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CONVERGENCE ANALYSIS OF PRECONDITIONED AOR ITERATIVE METHOD

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Abstract. In this paper, we consider a preconditioned accelerated overrelaxation (PAOR) method to solve systems of linear equations. We show the convergence of the PAOR method. We also give comparison results when the coefficient matrix is an L- or H-matrix. Finally, we provide some numerical experiments to show efficiency of PAOR method.

1. Introduction

Consider the following linear system of n equations

$$\mathcal{A}\mathbf{x} = \mathbf{b} \tag{1}$$

where $\mathcal{A} = (a_{ij}) \in \mathbb{R}^{n\times n}$ is an $n \times n$ nonsingular matrix, and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. If $\mathcal{A}$ is splitted into

$$\mathcal{A} = M - N,$$

where M is a nonsingular matrix, then the basic splitting iterative method can be expressed as:

$$\mathbf{x}^{(k+1)} = M^{-1}N\mathbf{x}^{(k)} + M^{-1}\mathbf{b}, \qquad k = 0, 1, 2, \cdots . \tag{2}$$

As it is well known, the above iterative method is convergent to the unique solution $x = \mathcal{A}^{-1}\mathbf{b}$ for each initial value $x^{(0)}$ if and only if the spectral radius of the iteration matrix $M^{-1}N$ satisfies $\rho(M^{-1}N) < 1$. To improve the convergence rate of the basic iterative method, several preconditioned iterative methods have been proposed (see, e.g., [1, 2, 3]).

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The main idea of these preconditioned iterative methods is to transform the original system into the preconditioned form

$$PA\mathbf{x} = P\mathbf{b} \tag{3}$$

where $P \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. Then the basic iterative scheme of the preconditioned system is given by

$$\mathbf{x}^{(k+1)} = M_P^{-1} N_P \mathbf{x}^{(k)} + M_P^{-1}\mathbf{b}, \qquad k = 0, 1, 2, \cdots, \tag{4}$$

where $P\mathcal{A} = M_P - N_P$ with a nonsingular matrix M_P.

Without loss of generality, suppose that the coefficient matrix $\mathcal{A}$ has the following splitting

$$\mathcal{A} = \mathcal{I} - \mathcal{L} - \mathcal{U}$$

where $\mathcal{I}$ is identity matrix, $-\mathcal{L}$ and $-\mathcal{U}$ are strictly lower and upper triangular matrix of $\mathcal{A}$, respectively. For this splitting, the AOR iterative method is as follows:

$$x^{(i+1)} = L_{rw} x^{(i)} + \omega(\mathcal{I} - r\mathcal{L})^{-1}\mathbf{b}, \qquad i = 0, 1, 2, \cdots, \tag{5}$$

where

$$L_{rw} = (\mathcal{I} - r\mathcal{L})^{-1}[(1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U}]$$

is the iteration matrix and r, ω are acceleration parameters with $\omega \neq 0$.

Liu et al. [1] considered $P = I + S_\beta$ as a preconditioner and gave the sufficient conditions for convergence of the Gauss-Seidel method when the coefficient matrix $\mathcal{A}$ is an H-matrix, where

$$S_\beta = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ -\beta_1 a_{21} & 0 & \cdots & 0 & 0 \\ 0 & -\beta_2 a_{32} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\beta_{n-1} a_{n,n-1} & 0 \end{pmatrix}$$

whereas $\beta_i \geq 0$, $i = 1, \cdots, n-1$.

Consider the preconditioned linear system

$$\mathcal{A}_\beta \mathbf{x} = \mathbf{b}_\beta \tag{6}$$

where $\mathcal{A}_\beta = (I + S_\beta)A$ and $\mathbf{b}_\beta = (I + S_\beta)\mathbf{b}$.

In this paper, we will show the convergence analysis for the preconditioned AOR method when the coefficient matrix $\mathcal{A}$ is an L- or an H-matrix.

2. Preliminaries

For convenience, some notations, definitions and some results that will be used in the next sections are given.

A matrix A is called nonnegative(positive) if each entry of A is nonnegative(positive). We denote it by $A \geq 0(> 0)$. Similarly, for $n-$dimensional vectors x, by identifying them with $n \times 1$ matrices, we can also define $\mathbf{x} \geq 0(> 0)$. Denote by $\rho(A)$ the spectral radius of A.

Definition 2.1. [6] *A real matrix A is called an M-matrix if $A = sI - B$, $B \geq 0$ and $s > \rho(B)$.*

Definition 2.2. [7, 9] *A matrix $A = (a_{ij})$ is called*

1. *a Z-matrix if $a_{ij} \leq 0$ for $i, j = 1, 2, \cdots, n$ such that $i \neq j$,*
2. *an L-matrix if $a_{ij} \leq 0$ for $i, j = 1, 2, \cdots, n$, $(i \neq j)$ and $a_{ii} > 0$, $i = 1, 2, \cdots, n$,*
3. *an H-matrix if its comparison matrix $\langle A \rangle = (\bar{a}_{ij})$ is a nonsingular M-matrix, where $\bar{a}_{ij}$ is*

$$\bar{a}_{ii} = |a_{i,i}|, \qquad \bar{a}_{ij} = -|a_{ij}|, \quad i \neq j.$$

It must be noted that an L-matrix A is a nonsingular M-matrix if A is nonsingular and $A^{-1} \geq 0$.

