Honam Mathematical J. **32** (2010), No. 3, pp. 389–397

CERTAIN INTEGRAL REPRESENTATIONS OF EULER TYPE FOR THE EXTON FUNCTION X_5

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Abstract. Exton introduced 20 distinct triple hypergeometric functions whose names are X_i $(i = 1, \ldots, 20)$ to investigate their twenty Laplace integral representations whose kernels include the confluent hypergeometric functions ${}_0F_1$, ${}_1F_1$, a Humbert function Ψ_2, a Humbert function Φ_2. The object of this paper is to present 25 (presumably new) integral representations of Euler types for the Exton hypergeometric function X_5 among his twenty X_i $(i = 1, \ldots, 20)$, whose kernels include the Exton function X_5 itself, the Exton function X_6, the Horn's functions H_3 and H_4, and the hypergeometric function $F = {}_2F_1$.

1. Introduction

Exton [4] introduced 20 distinct triple hypergeometric functions whose names are X_i $(i = 1, \ldots, 20)$ to investigate their twenty Laplace integral representations which include the confluent hypergeometric functions ${}_0F_1$, ${}_1F_1$, a Humbert function Ψ_2, a Humbert function Φ_2 in their kernels. The Exton functions X_i have been studied a lot until today, for example, see [2, 5, 6, 7, 8, 9, 10]. Here, we choose to investigate the Exton function X_5 to present (presumably new) 25 integral representations of Euler type which contain the Exton function X_5 itself, the Exton function X_6, the Horn's functions H_3 and H_4, and the hypergeometric function $F = {}_2F_1$ in their kernels.

Received July 12, 2010. Accepted August 10, 2010.

2000 Mathematics Subject Classification. Primary 33C20, 33C65; Secondary 33C05, 33C60, 33C70, 68Q40, 11Y35.

Key words and phrases. Generalized hypergeometric series; Multiple hypergeometric functions; Integrals of Euler type; Laplace integral; Exton functions X_i; Humbert function Ψ_2; Appell function F_4; Srivastava function $F^{(3)}$.

Exton [4] defined the function X_5 by the following triple series

$$X_5(a_1,a_2,a_3;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p}{(c)_{m+n+p}}\frac{x^m\,y^n\,z^p}{m!\,n!\,p!},\tag{1.1}$$

where $(\lambda)_m$ denotes the Pochhammer symbol defined by

$$(\lambda)_m:=\frac{\Gamma(\lambda+m)}{\Gamma(\lambda)}\quad\left(\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ m\in\mathbb{N}_0\right),$$

$\mathbb{C}$, $\mathbb{Z}_0^-$, and $\mathbb{N}_0$ being the set of complex numbers, the set of nonpositive integers, and the set of nonnegative integers, respectively. The precise three-dimensional region of convergence of (1.1) is given by Srivastava and Karlsson [10, p. 102, 44a]:

$$\left\{r<\frac{1}{4}\wedge\max(s,t)<\frac{1}{2}+\frac{1}{2}\sqrt{1-4r}\right\},\ |x|<r,\ |y|<s,\ |z|<t,$$

where the positive quantities r, s and t are associated radii of convergence. For more details about this function and many other three-variable hypergeometric functions, we also refer to Srivastava and Karlsson [10].

It may be recalled the Laplace integral representation of (1.1) (see [4]) in passing that

$$\begin{aligned}X_5(a_1,a_2,a_3;c;x,y,z)&=\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\\&\cdot\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-s-\xi-\eta}s^{a_1-1}\xi^{a_2-1}\eta^{a_3-1}{}_1F_0\left(-;c;xs^2+ys\xi+zs\eta\right)\\&\quad ds\,d\xi\,d\eta,\end{aligned}\tag{1.2}$$

provided $\Re(a_1)>0$, $\Re(a_2)>0$, and $\Re(a_3)>0$.

