but  $x \notin U(\mu; t_x)$ . This is a contradiction, and so  $\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$ . Therefore we have the following theorem.

**THEOREM 3.22.** *A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$  if and only if its nonempty level set  $U(\mu; t)$  is an  $h$ -ideal of  $R$  for all  $t \in (0, \frac{1-m}{2}]$ .*

**COROLLARY 3.23.** *A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q_m)$ -fuzzy  $k$ -ideal of  $R$  if and only if its nonempty level set  $U(\mu; t)$  is a  $k$ -ideal of  $R$  for all  $t \in (0, \frac{1-m}{2}]$ .*

**COROLLARY 3.24.** [9] *A fuzzy subset  $\mu$  of  $R$  is an  $(\in, \in \vee q)$ -fuzzy  $h$ -ideal ( $k$ -ideal) of  $R$  if and only if its nonempty level set  $U(\mu; t)$  is an  $h$ -ideal ( $k$ -ideal) of  $R$  for all  $t \in (0, 0.5]$ .*

**THEOREM 3.25.** *Let  $\{\mu_i \mid i \in \Lambda\}$  be a family of  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideals of  $R$ . Then  $\mu := \bigcap_{i \in \Lambda} \mu_i$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$ .*

*Proof.* We first show that  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left (resp. right) ideal of  $R$ . Let  $x, y \in R$  and  $t_1, t_2 \in (0, 1]$  be such that  $[x; t_1] \in \mu$  and

$[y, t_2] \in \mu$ . Assume that  $[x + y; \min\{t_1, t_2\}] \overline{\in \vee q_m} \mu$ . Then  $\mu(x + y) < \min\{t_1, t_2\}$  and  $\mu(x + y) + \min\{t_1, t_2\} \leq 1 - m$ , which imply that

$$(3.7) \quad \mu(x + y) < \frac{1-m}{2}.$$

Let  $\Omega_1 := \{i \in \Lambda \mid [x + y; \min\{t_1, t_2\}] \in \mu_i\}$  and

$$\Omega_2 := \{i \in \Lambda \mid [x+y; \min\{t_1, t_2\}] q_m \mu_i\} \cap \{j \in \Lambda \mid [x+y; \min\{t_1, t_2\}] \overline{\in} \mu_j\}.$$

Then  $\Lambda = \Omega_1 \cup \Omega_2$  and  $\Omega_1 \cap \Omega_2 = \emptyset$ . If  $\Omega_2 = \emptyset$ , then  $[x + y; \min\{t_1, t_2\}] \in \mu_i$  for all  $i \in \Lambda$ , that is,  $\mu_i(x+y) \geq \min\{t_1, t_2\}$  for all  $i \in \Lambda$ , which implies that  $\mu(x + y) \geq \min\{t_1, t_2\}$ . This is a contradiction. Hence  $\Omega_2 \neq \emptyset$ , and so for every  $i \in \Omega_2$  we have  $\mu_i(x + y) < \min\{t_1, t_2\}$  and

$$\mu_i(x + y) + \min\{t_1, t_2\} > 1 - m.$$

It follows that  $\min\{t_1, t_2\} > \frac{1-m}{2}$ . Now  $[x; t_1] \in \mu$  implies  $\mu(x) \geq t_1$  and thus  $\mu_i(x) \geq \mu(x) \geq t_1 \geq \min\{t_1, t_2\} > \frac{1-m}{2}$  for all  $i \in \Lambda$ . Similarly  $\mu_i(y) > \frac{1-m}{2}$  for all  $i \in \Lambda$ . Next suppose that  $t := \mu_i(x + y) < \frac{1-m}{2}$ . Taking  $t < r < \frac{1-m}{2}$ , we get  $[x; r] \in \mu_i$  and  $[y; r] \in \mu_i$ , but  $[x+y; \min\{r, r\}] = [x + y; r] \overline{\in \vee q_m} \mu_i$ . This contradicts (3.1). Hence  $\mu_i(x + y) \geq \frac{1-m}{2}$  for all  $i \in \Lambda$ , and so  $\mu(x + y) \geq \frac{1-m}{2}$  which contradicts (3.7). Therefore  $[x + y; \min\{t_1, t_2\}] \in \vee q_m \mu$ . Similarly, we can show that if  $[x; t] \in \mu$ , where  $t \in (0, 1]$ , then  $[xy; t] \in \vee q_m \mu$  and  $[yx; t] \in \vee q_m \mu$  for all  $y \in R$ . Now, let  $a, b, x, y \in R$  and  $t, r \in (0, 1]$  be such that  $x + a + y = b + y$ ,  $[a; t] \in \mu$  and  $[b; r] \in \mu$ . If  $[x; \min\{t, r\}] \overline{\in \vee q_m} \mu$ , then  $\mu(x) < \min\{t, r\}$  and  $\mu(x) + \min\{t, r\} + m \leq 1$ . It follows that

$$(3.8) \quad \mu(x) < \frac{1-m}{2}.$$

If  $\mu_i(x) \geq \min\{t, r\}$  for all  $\mu_i$ , then  $\mu(x) \geq \min\{t, r\}$  which is impossible. Hence  $\mu_i(x) + \min\{t, r\} + m > 1$  and  $\mu_i(x) < \min\{t, r\}$  for some  $\mu_i$  since each  $\mu_i$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal. Thus

$$2 \min\{t, r\} > \mu_i(x) + \min\{t, r\} > 1 - m,$$

and so  $\min\{t, r\} > \frac{1-m}{2}$ . Since  $[a; t] \in \mu$  and  $[b; r] \in \mu$ , we get  $\mu_i(a) \geq \mu(a) \geq t$  and  $\mu_i(b) \geq \mu(b) \geq r$ . It follow that

$$\min\{\mu_i(a), \mu_i(b)\} \geq \min\{\mu(a), \mu(b)\} \geq \min\{t, r\} > \frac{1-m}{2}.$$

Suppose that  $\mu_i(x) = s < \frac{1-m}{2}$  for some  $\mu_i$ . Then there exists  $s' \in (0, \frac{1-m}{2}]$  such that  $s < s' \leq \frac{1-m}{2}$ . Thus  $\mu_i(a) > \frac{1-m}{2} \geq s'$  and  $\mu_i(b) > \frac{1-m}{2} \geq s'$ , which imply that  $[a; s'] \in \mu_i$  and  $[b; s'] \in \mu_i$ . On the other hand,  $\mu_i(x) = s < s'$  and  $\mu_i(x) + s' < 2s' \leq 1 - m$ , that is,  $[x; s'] \overline{\in \vee q_m} \mu_i$ . This contradicts the assumption that  $\mu_i$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$ . Therefore  $\mu_i(x) \geq \frac{1-m}{2}$  for all  $\mu_i$ , and consequently  $\mu(x) \geq \frac{1-m}{2}$ .