Definition 2.3. [7] *A matrix A is irreducible if the directed graph associated to A is strongly connected.*

Definition 2.4. *Let A be a real matrix. The representation*

$$A = M - N$$

is called a splitting of A if M is a nonsingular matrix. The splitting is said to be

1. *convergent if $\rho(M^{-1}N) < 1$;*
2. *regular if $M^{-1} \geq 0$ and $N \geq 0$;*
3. *nonnegative if $M^{-1}N \geq 0$;*
4. *M-splitting if M is a nonsingular M-matrix and $N \geq 0$.*

It is obvious that an M-splitting is regular and a regular splitting is nonnegative.

Lemma 2.5. [7] *Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then*

1. *A has a positive real eigenvalue equal to its spectral radius $\rho(A)$,*
2. *for $\rho(A)$ there corresponds an eigenvector $x > 0$,*

3. *$\rho(A)$ is a simple eigenvalue of A,*
4. *$\rho(A)$ increases when any entry of A increases.*

Lemma 2.6. [6] *Let A be a nonnegative matrix.*

1. *If $\alpha x \le Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \le \rho(A)$.*
2. *If $Ax \le \beta x$ for some positive vector x, then $\rho(A) \le \beta$.*

Moreover, if A is irreducible and if $0 \neq \alpha x \le Ax \le \beta x$ for some nonnegative vector x, then $\alpha \le \rho(A) \le \beta$ and x is a positive vector.

Lemma 2.7. [8] *Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.*

Theorem 2.8. [6] *Let A be a Z-matrix. Then the following statements are equivalent:*

1. *A is nonsingular M-matrix.*
2. *There is a positive vector x such that $Ax > 0$.*
3. *All principal submatrices of A are M-matrices.*

Lemma 2.9. *Let $\mathcal{A}$ be a Z-matrix. Then $\mathcal{A}$ is a nonsingular M-matrix if and only if $\mathcal{A}_\beta$ is a nonsingular M-matrix for $\beta_i \in [0,1], i = 1, 2, \cdots, n-1$.*

Proof. Let $\mathcal{A}$ be a nonsingular M-matrix. We have

$$\mathcal{A}_\beta = (I + S_\beta)\mathcal{A} =$$

$$\begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} - \beta_1 a_{21} & 1 - \beta_1 a_{21} a_{12} & \cdots & a_{2n} - \beta_1 a_{21} a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} - \beta_{n-1} a_{n,n-1} a_{n-1,1} & a_{n2} - \beta_{n-1} a_{n,n-1} a_{n-1,2} & \cdots & 1 - \beta_{n-1} a_{n,n-1} a_{n-1,n} \end{pmatrix}.$$

Suppose that $\mathcal{A}$ is a nonsingular M-matrix, then by Theorem 2.8, there exists a positive vector x such that $\mathcal{A}\mathbf{x} > 0$. On the other hand, since $\mathcal{A}$ is a Z-matrix, $S_\beta > 0$. So for the above vector x, we have $\mathcal{A}_\beta \mathbf{x} = (I + S_\beta)\mathcal{A}\mathbf{x} > 0$. Hence, by Theorem 2.8, $\mathcal{A}_\beta$ is a nonsingular M-matrix. Note that if $\mathcal{A}_\beta$ be an M-matrix, then $\mathcal{A}_\beta^T$ is also an M-matrix. By Theorem 2.8, there exists a positive vector $\mathbf{x}$ such that $\mathcal{A}_\beta^T \mathbf{x} > 0$, so $\mathcal{A}^T(I + S_\beta^T)\mathbf{x} > 0$. Set $\mathbf{y} = (I + S_\beta^T)\mathbf{x}$. Then, we have $\mathbf{y} > 0$ and

$\mathcal{A}^T\mathbf{y} > 0$, which means that $\mathcal{A}^T$ is a nonsingular M-matrix, hence, $\mathcal{A}$ is also a nonsingular M-matrix. □

3. The preconditioned AOR method for L-matrices

In this section, we consider the preconditioned linear system

$$\mathcal{A}_\beta\mathbf{x} = \mathbf{b}_\beta$$

where $\mathcal{A}_\beta = (I + S_\beta)\mathcal{A}$ and $\mathbf{b}_\beta = (I + S_\beta)\mathbf{b}$. We split the coefficient matrix $\mathcal{A}_\beta$ as

$$\mathcal{A}_\beta = \mathcal{D}_\beta - \mathcal{L}_\beta - \mathcal{U}_\beta$$

where $\mathcal{D}_\beta, -\mathcal{L}_\beta$, and $-\mathcal{U}_\beta$ are the diagonal, strictly lower and strictly upper triangular matrices of $\mathcal{A}_\beta$, respectively. Then the preconditioned AOR iterative method is as follows:

$$\mathbf{x}^{(i+1)} = \tilde{L}_{rw}\mathbf{x}^{(i)} + \omega(\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}\mathbf{b}_\beta, \qquad i = 0, 1, 2, \cdots, \tag{7}$$

where

$$\tilde{L}_{rw} = (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(1-\omega)\mathcal{D}_\beta + (\omega - r)\mathcal{L}_\beta + \omega\mathcal{U}_\beta]$$

is the iteration matrix.