2. Integral representations of Euler type for X_5

Each of the following integral representations for X_5 holds true.

$$\begin{aligned}X_5(a_1,a_2,a_3;c;x,y,z)&=\frac{\Gamma(s)}{\Gamma(a_1)\Gamma(s-a_1)}\\&\cdot\int_0^1\xi^{a_1-1}(1-\xi)^{s-a_1-1}X_5\left(s,a_2,a_3;c;x\xi^2,y\xi,z\xi\right)d\xi\\&\quad(\Re(s)>\Re(a_1)>0);\end{aligned}\tag{2.1}$$

$$X_5\left(a_1,a_2,a_3;c;x,y,z\right)=\frac{\Gamma\left(s\right)\left(1+\lambda\right)^{a_1}}{\Gamma\left(a_1\right)\Gamma\left(s-a_1\right)}$$
$$\cdot\int_0^1 \xi^{a_1-1}\left(1-\xi\right)^{s-a_1-1}\left(1+\lambda\xi\right)^{-s}\ X_5\left(s,a_2,a_3;c;\sigma^2x,\sigma y,\sigma z\right)d\xi$$
$$\left(\sigma:=\frac{\left(1+\lambda\right)\xi}{1+\lambda\xi};\ \Re(s)>\Re(a_1)>0;\ \lambda>-1\right); \tag{2.2}$$

$$X_5\left(a_1,a_2,a_3;c;x,y,z\right)=\frac{\Gamma\left(s\right)\left(\beta-\gamma\right)^{a_1}\left(\alpha-\gamma\right)^{s-a_1}}{\Gamma\left(a_1\right)\Gamma\left(s-a_1\right)\left(\beta-\alpha\right)^{s-1}}$$
$$\cdot\int_\alpha^\beta\left(\beta-\xi\right)^{s-a_1-1}\left(\xi-\alpha\right)^{a_1-1}\left(\xi-\gamma\right)^{-s}\ X_5\left(s,a_2,a_3;c;\sigma^2x,\sigma y,\sigma z\right)d\xi$$
$$\left(\sigma:=\frac{\left(\beta-\gamma\right)\left(\xi-\alpha\right)}{\left(\beta-\alpha\right)\left(\xi-\gamma\right)};\ \Re(s)>\Re(a_1)>0;\ \gamma<\alpha<\beta\right); \tag{2.3}$$

$$X_5\left(a_1,a_2,a_3;c;x,y,z\right)=\frac{\Gamma\left(s\right)\left(\gamma-\beta\right)^{a_1}\left(\gamma-\alpha\right)^{s-a_1}}{\Gamma\left(a_1\right)\Gamma\left(s-a_1\right)\left(\beta-\alpha\right)^{s-1}}$$
$$\cdot\int_\alpha^\beta\left(\beta-\xi\right)^{s-a_1-1}\left(\xi-\alpha\right)^{a_1-1}\left(\gamma-\xi\right)^{-s}\ X_5\left(s,a_2,a_3;c;\sigma^2x,\sigma y,\sigma z\right)d\xi$$
$$\left(\sigma:=\frac{\left(\gamma-\beta\right)\left(\xi-\alpha\right)}{\left(\beta-\alpha\right)\left(\gamma-\xi\right)};\ \Re(s)>0>\Re(a_1)>0;\ \alpha<\beta<\gamma\right); \tag{2.4}$$

$$X_5\left(a_1,a_2,a_3;c;x,y,z\right)=\frac{2\Gamma\left(s\right)}{\Gamma\left(a_1\right)\Gamma\left(s-a_1\right)}\int_0^{\frac{\pi}{2}}\left(\sin^2\xi\right)^{a_1-\frac{1}{2}}\left(\cos^2\xi\right)^{s-a_1-\frac{1}{2}}$$
$$\cdot X_5\left(s,a_2,a_3;c;x\sin^4\xi,y\sin^2\xi,z\sin^2\xi\right)d\xi\quad\left(\Re(s)>\Re(a_1)>0\right); \tag{2.5}$$