This contradicts (3.8), and hence  $[x; \min\{t, r\}] \in \vee q_m \mu$ . This completes the proof.  $\square$

**COROLLARY 3.26.** *The intersection of any family of  $(\in, \in \vee q_m)$ -fuzzy  $k$ -ideals of  $R$  is an  $(\in, \in \vee q_m)$ -fuzzy  $k$ -ideal of  $R$ .*

If we take  $m = 0$  in Theorem 3.25 and Corollary 3.26, then we have the following corollary.

**COROLLARY 3.27.** [9] *The intersection of any family of  $(\in, \in \vee q)$ -fuzzy  $h$ -ideals (resp.  $k$ -ideals) of  $R$  is an  $(\in, \in \vee q)$ -fuzzy  $h$ -ideal (resp.  $k$ -ideal) of  $R$ .*

**LEMMA 3.28.** [9] *A fuzzy subset  $\mu$  of  $R$  is a fuzzy  $h$ -ideal (resp.  $k$ -ideal) of  $R$  if and only if  $\mu$  is an  $(\in, \in)$ -fuzzy  $h$ -ideal (resp.  $k$ -ideal) of  $R$ .*

We provide a condition for an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal (resp.  $k$ -ideal) to be an  $(\in, \in)$ -fuzzy  $h$ -ideal (resp.  $k$ -ideal)

**THEOREM 3.29.** *Any  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal (resp.  $k$ -ideal)  $\mu$  of  $R$  such that  $\mu(x) < \frac{1-m}{2}$  for all  $x \in R$  is an  $(\in, \in)$ -fuzzy  $h$ -ideal (resp.  $k$ -ideal) of  $R$ .*

*Proof.* Since  $\mu(x) < \frac{1-m}{2}$  for all  $x \in R$ , it is straightforward.  $\square$

**COROLLARY 3.30.** [9] *Any  $(\in, \in \vee q)$ -fuzzy  $h$ -ideal (resp.  $k$ -ideal)  $\mu$  of  $R$  such that  $\mu(x) < 0.5$  for all  $x \in R$  is an  $(\in, \in)$ -fuzzy  $h$ -ideal (resp.  $k$ -ideal) of  $R$ .*

For any fuzzy subset  $\mu$  of  $R$  and any  $t \in (0, 1]$ , we consider four subsets:

$$Q(\mu; t) := \{x \in R \mid [x; t]q\mu\}, \quad [\mu]_t := \{x \in R \mid [x; t] \in \vee q \mu\}, \\ Q^m(\mu; t) := \{x \in R \mid [x; t]q_m \mu\}, \quad [\mu]_t^m := \{x \in R \mid [x; t] \in \vee q_m \mu\}.$$

It is clear that  $[\mu]_t = U(\mu; t) \cup Q(\mu; t)$  and  $[\mu]_t^m = U(\mu; t) \cup Q^m(\mu; t)$ .

**THEOREM 3.31.** *If  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$ , then*

$$(3.9) \quad \left(\forall t \in \left(\frac{1-m}{2}, 1\right]\right) \left(Q^m(\mu; t) \neq \emptyset \Rightarrow Q^m(\mu; t) \text{ is an } h\text{-ideal of } R\right).$$

*Proof.* Assume that  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$  and let  $t \in (\frac{1-m}{2}, 1]$  be such that  $Q^m(\mu; t) \neq \emptyset$ . Let  $x, y \in Q^m(\mu; t)$ . Then  $[x; t]q_m \mu$  and  $[y; t]q_m \mu$ , i.e.,  $\mu(x) + t + m > 1$  and  $\mu(y) + t + m > 1$ .

Using Theorem 3.12(1), we have

$$\begin{aligned} \mu(x + y) &\geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \\ &= \begin{cases} \min\{\mu(x), \mu(y)\} & \text{if } \min\{\mu(x), \mu(y)\} < \frac{1-m}{2}, \\ \frac{1-m}{2} & \text{if } \min\{\mu(x), \mu(y)\} \geq \frac{1-m}{2} \end{cases} \\ &> 1 - t - m, \end{aligned}$$

that is,  $[x + y; t]_{\text{qm}} \mu$ . Hence  $x + y \in Q^m(\mu; t)$ . Let  $x \in Q^m(\mu; t)$  and  $y \in R$ . Then  $\mu(x) + t + m > 1$ . Theorem 3.12(2) implies that

$$\begin{aligned} \mu(yx) &\geq \min\{\mu(x), \frac{1-m}{2}\} \\ &= \begin{cases} \frac{1-m}{2} & \text{if } \mu(x) \geq \frac{1-m}{2}, \\ \mu(x) & \text{if } \mu(x) < \frac{1-m}{2}, \end{cases} \\ &> 1 - t - m, \end{aligned}$$

that is,  $[yx; t]_{\text{qm}} \mu$ . Hence  $yx \in Q^m(\mu; t)$ . Similarly,  $xy \in Q^m(\mu; t)$ . Hence  $Q^m(\mu; t)$  is an ideal of  $R$ . Now, let  $a, b \in Q^m(\mu; t)$  and  $x, y \in R$  be such that  $x + a + y = b + y$ . Then  $\mu(a) + t + m > 1$  and  $\mu(b) + t + m > 1$ . Using (3.5), we get

$$\begin{aligned} \mu(x) &\geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} \\ &= \begin{cases} \min\{\mu(a), \mu(b)\} & \text{if } \min\{\mu(a), \mu(b)\} < \frac{1-m}{2}, \\ \frac{1-m}{2} & \text{if } \min\{\mu(a), \mu(b)\} \geq \frac{1-m}{2} \end{cases} \\ &> 1 - t - m, \end{aligned}$$

that is,  $[x; t]_{\text{qm}} \mu$ . Therefore  $x \in Q^m(\mu; t)$ , and consequently  $Q^m(\mu; t)$  is an  $h$ -ideal of  $R$ . □