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{A}_\beta$ be the coefficient matrices of linear system (1) and (6), respectively. Suppose that $\mathcal{A}$ is irreducible L-matrix and $0 \le r \le \omega \le 1 (r \ne 1, \omega \ne 0)$.*

1. *The iterative matrix $L_{r\omega}$ in (5) is a nonnegative irreducible matrix.*
2. *If there exists a nonempty set $\alpha \subset Q = \{2, 3, \cdots, n\}$ such that*

$$0 < a_{i,i-1}a_{i-1,i} < 1, \quad i \in \alpha \quad \text{and} \quad a_{i,i-1}a_{i-1,i} = 0, \quad i \in Q\backslash\alpha,$$

then $\tilde{L}_{rw}$ in (7) is a nonnegative irreducible matrix.

Proof. (a) Note that

$$\begin{aligned} L_{rw} &= (\mathcal{I} - r\mathcal{L})^{-1}[(1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U}] \\ &= (\mathcal{I} + r\mathcal{L} + r^2\mathcal{L}^2 + \cdots + r^{n-1}\mathcal{L}^{n-1})[(1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U}] \\ &= (1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U} + r\mathcal{L}[(1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U}] \\ &\qquad + [r^2\mathcal{L}^2 + \cdots + r^{n-1}\mathcal{L}^{n-1}][(1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U}] \\ &= (1-\omega)\mathcal{I} + \omega(1-r)\mathcal{L} + \omega\mathcal{U} + T \end{aligned}$$

where

$$T = r\mathcal{L}[(\omega - r)\mathcal{L} + \omega\mathcal{U}] + [r^2\mathcal{L}^2 + \cdots + r^{n-1}\mathcal{L}^{n-1}][(1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U}] \ge 0.$$

Since $\mathcal{A}$ is an L-matrix, it holds that $\mathcal{I} \ge 0$, $\mathcal{L} \ge 0$ and $\mathcal{U} \ge 0$, using the fact that $0 \le r \le \omega \le 1 (r \ne 1, \omega \ne 0)$, we have $L_{r\omega} \ge 0$. Since $\mathcal{A}$ is

an irreducible matrix, so is $(1-\omega)\mathcal{I}+\omega(1-r)\mathcal{L}+\omega\mathcal{U}$. Thus L_{rw} is an irreducible matrix.
(b) Note that

$$\begin{aligned}\tilde{L}_{rw} &= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(1-\omega)\mathcal{D}_\beta+(\omega-r)\mathcal{L}_\beta+\omega\mathcal{U}_\beta]\\ &= (\mathcal{I}-r\mathcal{D}_\beta^{-1}\mathcal{L}_\beta)^{-1}[(1-\omega)\mathcal{I}+(\omega-r)\mathcal{D}_\beta^{-1}\mathcal{L}_\beta+\omega\mathcal{D}_\beta^{-1}\mathcal{U}_\beta]\\ &= (1-\omega)\mathcal{I}+\omega(1-r)\mathcal{D}_\beta^{-1}\mathcal{L}_\beta+\omega\mathcal{D}_\beta^{-1}\mathcal{U}_\beta+T_\beta\end{aligned}$$

where

$$\begin{aligned}T_\beta = \ & r\mathcal{D}_\beta^{-1}\mathcal{L}_\beta[(\omega-r)\mathcal{D}_\beta^{-1}\mathcal{L}_\beta+\omega\mathcal{D}_\beta^{-1}\mathcal{U}_\beta]\\ &+[r^2(\mathcal{D}_\beta^{-1}\mathcal{L}_\beta)^2+\cdots+r^{n-1}(\mathcal{D}_\beta^{-1}\mathcal{L}_\beta)^{n-1}]\times\\ &[(1-\omega)\mathcal{D}_\beta^{-1}+(\omega-r)(\mathcal{D}_\beta^{-1}\mathcal{L}_\beta)+\omega(\mathcal{D}_\beta^{-1}\mathcal{U}_\beta)]\geq 0.\end{aligned}$$

By similar arguments given in proof of (a), we can easily show that $\tilde{L}_{rw}\geq 0$. Since $0<\beta_i\leq 1$, it is obvious that $\mathcal{A}_\beta$ is an irreducible matrix. Thus $\tilde{L}_{rw}$ is a nonnegative irreducible matrix. □

Theorem 3.2. *Let $\mathcal{A}\in\mathbb{R}^{n\times n}$ be a nonsingular L-matrix. Assume that $0\leq r\leq\omega\leq 1(r\neq 1,\omega\neq 0)$, and $0<\beta_i\leq 1$ but $\beta_{i-1}a_{i,i-1}\neq 0$ for some $i=1,2,\cdots,n-1$.*

(a) If $\rho(L_{r\omega})<1$, then $\rho(\tilde{L}_{r\omega})<\rho(L_{r\omega})<1$.
(b) If $\mathcal{A}$ is an irreducible matrix and if there exists a non-empty set $\alpha\subset Q=\{2,3,\cdots\}$ such that

$$0<a_{i,i-1}a_{i-1,i}<1,\quad i\in\alpha,\quad \text{and}\quad a_{i,i-1}a_{i-1,i}=0,\quad i\in Q\backslash\alpha,$$

then it holds that

$$\begin{cases}\rho(\tilde{L}_{r\omega})<\rho(L_{r\omega}) & \text{if} \quad \rho(L_{r\omega})<1;\\ \rho(\tilde{L}_{r\omega})=\rho(L_{r\omega}) & \text{if} \quad \rho(L_{r\omega})=1;\\ \rho(\tilde{L}_{r\omega})>\rho(L_{r\omega}) & \text{if} \quad \rho(L_{r\omega})>1.\end{cases}$$

Proof. Let

$$\begin{aligned}&M=\tfrac{1}{\omega}(\mathcal{I}-r\mathcal{L}),\\ &N=\tfrac{1}{\omega}[(1-\omega)\mathcal{I}+(\omega-r)\mathcal{L}+\omega\mathcal{U}),\\ &E_\beta=\tfrac{1}{\omega}(\mathcal{D}_\beta-r\mathcal{L}_\beta),\\ &F_\beta=\tfrac{1}{\omega}[(1-\omega)\mathcal{D}_\beta+(\omega-r)\mathcal{L}_\beta+\omega\mathcal{U}_\beta),\\ &M_\beta=\tfrac{1}{\omega}(\mathcal{I}+S_\beta)(\mathcal{I}-r\mathcal{L}),\\ &N_\beta=\tfrac{1}{\omega}(\mathcal{I}+S_\beta)[(1-\omega)\mathcal{I}+(\omega-r)\mathcal{L}+\omega\mathcal{U}).\end{aligned}$$

Then we have the following splitting

$$\mathcal{A}=M-N\quad\text{and}\quad\mathcal{A}_\beta=E_\beta-F_\beta=M_\beta-N_\beta.$$

(a) Since $\mathcal{A}$ is an L-matrix and $0 \leq r \leq \omega \leq 1 (r \neq 1, \omega \neq 0)$, $M = \frac{1}{\omega}(\mathcal{I} - r\mathcal{L})$ is a nonsingular M-matrix and $N \geq 0$ so that $\mathcal{A} = M - N$ is an M-splitting. By the fact that $\rho(L_{rw}) < 1$ and Lemma 2.7, $\mathcal{A}$ is a nonsingular M-matrix. We also show that $\mathcal{A}_\beta$ is a nonsingular M-matrix using Lemma 2.9.

Since $\mathcal{A}_\beta$ is a nonsingular M-matrix, $(\mathcal{D}_\beta)_{i,i} > 0$ and D_β is invertible. Using the fact that $(\mathcal{L}_\beta)_{ij} = -a_{ij} + \beta_{i-1} a_{i,i-1} a_{i-1j} \geq 0$, for $i = 3, \cdots, n$, $j < i - 2$, and $(\mathcal{L}_\beta)_{i,i-1} = -a_{i,i-1}(1 - \beta_{i-1}) \geq 0$, $i = 2, \cdots, n$, we have $\mathcal{L}_\beta \geq 0$ so that $E_\beta = \frac{1}{\omega}(\mathcal{D}_\beta - r\mathcal{L}_\beta)$ is a Z-matrix and it is also a nonsingular M-matrix. By our assumptions, F_β is a nonnegative matrix so that $A_\beta = E_\beta - F_\beta$ is an M-splitting. Thus by Lemma 2.7, we have

$$\rho(\tilde{L}_{rw}) = \rho(E_\beta^{-1} F_\beta) < 1.$$

Using the fact that $\mathcal{A} = M - N$, $\mathcal{A}_\beta = E_\beta - F_\beta$ are M-splitting and $M^{-1}N = M_\beta^{-1} N_\beta$ yields that two splittings are regular and nonnegative.

On the other hand, $\mathcal{A}_\beta$ can be represented as

$$\mathcal{A}_\beta = (\mathcal{I} + S_\beta)\mathcal{A} = \mathcal{I} - \mathcal{L} - \mathcal{U} + S_\beta - S_\beta\mathcal{L} - S_\beta\mathcal{U}.$$

Denote by $S_2 = S_\beta\mathcal{L}$. Then S_2 is a strictly lower triangular matrix. Let $S_\beta\mathcal{U} = S_1 + S_3$ where S_1 and S_3 are diagonal and strictly upper triangular matrix of $S_\beta\mathcal{U}$, respectively. Then $\mathcal{D}_\beta = \mathcal{I} - S_1$, $\mathcal{L}_\beta = \mathcal{L} - S_\beta + S_2$, $\mathcal{U}_\beta = \mathcal{U} + S_3$, and

$$\mathcal{A}_\beta = \mathcal{I} - \mathcal{L} - \mathcal{U} + S_\beta - S_1 - S_2 - S_3 = \mathcal{D}_\beta - \mathcal{L}_\beta - \mathcal{U}_\beta.$$