**COROLLARY 3.32.** *If  $\mu$  is an  $(\in, \in \vee \text{qm})$ -fuzzy  $k$ -ideal of  $R$ , then*

$$\left(\forall t \in \left(\frac{1-m}{2}, 1\right]\right) \left(Q^m(\mu; t) \neq \emptyset \Rightarrow Q^m(\mu; t) \text{ is a } k\text{-ideal of } R\right).$$

If we take  $m = 0$  in Theorem 3.31 and Corollary 3.32, then we have the following corollaries.

**COROLLARY 3.33.** [18] *If  $\mu$  is an  $(\in, \in \vee \text{q})$ -fuzzy  $h$ -ideal of  $R$ , then*

$$\left(\forall t \in (0.5, 1]\right) \left(Q(\mu; t) \neq \emptyset \Rightarrow Q(\mu; t) \text{ is an } h\text{-ideal of } R\right).$$

**COROLLARY 3.34.** [18] *If  $\mu$  is an  $(\in, \in \vee \text{q})$ -fuzzy  $k$ -ideal of  $R$ , then*

$$\left(\forall t \in (0.5, 1]\right) \left(Q(\mu; t) \neq \emptyset \Rightarrow Q(\mu; t) \text{ is a } k\text{-ideal of } R\right).$$

**THEOREM 3.35.** *For any fuzzy subset  $\mu$  of  $R$ , the following are equivalent:*

- (1)  $\mu$  is an  $(\in, \in \vee \text{qm})$ -fuzzy  $h$ -ideal of  $R$ .
- (2)  $(\forall t \in (0, 1]) \left([\mu]_t^m \neq \emptyset \Rightarrow [\mu]_t^m \text{ is an } h\text{-ideal of } R\right)$ .

*Proof.* Assume that  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$  and let  $t \in (0, 1]$  be such that  $[\mu]_t^m \neq \emptyset$ . Let  $x, y \in [\mu]_t^m$ . Then  $\mu(x) \geq t$  or  $\mu(x) + t + m > 1$ , and  $\mu(y) \geq t$  or  $\mu(y) + t + m > 1$ . We can consider four cases:

$$(3.10) \quad \mu(x) \geq t \text{ and } \mu(y) \geq t,$$

$$(3.11) \quad \mu(x) \geq t \text{ and } \mu(y) + t + m > 1,$$

$$(3.12) \quad \mu(x) + t + m > 1 \text{ and } \mu(y) \geq t,$$

$$(3.13) \quad \mu(x) + t + m > 1 \text{ and } \mu(y) + t + m > 1.$$

For the first case, Theorem 3.12(1) implies that

$$\mu(x+y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \geq \min\{t, \frac{1-m}{2}\} = \begin{cases} \frac{1-m}{2} & \text{if } t > \frac{1-m}{2}, \\ t & \text{if } t \leq \frac{1-m}{2}, \end{cases}$$

and so  $\mu(x+y) + t + m > \frac{1-m}{2} + \frac{1-m}{2} + m = 1$ , i.e.  $(x+y)_t q_m \mu$ , or  $x+y \in U(\mu; t)$ . Therefore  $x+y \in U(\mu; t) \cup Q^m(\mu; t) = [\mu]_t^m$ . For the case (3.11), assume that  $t > \frac{1-m}{2}$ . Then  $1-t-m \leq 1-t < \frac{1-m}{2}$ . Theorem 3.12(1) implies that

$$\begin{aligned} & \mu(x+y) \\ & \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \\ & = \begin{cases} \min\{\mu(y), \frac{1-m}{2}\} > 1-t-m & \text{if } \min\{\mu(y), \frac{1-m}{2}\} \leq \mu(x), \\ \mu(x) \geq t & \text{if } \min\{\mu(y), \frac{1-m}{2}\} > \mu(x). \end{cases} \end{aligned}$$

Thus  $x+y \in U(\mu; t) \cup Q^m(\mu; t) = [\mu]_t^m$ . Suppose that  $t \leq \frac{1-m}{2}$ . Then  $1-t \geq \frac{1-m}{2}$ . Using Theorem 3.12(1), we obtain

$$\begin{aligned} \mu(x+y) & \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \\ & = \begin{cases} \min\{\mu(x), \frac{1-m}{2}\} \geq t & \text{if } \min\{\mu(x), \frac{1-m}{2}\} \leq \mu(y), \\ \mu(y) > 1-t-m & \text{if } \min\{\mu(x), \frac{1-m}{2}\} > \mu(y), \end{cases} \end{aligned}$$

which implies that  $x+y \in U(\mu; t) \cup Q^m(\mu; t) = [\mu]_t^m$ . We have similar result for the case (3.12). For the final case, if  $t > \frac{1-m}{2}$  then  $1-t-m \leq 1-t < \frac{1-m}{2}$ . Hence

$$\begin{aligned} & \mu(x+y) \\ & \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \\ & = \begin{cases} \frac{1-m}{2} > 1-t-m & \text{if } \min\{\mu(x), \mu(y)\} \geq \frac{1-m}{2}, \\ \min\{\mu(x), \mu(y)\} > 1-t-m & \text{if } \min\{\mu(x), \mu(y)\} < \frac{1-m}{2}, \end{cases} \end{aligned}$$

and so  $x + y \in Q^m(\mu; t) \subseteq [\mu]_t^m$ . If  $t \leq \frac{1-m}{2}$ , then

$$\begin{aligned} & \mu(x + y) \\ & \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \\ & = \begin{cases} \frac{1-m}{2} \geq t & \text{if } \min\{\mu(x), \mu(y)\} \geq \frac{1-m}{2}, \\ \min\{\mu(x), \mu(y)\} > 1 - t - m & \text{if } \min\{\mu(x), \mu(y)\} < \frac{1-m}{2}, \end{cases} \end{aligned}$$