Note that

$$\begin{aligned} N_\beta - F_\beta == & \frac{1}{\omega}(\mathcal{I} + S_\beta)[(1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U}] \\ & - \frac{1}{\omega}[(1-\omega)\mathcal{D}_\beta + (\omega - r)\mathcal{L}_\beta + \omega\mathcal{U}_\beta] \\ = & \frac{1}{\omega}[(1-\omega)(\mathcal{I} - \mathcal{D}_\beta) + (\omega - r)(\mathcal{L} - \mathcal{L}_\beta) + \omega(\mathcal{U} - \mathcal{U}_\beta) \\ & + (1-\omega)S_\beta + (\omega - r)S_2 + \omega(S_1 + S_3)] \\ = & \frac{1}{\omega}[(1-\omega)(\mathcal{I} - \mathcal{D}_\beta) + (\omega - r)(S_\beta - S_2) - \omega S_3 \\ & + (1-\omega)S_\beta + (\omega - r)S_2 + \omega(S_1 + S_3)] \\ = & \frac{1}{\omega}[(1-\omega)(\mathcal{I} - \mathcal{D}_\beta) + (1-r)S_\beta + \omega S_1] \geq 0. \end{aligned}$$

Thus $N_\beta \geq F_\beta$ and $\mathcal{A}_\beta + N_\beta \geq \mathcal{A}_\beta + F_\beta$. Furthermore we have

$$M_\beta \geq E_\beta \quad \text{and} \quad \mathcal{A}_\beta^{-1} M_\beta \geq \mathcal{A}_\beta^{-1} E_\beta \geq 0.$$

By Theorem 1.1 in [10], we have

$$\rho(E_\beta^{-1}F_\beta) \leq \rho(M_\beta^{-1}N_\beta).$$

Hence

$$\rho(\tilde{L}_{rw}) < \rho(L_{rw}) < 1.$$

(b) Let $\mathcal{A}$ be an irreducible matrix. By Lemma 3.1, $L_{r\omega}$ is a nonnegative and irreducible matrix, and by Lemma 2.5, there exists a positive vector x such that

$$L_{r\omega}\mathbf{x} = \lambda\mathbf{x}$$

where $\lambda = \rho(L_{rw})$. Thus we can easily show that

$$[(1-\omega)\mathcal{I} + (\omega - r)\mathcal{L} + \omega\mathcal{U}]\mathbf{x} = \lambda(\mathcal{I} - r\mathcal{L})\mathbf{x}$$

or equivalently

$$[(1-\omega-\lambda)\mathcal{I} + (\omega - r + r\lambda)\mathcal{L} + \omega\mathcal{U}]\mathbf{x} = 0$$

and

$$(\lambda - 1)(\mathcal{I} - r\mathcal{L})\mathbf{x} = \omega(\mathcal{L} + \mathcal{U} - \mathcal{I})\mathbf{x}.$$

For the above λ and $\mathbf{x}$ we have

$$\begin{aligned}
\tilde{L}_{rw}\mathbf{x} - \lambda\mathbf{x} =& \\
&= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(1-\omega)\mathcal{D}_\beta + (\omega - r)\mathcal{L}_\beta + \omega\mathcal{U}_\beta - \lambda(\mathcal{D}_\beta - r\mathcal{L}_\beta)]\mathbf{x} \\
&= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(1-\omega-\lambda)\mathcal{D}_\beta + (\omega - r + r\lambda)\mathcal{L}_\beta + \omega\mathcal{U}_\beta]\mathbf{x} \\
&= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(1-\omega-\lambda)(\mathcal{I} - S_1) + (\omega - r + r\lambda)(\mathcal{L} - S_\beta + S_2) \\
&\quad + \omega(\mathcal{U} + S_3)]\mathbf{x} \\
&= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(\lambda + \omega - 1)S_1 + (\omega - r + r\lambda)(S_2 - S_\beta) + \omega S_3]\mathbf{x} \\
&= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(\lambda - 1)S_1 + \omega(S_1 + S_2 - S_\beta + S_3) + (\lambda r - r) \\
&\quad (S_2 - S_\beta)]\mathbf{x} \\
&= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(\lambda - 1)S_1 + \omega(S_\beta\mathcal{U} + S_\beta\mathcal{L} - S_\beta) + r(1-\lambda) \\
&\quad (S_\beta - S_2)]\mathbf{x} \\
&= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(\lambda - 1)S_1 + \omega S_\beta(\mathcal{U} + \mathcal{L} - \mathcal{I}) + r(1-\lambda)(S_\beta - S_2)]\mathbf{x} \\
&= (\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[(\lambda - 1)S_1 + (\lambda - 1)(1 - r)S_\beta]\mathbf{x},
\end{aligned}$$

where $S_1 \geq 0$, $S_\beta \geq 0$ and $1 - r \geq 0$. Therefore

$$\tilde{L}_{rw}\mathbf{x} - \lambda\mathbf{x} = (\lambda - 1)(\mathcal{D}_\beta - r\mathcal{L}_\beta)^{-1}[S_1 + (1-r)S_\beta]\mathbf{x}.$$

If $\lambda < 1$, then $\tilde{L}_{rw}\mathbf{x} - \lambda\mathbf{x} \leq 0$ and $\tilde{L}_{rw}\mathbf{x} - \lambda\mathbf{x} \neq 0$, i.e. $\tilde{L}_{rw}\mathbf{x} \leq \lambda\mathbf{x}$ and $\tilde{L}_{rw}\mathbf{x} \neq \lambda\mathbf{x}$. If $\lambda = 1$, then $\tilde{L}_{rw}\mathbf{x} - \lambda\mathbf{x} = 0$, i.e. $\tilde{L}_{rw}\mathbf{x} = \lambda\mathbf{x}$. If $\lambda > 1$ then $\tilde{L}_{rw}\mathbf{x} - \lambda\mathbf{x} \geq 0$ and $\tilde{L}_{rw}\mathbf{x} - \lambda\mathbf{x} \neq 0$, i.e. $\tilde{L}_{rw}\mathbf{x} \geq \lambda\mathbf{x}$ and

$\tilde{L}_{rw}\mathbf{x} \neq \lambda\mathbf{x}$. Using the above estimates and Lemma 2.6, we can easily prove the conclusion of (b) and it completes the proof. □

It is well known that by taking a special value $\omega = r$ in AOR method, we obtain the SOR iteration. Hence we have the following result.

Corollary 3.3. *Let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be a nonsingular L-matrix. Assume that $0 \le \omega \le 1 (\omega \neq 0)$ and $0 < \beta_i \le 1$ but $\beta_{i-1}a_{i,i-1} \neq 0$ for some $i = 1, 2, \cdots, n-1$.*

(a) If $\rho(L_\omega) < 1$, then $\rho(\tilde{L}_\omega) < \rho(L_\omega) < 1$.

(b) If $\mathcal{A}$ is an irreducible matrix, and if there exists a non-empty set $\alpha \subset Q = \{2, 3, \cdots, n\}$ such that

$$0 < a_{i,i-1}a_{i-1,i} < 1, \quad i \in \alpha \quad \textit{and} \quad a_{i,i-1}a_{i-1,i} = 0, \quad i \in Q\backslash\alpha,$$

then

$$\begin{cases} \rho(\tilde{L}_\omega) < \rho(L_\omega) & \text{if} \quad \rho(L_\omega) < 1; \\ \rho(\tilde{L}_\omega) = \rho(L_\omega) & \text{if} \quad \rho(L_\omega) = 1; \\ \rho(\tilde{L}_\omega) > \rho(L_\omega) & \text{if} \quad \rho(L_\omega) > 1. \end{cases}$$

4. The preconditioned AOR method for H-matrices

In this section, we show the convergence of the preconditioned AOR method for the system (1) when the coefficient matrix $\mathcal{A}$ is an H-matrix. First, we give some lemmas which are useful in the sequel.

Lemma 4.1. [4] *Let $\mathcal{A}$ be an H-matrix. Then $|\mathcal{A}^{-1}| \le \langle\mathcal{A}\rangle$, where $\langle\mathcal{A}\rangle$ denotes the comparison matrix given in Definition 2.*

Lemma 4.2. [7] *Let $\mathcal{A}$ and $\mathbf{B}$ be two $n\times n$ matrices with $0 \le |\mathbf{B}| \le \mathcal{A}$. Then $\rho(\mathbf{B}) \le \rho(\mathcal{A})$.*

By above lemmas we state and prove some lemmas and theorems.

Lemma 4.3. *If $\mathcal{A}$ is a nonsingular M-matrix, then $\rho(L_{r\omega}) < 1$.*

Proof. Since $\mathcal{A}$ is an M-matrix, it is an L-matrix so that $M = \frac{1}{\omega}(\mathcal{I} - r\mathcal{L})$ is a nonsingular M-matrix and $N = \frac{1}{\omega}[(1-\omega)\mathcal{I}(\omega - r)\mathcal{L} + \omega\mathcal{U}] \ge 0$. Hence $\mathcal{A} = M - N$ is an M-splitting, and by Lemma 2.7, $\rho(M^{-1}N) = \rho(L_{r\omega}) < 1$. □

Lemma 4.4. *If $\mathcal{A}$ is an H-matrix, then $\rho(L_{r\omega}) < 1$.*

Proof. Let $\mathcal{A}$ be an H-matrix, then $\langle\mathcal{A}\rangle$ is an M-matrix. By Lemma 4.3, $\rho(\bar{L}_{r\omega}) < 1$ where

$$\bar{L}_{r\omega} = (I - r|L|)^{-1}[(1-\omega)I + (\omega - r)|L| + \omega|U|].$$

Let

$$X = (I - rL)^{-1},$$
$$Y = (1-\omega)I + (\omega - r)L + \omega U,$$
$$Z = (I - r|L|)^{-1},$$
$$T = (1-\omega)I + (\omega - r)|L| + \omega|U|.$$

Obviously, X is an H-matrix and $\langle X\rangle = Z$. By Lemma 4.1, $|X^{-1}| \le Z^{-1}$. Hence, we have

$$|X^{-1}Y| \le |X^{-1}||Y| \le |X^{-1}|T \le Z^{-1}T$$

and $|L_{r\omega}| \le \bar{L}_{r\omega}$. From Lemma 4.2, we have $\rho(L_{r\omega}) \le \rho(\bar{L}_{r\omega})$ and this completes the proof. □