which implies that  $x + y \in U(\mu; t) \cup Q^m(\mu; t) = [\mu]_t^m$ . Let  $x \in [\mu]_t^m$  and  $y \in R$ . Then  $\mu(x) \geq t$  or  $\mu(x) + t + m > 1$ . Assume that  $\mu(x) \geq t$ . Theorem 3.12(2) implies that

$$\begin{aligned} \mu(yx) & \geq \min\{\mu(x), \frac{1-m}{2}\} \geq \min\{t, \frac{1-m}{2}\} \\ & = \begin{cases} t & \text{if } t \leq \frac{1-m}{2}, \\ \frac{1-m}{2} > 1 - t - m & \text{if } t > \frac{1-m}{2} \end{cases} \end{aligned}$$

so that  $yx \in U(\mu; t) \cup Q^m(\mu; t) = [\mu]_t^m$ . Suppose that  $\mu(x) + t + m > 1$ . If  $t > \frac{1-m}{2}$ , then

$$\mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\} = \begin{cases} \frac{1-m}{2} > 1 - t - m & \text{if } \mu(x) \geq \frac{1-m}{2}, \\ \mu(x) > 1 - t - m & \text{if } \mu(x) < \frac{1-m}{2} \end{cases}$$

and thus  $yx \in Q^m(\mu; t) \subseteq [\mu]_t^m$ . If  $t \leq \frac{1-m}{2}$  then

$$\mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\} = \begin{cases} \frac{1-m}{2} \geq t & \text{if } \mu(x) \geq \frac{1-m}{2}, \\ \mu(x) > 1 - t - m & \text{if } \mu(x) < \frac{1-m}{2} \end{cases}$$

which implies that  $yx \in U(\mu; t) \cup Q^m(\mu; t) = [\mu]_t^m$ . Similarly,  $xy \in [\mu]_t^m$ . Now, let  $a, b \in [\mu]_t^m$  and  $x, y \in R$  be such that  $x + a + y = b + y$ . Then we have the following four cases:

- (3.14)  $\mu(a) \geq t$  and  $\mu(b) \geq t$ ,
- (3.15)  $\mu(a) \geq t$  and  $\mu(b) + t + m > 1$ ,
- (3.16)  $\mu(a) + t + m > 1$  and  $\mu(b) \geq t$ ,
- (3.17)  $\mu(a) + t + m > 1$  and  $\mu(b) + t + m > 1$ ,

and we have

$$(3.18) \quad \mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$$

by (3.5). Using (3.14) and (3.18), we get  $\mu(x) \geq \min\{t, \frac{1-m}{2}\}$ . If  $t \leq \frac{1-m}{2}$ , then  $\mu(x) \geq t$ , i.e.,  $x \in U(\mu; t) \subseteq [\mu]_t^m$ . If  $t > \frac{1-m}{2}$ , then  $\mu(x) \geq \frac{1-m}{2} > 1 - t - m$ , and so  $x \in Q^m(\mu; t) \subseteq [\mu]_t^m$ . For the case (3.15), if  $t > \frac{1-m}{2}$  then  $1 - t - m < \frac{1-m}{2}$  and  $\mu(a) \geq t > \frac{1-m}{2}$  and so

$$\begin{aligned} \mu(x) & \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} = \min\{\mu(b), \frac{1-m}{2}\} \\ & = \begin{cases} \frac{1-m}{2} > 1 - t - m & \text{if } \mu(b) \geq \frac{1-m}{2}, \\ \mu(b) > 1 - t - m & \text{if } \mu(b) < \frac{1-m}{2}. \end{cases} \end{aligned}$$

Hence  $x \in Q^m(\mu; t) \subseteq [\mu]_t^m$ . If  $t \leq \frac{1-m}{2}$  then

$$\begin{aligned} \mu(x) &\geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} \\ &= \begin{cases} \min\{\mu(a), \frac{1-m}{2}\} \geq t & \text{if } \min\{\mu(a), \frac{1-m}{2}\} < \mu(b), \\ \mu(b) > 1-t-m & \text{if } \min\{\mu(a), \frac{1-m}{2}\} \geq \mu(b) \end{cases} \end{aligned}$$

which implies that  $x \in U(\mu; t) \cup Q^m(\mu; t) \subseteq [\mu]_t^m$ . Similarly, we have  $x \in [\mu]_t^m$  from (3.16) and (3.18). For the case (3.17), assume first that  $t > \frac{1-m}{2}$ . Then  $1-t-m < \frac{1-m}{2}$ . If  $\min\{\mu(a), \mu(b)\} \geq \frac{1-m}{2}$ , then

$$\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} = \frac{1-m}{2} > 1-t-m,$$

and if  $\min\{\mu(a), \mu(b)\} < \frac{1-m}{2}$  then

$$\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} = \min\{\mu(a), \mu(b)\} > 1-t-m.$$

Therefore  $x \in Q^m(\mu; t) \subseteq [\mu]_t^m$ . Now suppose that  $t \leq \frac{1-m}{2}$ . Then  $1-t \geq \frac{1-m}{2}$ . If  $\min\{\mu(a), \mu(b)\} \geq \frac{1-m}{2}$ , then

$$\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} = \frac{1-m}{2} \geq t.$$

If  $\min\{\mu(a), \mu(b)\} < \frac{1-m}{2}$  then

$$\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\} = \min\{\mu(a), \mu(b)\} > 1-t-m.$$

Thus  $x \in U(\mu; t) \cup Q^m(\mu; t) = [\mu]_t^m$ . Consequently,  $[\mu]_t^m$  is an  $h$ -ideal of  $R$ .

Conversely, suppose that (2) is valid. If there exist  $x_0, y_0 \in R$  such that  $\mu(x_0 + y_0) < \min\{\mu(x_0), \mu(y_0), \frac{1-m}{2}\}$ , then

$$\mu(x_0 + y_0) < t_0 \leq \min\{\mu(x_0), \mu(y_0), \frac{1-m}{2}\}$$

for some  $t_0 \in (0, \frac{1-m}{2}]$ . It follows that  $x_0, y_0 \in U(\mu; t_0) \subseteq [\mu]_{t_0}^m$  so that  $x_0 + y_0 \in [\mu]_{t_0}^m$ . Thus  $\mu(x_0 + y_0) \geq t_0$  or  $\mu(x_0 + y_0) + t_0 + m > 1$ , a contradiction. Therefore  $\mu(x + y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$  for all  $x, y \in R$ . Assume that  $\mu(ba) < \min\{\mu(a), \frac{1-m}{2}\}$  for some  $a, b \in R$ . Then there exists  $t_a \in (0, 1]$  such that  $\mu(ba) < t_a \leq \min\{\mu(a), \frac{1-m}{2}\}$ . Then  $t_a \in (0, \frac{1-m}{2}]$  and  $a \in U(\mu; t_a) \subseteq [\mu]_{t_a}^m$ , but  $ba \notin U(\mu; t_a)$ . Also,  $\mu(ba) + t_a + m < 2t_a + m \leq 1$ , and so  $ba \notin Q^m(\mu; t_a)$ . Therefore  $ba \notin U(\mu; t_a) \cup Q^m(\mu; t_a) = [\mu]_{t_a}^m$ , a contradiction. Hence  $\mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\}$  for all  $x, y \in R$ . Similarly,  $\mu(xy) \geq \min\{\mu(x), \frac{1-m}{2}\}$  for all  $x, y \in R$ . Finally, let  $a, b, x, y \in R$  be such that  $x + a + y = b + y$ . Suppose that  $\mu(x) < \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$ . Then there exists  $t_x \in (0, 1]$  such that  $\mu(x) < t_x \leq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$ . Then  $t_x \in (0, \frac{1-m}{2}]$  and  $a, b \in U(\mu; t_x) \subseteq [\mu]_{t_x}^m$ , but  $x \notin U(\mu; t_x)$ . Also,  $\mu(x) + t_x + m < 2t_x + m \leq 1$ , that is,  $x \notin Q^m(\mu; t_x)$ . Consequently,  $x \notin [\mu]_{t_x}^m$ , a contradiction. Therefore  $\mu(x) \geq$

$\min\{\mu(a), \mu(b), \frac{1-m}{2}\}$ . Using Theorem 3.17, we know that  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy  $h$ -ideal of  $R$ .  $\square$

**COROLLARY 3.36.** *For any fuzzy subset  $\mu$  of  $R$ , the following are equivalent:*

- (1)  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy  $k$ -ideal of  $R$ .
- (2)  $(\forall t \in (0, 1]) \left( [\mu]_t^m \neq \emptyset \implies [\mu]_t^m \text{ is a } k\text{-ideal of } R \right)$ .

If we take  $m = 0$  in Theorem 3.35 and Corollary 3.36, then we have the following corollary.

**COROLLARY 3.37.** [18] *For any fuzzy subset  $\mu$  of  $R$ , the following are equivalent:*

- (1)  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy  $h$ -ideal ( $k$ -ideal) of  $R$ .
- (2)  $(\forall t \in (0, 1]) \left( [\mu]_t \neq \emptyset \implies [\mu]_t \text{ is an } h\text{-ideal } (k\text{-ideal}) \text{ of } R \right)$ .

#### 4. Implication-based fuzzy $h$ -ideals

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example  $\wedge, \vee, \neg, \rightarrow$  in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition  $\Phi$  is denoted by  $[\Phi]$ . For a universe  $U$  of discourse, we display the fuzzy logical and corresponding set-theoretical notations used in this paper

- (4.1)  $[x \in \mu] = \mu(x),$
- (4.2)  $[\Phi \wedge \Psi] = \min\{[\Phi], [\Psi]\},$
- (4.3)  $[\Phi \rightarrow \Psi] = \min\{1, 1 - [\Phi] + [\Psi]\},$
- (4.4)  $[\forall x \Phi(x)] = \inf_{x \in U} [\Phi(x)],$
- (4.5)  $\models \Phi$  if and only if  $[\Phi] = 1$  for all valuations.

The truth valuation rules given in (4.3) are those in the Łukasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a selection of them in the following.

- (a) Gaines-Rescher implication operator ( $I_{GR}$ ):

$$I_{GR}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gödel implication operator ( $I_G$ ):

$$I_G(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gödel implication operator ( $I_{cG}$ ):

$$I_{cG}(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ 1 - a & \text{otherwise.} \end{cases}$$

Ying [21] introduced the concept of fuzzifying topology. We can expand his/her idea to hemirings, and we define a fuzzifying  $h$ -ideal ( $k$ -ideal) as follows.

**DEFINITION 4.1.** A fuzzy subset  $\mu$  of  $R$  is called a *fuzzifying left* (resp. *right*) *ideal* of  $R$  if it satisfies the following conditions:

(1) for all  $x, y \in R$ , we have

$$(4.6) \quad \models [x \in \mu] \wedge [y \in \mu] \rightarrow [x + y \in \mu].$$

(2) for all  $x, y \in R$ , we get

$$(4.7) \quad \models [x \in \mu] \rightarrow [yx \in \mu] \text{ (resp. } \models [x \in \mu] \rightarrow [xy \in \mu])$$

**DEFINITION 4.2.** A fuzzy subset  $\mu$  of  $R$  is called a *fuzzifying left* (resp. *right*)  *$h$ -ideal* of  $R$  if it is a fuzzifying left (resp. right) ideal of  $R$  such that for all  $a, b, x, y \in R$

$$(4.8) \quad x + a + y = b + y \Rightarrow \left( \models [a \in \mu] \wedge [b \in \mu] \rightarrow [x \in \mu] \right).$$

**DEFINITION 4.3.** A fuzzy subset  $\mu$  of  $R$  is called a *fuzzifying left* (resp. *right*)  *$k$ -ideal* of  $R$  if it is a fuzzifying left (resp. right) ideal of  $R$  such that for all  $a, b, x \in R$

$$(4.9) \quad x + a = b \Rightarrow \left( \models [a \in \mu] \wedge [b \in \mu] \rightarrow [x \in \mu] \right).$$

Obviously, conditions (2.1) and (2.2) are equivalent to (4.6) and (4.7), respectively. Therefore a fuzzifying left (resp. right) ideal is an ordinary fuzzy left (resp. right) ideal. Also, conditions (2.3) and (2.4) are equivalent to (4.8) and (4.9), respectively. Therefore a fuzzifying left (resp. right)  $h$ -ideal ( $k$ -ideal) is an ordinary fuzzy left (resp. right)  $h$ -ideal ( $k$ -ideal).