Lemma 4.5. *Let $\mathcal{A}$ be an H-matrix. Then*

$$\beta_i' = 1 + \frac{|a_{i,i-1}| + 1}{|a_{i,i-1}\big(2\|\langle\mathcal{A}\rangle^{-1}\|_\infty - 1\big)} > 1.$$

Proof. Since $\langle\mathcal{A}\rangle = \mathcal{I} - |\mathcal{L}| - |\mathcal{U}| \le \mathcal{I}$ is a nonsingular M-matrix, $\langle\mathcal{A}\rangle^{-1} \ge 0$ and $0 \le \mathcal{I} \le \langle\mathcal{A}\rangle^{-1}$. Thus $\|\langle\mathcal{A}\rangle^{-1}\|_\infty \ge 1$ and then $\beta_i' > 1$. □

Theorem 4.6. *Let $\mathcal{A}$ be an H-matrix. Then $\mathcal{A}_\beta$ is an H-matrix and $\rho(\tilde{L}_{r\omega}) < 1$ for $\beta_i \in [0, \beta_i')$, $i = 1, 2, \cdots, n-1$.*

Proof. Let $r = \langle\mathcal{A}\rangle^{-1}\mathbf{e}$ where $\mathbf{e} = (1, 1, \cdots, 1)^T$. Since $\mathcal{A}$ is an H-matrix, by Theorem 2.8, there exists a vector $r > 0$ such that $\langle\mathcal{A}\rangle r = \mathbf{e} > 0$. We show that $\langle\mathcal{A}_\beta\rangle$ is an M-matrix. Note that

$$(\langle\mathcal{A}_\beta\rangle r)_1 = r_1 - \sum_{j=2}^{n} |a_{1j}| r_j = (\langle\mathcal{A}\rangle r)_1 > 0.$$

For $i = 2, \cdots, n$, we have

$$
\begin{aligned}
(\langle \mathcal{A}_\beta \rangle r)_i &= |1 - \beta_{i-1} a_{i,i-1} a_{i-1,i}| r_i - \sum_{j \neq i} |a_{ij} - \beta_{i-1} a_{i,i-1} a_{i-1,j}| r_j \\
&\geq r_i - \beta_{i-1} |a_{i,i-1} a_{i-1,i}| r_i - \sum_{j \neq i, i-1} |a_{ij}| r_j \\
&- \sum_{j \neq i, i-1} \beta_{i-1} |a_{i,i-1} a_{i-1,j}| r_j - |a_{i,i-1}||1 - \beta_{i-1}| r_{i-1} \\
&\geq (\langle \mathcal{A} \rangle r)_i + |a_{i,i-1}| r_{i-1} - \beta_{i-1} |a_{i,i-1}| [|a_{i-1,i}| r_i \\
&+ \sum_{j \neq i, i-1} |a_{i-1j}| r_j] - |a_{i,i-1}||1 - \beta_{i-1}| r_{i-1} \\
&= 1 + |a_{i,i-1}| r_{i-1} - \beta_{i-1} |a_{i,i-1}| [-(\langle \mathcal{A} \rangle r)_i + r_{i-1}] \\
&- |a_{i,i-1}||1 - \beta_{i-1}| r_{i-1} \\
&= 1 + |a_{i,i-1}| r_{i-1} - \beta_{i-1} |a_{i,i-1}| [-1 + r_{i-1}] - |a_{i,i-1}||1 - \beta_{i-1}| r_{i-1} \\
&= 1 + \beta_{i-1} |a_{i,i-1}| + [1 - \beta_{i-1} - |1 - \beta_{i-1}|] |a_{i,i-1}| r_{i-1}.
\end{aligned}
$$

If $0 \leq \beta_i \leq 1$, then

$$
(\langle \mathcal{A}_\beta \rangle r)_i = 1 + \beta_{i-1} |a_{i,i-1}| > 0.
$$

Therefore, $\langle \mathcal{A}_\beta \rangle$ is an M-matrix and $\mathcal{A}_\beta$ is an H-matrix.
If $\beta_i > 1$, then

$$
\begin{aligned}
(\langle \mathcal{A}_\beta \rangle r)_i &= 1 + 2|a_{i,i-1}| r_{i-1} - (2r_{i-1} - 1) \beta_i |a_{i,i-1}| \\
&> 1 + 2|a_{i,i-1}| r_{i-1} - (2r_{i-1} - 1) [1 + \frac{|a_{i,i-1}| + 1}{|a_{i,i-1}|(2\|\langle \mathcal{A} \rangle^{-1}\|_\infty - 1)}] \\
&\quad |a_{i,i-1}| \\
&\geq 1 + |a_{i,i-1}| - (2\|\langle \mathcal{A} \rangle^{-1}\|_\infty - 1) |a_{i,i-1}| \frac{|a_{i,i-1}| + 1}{|a_{i,i-1}|(2\|\langle \mathcal{A} \rangle^{-1}\|_\infty - 1)} \\
&= 0.
\end{aligned}
$$

Therefore, $\langle \mathcal{A}_\beta \rangle$ is an M-matrix and $\mathcal{A}_\beta$ is an H-matrix, and by Lemma 4.4, $\rho(\tilde{L}_{r\omega}) < 1$. □

For the SOR iterative method, the following corollary holds.