In [22], the concept of  $t$ -tautology is introduced, i.e.,

$$(4.10) \quad \models_t \Phi \text{ if and only if } [\Phi] \geq t \text{ for all valuations.}$$

**DEFINITION 4.4.** Let  $\mu$  be a fuzzy subset of  $R$  and  $t \in (0, 1]$ .  $\mu$  is called a  *$t$ -implication-based fuzzy left* (resp. *right*) *ideal* of  $R$  if it satisfies the following conditions:

(1) for all  $x, y \in R$ , we have

$$(4.11) \quad \models_t [x \in \mu] \wedge [y \in \mu] \rightarrow [x + y \in \mu].$$

(2) for all  $x, y \in R$ , we get

$$(4.12) \quad \models_t [x \in \mu] \rightarrow [yx \in \mu] \text{ (resp. } \models_t [x \in \mu] \rightarrow [xy \in \mu])$$

**DEFINITION 4.5.** Let  $\mu$  be a fuzzy subset of  $R$  and  $t \in (0, 1]$ .  $\mu$  is called a  $t$ -implication-based fuzzy left (resp. right)  $h$ -ideal of  $R$  if it is a  $t$ -implication-based fuzzy left (resp. right) ideal of  $R$  such that for all  $a, b, x, y \in R$ ,

$$(4.13) \quad x + a + y = b + y \Rightarrow \left( \models_t [a \in \mu] \wedge [b \in \mu] \rightarrow [x \in \mu] \right).$$

**DEFINITION 4.6.** Let  $\mu$  be a fuzzy subset of  $R$  and  $t \in (0, 1]$ .  $\mu$  is called a  $t$ -implication-based fuzzy left (resp. right)  $k$ -ideal of  $R$  if it is a  $t$ -implication-based fuzzy left (resp. right) ideal of  $R$  such that for all  $a, b, x \in R$ ,

$$(4.14) \quad x + a = b \Rightarrow \left( \models_t [a \in \mu] \wedge [b \in \mu] \rightarrow [x \in \mu] \right).$$

Let  $I$  be an implication operator. Clearly,  $\mu$  is a  $t$ -implication-based fuzzy left  $h$ -ideal (resp.  $k$ -ideal) of  $R$  if and only if it satisfies:

- (1)  $(\forall x, y \in R) (I(\min\{\mu(x), \mu(y)\}, \mu(x + y)) \geq t)$ .
- (2)  $(\forall x, y \in R) (I(\mu(x), \mu(yx)) \geq t)$ .
- (3) for all  $a, b, x, y \in R$

$$x + a + y = b + y \implies I(\min\{\mu(a), \mu(b)\}, \mu(x)) \geq t$$

$$\left( \text{resp. } x + a = b \implies I(\min\{\mu(a), \mu(b)\}, \mu(x)) \geq t \right).$$

**THEOREM 4.7.** For any fuzzy subset  $\mu$  of  $R$ , we have

- (1) If  $I = I_{GR}$ , then  $\mu$  is a 0.5-implication-based fuzzy  $h$ -ideal of  $R$  if and only if  $\mu$  is a fuzzy  $h$ -ideal of  $R$ .
- (2) If  $I = I_G$ , then  $\mu$  is a  $\frac{1-m}{2}$ -implication-based fuzzy left  $h$ -ideal of  $R$  if and only if  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left  $h$ -ideal of  $R$ .
- (3) If  $I = I_{cG}$ , then  $\mu$  is a  $\frac{1-m}{2}$ -implication-based fuzzy left  $h$ -ideal of  $R$  if and only if  $\mu$  satisfies the following conditions:
  - (3.1)  $(\forall x, y \in R) (\max\{\mu(x + y), \frac{1-m}{2}\} \geq \min\{\mu(x), \mu(y), 1\})$ .
  - (3.2)  $(\forall x, y \in R) (\max\{\mu(yx), \frac{1-m}{2}\} \geq \min\{\mu(x), 1\})$ .
  - (3.3) for every  $a, b, x, y \in R$

$$x + a + y = b + y \implies \max\{\mu(x), \frac{1-m}{2}\} \geq \min\{\mu(a), \mu(b), 1\}.$$

*Proof.* (1) Straightforward.

(2) Assume that  $\mu$  is a  $\frac{1-m}{2}$ -implication-based fuzzy left  $h$ -ideal of  $R$ . Then

$$(a1) (\forall x, y \in R) \left( I_G(\min\{\mu(x), \mu(y)\}, \mu(x+y)) \geq \frac{1-m}{2} \right),$$

$$(a2) (\forall x, y \in R) \left( I_G(\mu(x), \mu(yx)) \geq \frac{1-m}{2} \right),$$

(a3) for all  $a, b, x, y \in R$

$$x + a + y = b + y \implies I_G(\min\{\mu(a), \mu(b)\}, \mu(x)) \geq \frac{1-m}{2}.$$

From (a1), we have  $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$  or

$$\min\{\mu(x), \mu(y)\} > \mu(x+y) \geq \frac{1-m}{2}.$$

It follows that  $\mu(x+y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\}$ . From (a2), we get

$$\mu(yx) \geq \mu(x) \text{ or } \mu(x) > \mu(yx) \geq \frac{1-m}{2},$$

and so  $\mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\}$ . Let  $a, b, x, y \in R$  be such that  $x+a+y = b+y$ . It follows from (a3) that  $\mu(x) \geq \min\{\mu(a), \mu(b)\}$  or

$$\min\{\mu(a), \mu(b)\} > \mu(x) \geq \frac{1-m}{2}$$

so that  $\mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}$ . Using Theorem 3.17, we know that  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left  $h$ -ideal of  $R$ .