Corollary 4.7. *Let $\mathcal{A}$ be an H-matrix. Then $\mathcal{A}_\beta$ is an H-matrix and $\rho(\tilde{L}_\omega) < 1$ for $\beta_i \in [0, \beta_i')$, $i = 1, 2, \cdots, n-1$.*

5. Numerical experiments

In this section, we give some numerical examples to show efficiency of the preconditioned AOR method.

Example 1. Suppose that the coefficient matrix $\mathcal{A}$ is as follows:

$$\mathcal{A} = \begin{pmatrix} 1 & -0.1 & -0.06 & -0.35 & -0.22 \\ -0.16 & 1 & -0.04 & -0.08 & -0.28 \\ -0.2 & -0.1 & 1 & -0.12 & -0.2 \\ -0.06 & -0.24 & -0.17 & 1 & -0.05 \\ -0.32 & -0.22 & -0.1 & -0.15 & 1 \end{pmatrix}.$$

The coefficient matrix $\mathcal{A}$ is an L-matrix. Let $\beta_1 = 0.86$, $\beta_2 = 0.79$, $\beta_3 = 0.95$ and $\beta_4 = 0.92$. For $r = 0.3093$, $\omega = 0.9827$, we have $\rho(\tilde{L}_{r\omega}) = 0.5760 < \rho(L_{r\omega}) = 0.6107$, and for $r = \omega = 0.66$ we get $\rho(\tilde{L}_{\omega}) = 0.6735 < \rho(L_{\omega}) = 0.6916$.
The matrix $\mathcal{A}$ is also an H-matrix. Let $\beta_1 = 3$, $\beta_2 = 2.6$, $\beta_3 = 4$ and $\beta_4 = 2$. For $r = 0.3093$ and $\omega = 0.9827$ we get $\rho(\tilde{L}_{r\omega}) = 0.4892 < \rho(L_{r\omega}) = 0.6107$ and for $r = \omega = 0.85$ we get $\rho(\tilde{L}_{\omega}) = 0.5009 < \rho(L_{\omega}) = 0.5536$.
Example 2. Suppose that the coefficient matrix $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 1 & -\frac{1}{2\times10+1} & -\frac{1}{3\times10+1} & \cdots & -\frac{1}{n\times10+1} \\ -\frac{1}{2\times10+2} & 1 & -\frac{1}{3\times10+2} & \cdots & -\frac{1}{n\times10+2} \\ -\frac{1}{3\times10+3} & -\frac{1}{2\times10+3} & 1 & \cdots & -\frac{1}{n\times10+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n\times10+n} & -\frac{1}{(n-1)\times10+n} & -\frac{1}{(n-2)\times10+n} & \cdots & 1 \end{pmatrix}$$

We take $\beta_i = 0.98$ for all $i = 1, 2, \cdots, n-1$. Table 1 shows the spectral radii, $\rho(L_{r\omega})$ and $\rho(\tilde{L}_{r\omega})$, of AOR method and the preconditioned AOR method, respectively, for different values of n and various r and ω. As this table shows, we have $\rho(L_{r\omega}) < \rho(\tilde{L}_{r\omega})$ for each case.
Example 3. Suppose that the coefficient matrix $\mathcal{A}$ is as

$$\mathcal{A} = \begin{pmatrix} 1 & 0.2 & -0.2 & 0.2 & 0.1 \\ 0.4 & 1 & 0.2 & -0.2 & 0.1 \\ -0.5 & 0.2 & 1 & 0.1 & -0.1 \\ 0.3 & -0.6 & 0.3 & 1 & 0.1 \\ 0.8 & 0.3 & -0.2 & 0.4 & 1 \end{pmatrix}.$$

TABLE 1. Numerical results for Example 2.

n	r	ω	$\rho(L_{r\omega})$	$\rho(\tilde{L}_{r\omega})$
50	0.45	0.78	0.3902	0.3799
100	0.38	0.96	0.2903	0.2784
150	0.37	0.96	0.3076	0.2971
200	0.28	0.95	0.3399	0.3294

The coefficient matrix $\mathcal{A}$ is an H-matrix. Let $\beta_1 = 0.99$, $\beta_2 = 0.8$, $\beta_3 = 0.56$ and $\beta_4 = 0.87$. For $r = 0.35$, $\omega = 0.98$, we have $\rho(\tilde{L}_{r\omega}) = 0.7533 < \rho(L_{r\omega}) = 0.7936$, and for $r = \omega = 0.88$ we get $\rho(\tilde{L}_{\omega}) = 0.7043 < \rho(L_{\omega}) = 0.7323$.
Let $\beta_1 = 2.5$, $\beta_2 = 2.01$, $\beta_3 = 2.92$ and $\beta_4 = 2.21$. For $r = 0.58$, $\omega = 0.95$, we have $\rho(\tilde{L}_{r\omega}) = 0.6346 < \rho(L_{r\omega}) = 0.7706$, and for $r = \omega = 0.89$ we get $\rho(\tilde{L}_{\omega}) = 0.6440 < \rho(L_{\omega}) = 0.7267$.

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