Conversely, suppose that  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left  $h$ -ideal of  $R$ . Then

$$(b1) (\forall x, y \in R) \left( \mu(x+y) \geq \min\{\mu(x), \mu(y), \frac{1-m}{2}\} \right),$$

$$(b2) (\forall x, y \in R) \left( \mu(yx) \geq \min\{\mu(x), \frac{1-m}{2}\} \right),$$

(b3) for all  $a, b, x, y \in R$

$$x + a + y = b + y \implies \mu(x) \geq \min\{\mu(a), \mu(b), \frac{1-m}{2}\}.$$

From (b1), if  $\min\{\mu(x), \mu(y), \frac{1-m}{2}\} = \min\{\mu(x), \mu(y)\}$  then

$$\mu(x+y) \geq \min\{\mu(x), \mu(y)\},$$

and thus  $I_G(\min\{\mu(x), \mu(y)\}, \mu(x+y)) = 1 \geq \frac{1-m}{2}$ . If  $\min\{\mu(x), \mu(y), \frac{1-m}{2}\} = \frac{1-m}{2}$ , then  $\mu(x+y) \geq \frac{1-m}{2}$  and so

$$I_G(\min\{\mu(x), \mu(y)\}, \mu(x+y)) \geq \frac{1-m}{2}.$$

From (b2), if  $\min\{\mu(x), \frac{1-m}{2}\} = \mu(x)$ , then  $I_G(\mu(x), \mu(yx)) = 1 \geq \frac{1-m}{2}$ . Otherwise,  $I_G(\mu(x), \mu(yx)) \geq \frac{1-m}{2}$ . Let  $a, b, x, y \in R$  be such that  $x +$

$a + y = b + y$ . It follows from (b3) that

$$I_G(\min\{\mu(a), \mu(b)\}, \mu(x)) = \begin{cases} 1 \geq \frac{1-m}{2} & \text{if } \min\{\mu(a), \mu(b), \frac{1-m}{2}\} = \min\{\mu(a), \mu(b)\}, \\ \mu(x) \geq \frac{1-m}{2} & \text{if } \min\{\mu(a), \mu(b), \frac{1-m}{2}\} = \frac{1-m}{2}. \end{cases}$$

Therefore  $\mu$  is a  $\frac{1-m}{2}$ -implication-based fuzzy left  $h$ -ideal of  $R$ .

(3) Suppose that  $\mu$  satisfies conditions (3.1), (3.2) and (3.3). In (3.1), if  $\min\{\mu(x), \mu(y), 1\} = 1$ , then  $\max\{\mu(x + y), \frac{1-m}{2}\} = 1$  and so

$$\mu(x + y) = 1 \geq \min\{\mu(x), \mu(y)\}.$$

Hence  $I_{cG}(\min\{\mu(x), \mu(y)\}, \mu(x + y)) = 1 \geq \frac{1-m}{2}$ . If  $\min\{\mu(x), \mu(y), 1\} = \min\{\mu(x), \mu(y)\}$ , then

$$(4.15) \quad \max\{\mu(x + y), \frac{1-m}{2}\} \geq \min\{\mu(x), \mu(y)\}.$$

Thus, if  $\max\{\mu(x + y), \frac{1-m}{2}\} = \frac{1-m}{2}$  in (4.15), then

$$\mu(x + y) \leq \frac{1-m}{2} \text{ and } \min\{\mu(x), \mu(y)\} \leq \frac{1-m}{2}.$$

Therefore

$$I_{cG}(\min\{\mu(x), \mu(y)\}, \mu(x + y)) = \begin{cases} 1 \geq \frac{1-m}{2} & \text{if } \mu(x + y) \geq \min\{\mu(x), \mu(y)\}, \\ 1 - \min\{\mu(x), \mu(y)\} \geq \frac{1-m}{2} & \text{otherwise.} \end{cases}$$

If  $\max\{\mu(x + y), \frac{1-m}{2}\} = \mu(x + y)$  in (4.15), then  $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$  and so  $I_{cG}(\min\{\mu(x), \mu(y)\}, \mu(x + y)) = 1 \geq \frac{1-m}{2}$ . In (3.2), if  $\mu(x) = 1$ , then  $\max\{\mu(yx), \frac{1-m}{2}\} = 1$  and thus  $I_{cG}(\mu(x), \mu(yx)) = 1 \geq \frac{1-m}{2}$ . If  $\mu(x) < 1$ , then

$$(4.16) \quad \max\{\mu(yx), \frac{1-m}{2}\} \geq \mu(x).$$

Thus, if  $\max\{\mu(yx), \frac{1-m}{2}\} = \mu(yx)$  in (4.16), then  $\mu(yx) \geq \mu(x)$ . Hence

$$I_{cG}(\mu(x), \mu(yx)) = 1 \geq \frac{1-m}{2}.$$

If  $\max\{\mu(yx), \frac{1-m}{2}\} = \frac{1-m}{2}$  in (4.16), then  $\mu(x) \leq \frac{1-m}{2}$  which implies that

$$I_{cG}(\mu(x), \mu(yx)) = \begin{cases} 1 \geq \frac{1-m}{2} & \text{if } \mu(yx) \geq \mu(x), \\ 1 - \mu(x) \geq \frac{1-m}{2} & \text{otherwise.} \end{cases}$$

Let  $a, b, x, y \in R$  be such that  $x + a + y = b + y$ . In (3.3), if  $\min\{\mu(a), \mu(b), 1\} = 1$ , then  $\max\{\mu(x), \frac{1-m}{2}\} = 1$  and thus  $\mu(x) = 1 \geq \min\{\mu(a), \mu(b)\}$ . Therefore  $I_{cG}(\min\{\mu(a), \mu(b)\}, \mu(x)) = 1 \geq \frac{1-m}{2}$ . If  $\min\{\mu(a), \mu(b)\} < 1$ , then

$$(4.17) \quad \max\{\mu(x), \frac{1-m}{2}\} \geq \min\{\mu(a), \mu(b)\}.$$

Thus, if  $\mu(x) > \frac{1-m}{2}$  in (4.17), then  $\mu(x) \geq \min\{\mu(a), \mu(b)\}$  and hence  $I_{cG}(\min\{\mu(a), \mu(b)\}, \mu(x)) = 1 \geq \frac{1-m}{2}$ . If  $\mu(x) \leq \frac{1-m}{2}$  in (4.17), then  $\min\{\mu(a), \mu(b)\} \leq \frac{1-m}{2}$ . Hence

$$I_{cG}(\min\{\mu(a), \mu(b)\}, \mu(x)) = \begin{cases} 1 \geq \frac{1-m}{2} & \text{if } \mu(x) \geq \min\{\mu(a), \mu(b)\}, \\ 1 - \min\{\mu(a), \mu(b)\} \geq \frac{1-m}{2} & \text{otherwise.} \end{cases}$$

Consequently  $\mu$  is a  $\frac{1-m}{2}$ -implication-based fuzzy left  $h$ -ideal of  $R$ .

Conversely, assume that  $\mu$  is a  $\frac{1-m}{2}$ -implication-based fuzzy left  $h$ -ideal of  $R$ . Then

- (c1)  $(\forall x, y \in R) (I_{cG}(\min\{\mu(x), \mu(y)\}, \mu(x+y)) \geq \frac{1-m}{2})$ .
- (c2)  $(\forall x, y \in R) (I_{cG}(\mu(x), \mu(yx)) \geq \frac{1-m}{2})$ .
- (c3) for all  $a, b, x, y \in R$

$$(4.18) x + a + y = b + y \Rightarrow I_{cG}(\min\{\mu(a), \mu(b)\}, \mu(x)) \geq \frac{1-m}{2}.$$

Let  $x, y \in R$ . (c1) implies that  $I_{cG}(\min\{\mu(x), \mu(y)\}, \mu(x+y)) = 1$ , i.e.,  $\min\{\mu(x), \mu(y)\} \leq \mu(x+y)$ , or  $1 - \min\{\mu(x), \mu(y)\} \geq \frac{1-m}{2}$ . Hence

$$\max\{\mu(x+y), \frac{1-m}{2}\} \geq \min\{\mu(x), \mu(y)\} = \min\{\mu(x), \mu(y), 1\}.$$

From (c2), we have  $I_{cG}(\mu(x), \mu(yx)) = 1$ , i.e.,  $\mu(x) \leq \mu(yx)$ , or  $1 - \mu(x) \geq \frac{1-m}{2}$  and so  $\mu(x) \leq \frac{1-m}{2}$ . It follows that

$$\max\{\mu(yx), \frac{1-m}{2}\} \geq \mu(x) = \min\{\mu(x), 1\}.$$

Let  $a, b, x, y \in R$  be such that  $x + a + y = b + y$ . (4.18) implies that  $I_{cG}(\min\{\mu(a), \mu(b)\}, \mu(x)) = 1$  or  $1 - \min\{\mu(a), \mu(b)\} \geq \frac{1-m}{2}$  so that

$$\min\{\mu(a), \mu(b)\} \leq \mu(x) \text{ or } \min\{\mu(a), \mu(b)\} \leq \frac{1-m}{2}.$$

Therefore  $\max\{\mu(x), \frac{1-m}{2}\} \geq \min\{\mu(a), \mu(b)\} = \min\{\mu(a), \mu(b), 1\}$ . Consequently,  $\mu$  satisfies conditions (3.1), (3.2) and (3.3).  $\square$

**COROLLARY 4.8.** For any fuzzy subset  $\mu$  of  $R$ , we have

- (1) If  $I = I_{GR}$ , then  $\mu$  is a 0.5-implication-based fuzzy  $k$ -ideal of  $R$  if and only if  $\mu$  is a fuzzy ideal  $k$ -ideal of  $R$ .
- (2) If  $I = I_G$ , then  $\mu$  is a  $\frac{1-m}{2}$ -implication-based fuzzy left  $k$ -ideal of  $R$  if and only if  $\mu$  is an  $(\in, \in \vee q_m)$ -fuzzy left  $k$ -ideal of  $R$ .
- (3) If  $I = I_{cG}$ , then  $\mu$  is a  $\frac{1-m}{2}$ -implication-based fuzzy left  $k$ -ideal of  $R$  if and only if  $\mu$  satisfies the following conditions:
  - (3.1)  $(\forall x, y \in R) (\max\{\mu(x+y), \frac{1-m}{2}\} \geq \min\{\mu(x), \mu(y), 1\})$ .
  - (3.2)  $(\forall x, y \in R) (\max\{\mu(yx), \frac{1-m}{2}\} \geq \min\{\mu(x), 1\})$ .

(3.3) for every  $a, b, x \in R$

$$x + a = b \implies \max\{\mu(x), \frac{1-m}{2}\} \geq \min\{\mu(a), \mu(b), 1\}.$$

**COROLLARY 4.9.** For any fuzzy subset  $\mu$  of  $R$ , we have

- (1) If  $I = I_G$ , then  $\mu$  is a 0.5-implication-based fuzzy left  $h$ -ideal (resp.  $k$ -ideal) of  $R$  if and only if  $\mu$  is an  $(\in, \in \vee q)$ -fuzzy left  $h$ -ideal (resp.  $k$ -ideal) of  $R$ .
- (2) If  $I = I_{cG}$ , then  $\mu$  is a 0.5-implication-based fuzzy left  $h$ -ideal (resp.  $k$ -ideal) of  $R$  if and only if  $\mu$  satisfies the following conditions:
  - (2.1)  $(\forall x, y \in R) (\max\{\mu(x+y), 0.5\} \geq \min\{\mu(x), \mu(y), 1\})$ .
  - (2.2)  $(\forall x, y \in R) (\max\{\mu(yx), 0.5\} \geq \min\{\mu(x), 1\})$ .
  - (2.3) for every  $a, b, x, y \in R$

$$x + a + y = b + y \implies \max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b), 1\}$$

$$\left( \text{resp. } x + a = b \implies \max\{\mu(x), 0.5\} \geq \min\{\mu(a), \mu(b), 1\} \right).$$

## 5. Acknowledgements

The authors wish to thank the anonymous reviewers for their valuable suggestions. The first author, Y. B. Jun, is an Executive Research Worker of Educational Research Institute in Gyeongsang National University.

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