$$X_5\left(a_1,a_2,a_3;c;x,y,z\right)=\frac{2\Gamma\left(s\right)\left(1+\lambda\right)^{a_1}}{\Gamma\left(a_1\right)\Gamma\left(s-a_1\right)}$$
$$\cdot\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{a_1-\frac{1}{2}}\left(\cos^2\xi\right)^{s-a_1-\frac{1}{2}}}{\left(1+\lambda\sin^2\xi\right)^s}X_5\left(s,a_2,a_3;c;\sigma^2x,\sigma y,\sigma z\right)d\xi$$
$$\left(\sigma:=\frac{\left(1+\lambda\right)\sin^2\xi}{1+\lambda\sin^2\xi};\ \Re(s)>\Re(a_1)>0;\ \lambda>-1\right); \tag{2.6}$$

$$X_5(a_1,a_2,a_3;c;x,y,z)=\frac{2\Gamma(s)\,\lambda^{a_1}}{\Gamma(a_1)\,\Gamma(s-a_1)}$$
$$\cdot\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{a_1-\frac{1}{2}}\left(\cos^2\xi\right)^{s-a_1-\frac{1}{2}}}{\left(\cos^2\xi+\lambda\sin^2\xi\right)^{s}}\times X_5\left(s,a_2,a_3;c;\sigma^2x,\sigma y,\sigma z\right)d\xi$$
$$\left(\sigma:=\frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi};\ \Re(s)>\Re(a_1)>0;\ \lambda>0\right);\tag{2.7}$$

$$X_5(a_1,a_2,a_3;c;x,y,z)=\frac{\Gamma(c)}{\Gamma(s)\,\Gamma(c-s)}\int_0^1\xi^{s-1}(1-\xi)^{c-s-1}$$
$$\cdot X_5(a_1,a_2,a_3;s;x\xi,y\xi,z\xi)\,d\xi\quad(\Re(c)>\Re(s)>0);\tag{2.8}$$

$$X_5(a_1,a_2,a_3;c;x,y,z)=\frac{\Gamma(c)\,(1+\lambda)^s}{\Gamma(s)\,\Gamma(c-s)}$$
$$\cdot\int_0^1\xi^{s-1}(1-\xi)^{c-s-1}(1+\lambda\xi)^{-c}\,X_5(a_1,a_2,a_3;s;\sigma x,\sigma y,\sigma z)\,d\xi$$
$$\left(\sigma:=\frac{(1+\lambda)\,\xi}{1+\lambda\xi};\ \Re(c)>\Re(s)>0,\ \lambda>-1\right);\tag{2.9}$$

$$X_5(a_1,a_2,a_3;c;x,y,z)=\frac{\Gamma(c)\,(\beta-\gamma)^s\,(\alpha-\gamma)^{c-s}}{\Gamma(s)\,\Gamma(c-s)\,(\beta-\alpha)^{c-1}}$$
$$\cdot\int_\alpha^\beta(\beta-\xi)^{c-s-1}(\xi-\alpha)^{s-1}(\xi-\gamma)^{-c}X_5(a_1,a_2,a_3;s;\sigma x,\sigma y,\sigma z)\,d\xi$$
$$\left(\sigma:=\frac{(\beta-\gamma)\,(\xi-\alpha)}{(\beta-\alpha)\,(\xi-\gamma)};\ \Re(c)>\Re(s)>0;\ \gamma<\alpha<\beta\right);\tag{2.10}$$

$$X_5(a_1,a_2,a_3;c;x,y,z)=\frac{\Gamma(c)\,(\gamma-\beta)^s\,(\gamma-\alpha)^{c-s}}{\Gamma(s)\,\Gamma(c-s)\,(\beta-\alpha)^{c-1}}$$
$$\cdot\int_\alpha^\beta(\beta-\xi)^{c-s-1}(\xi-\alpha)^{s-1}(\gamma-\xi)^{-c}\,X_5(a_1,a_2,a_3;s;\sigma x,\sigma y,\sigma z)\,d\xi$$
$$\left(\sigma:=\frac{(\gamma-\beta)\,(\xi-\alpha)}{(\beta-\alpha)\,(\gamma-\xi)};\ \Re\,(c)>\Re\,(s)>0;\ \alpha<\beta<\gamma\right);\tag{2.11}$$

$$X_5(a_1,a_2,a_3;c;x,y,z)=\frac{2\Gamma(c)}{\Gamma(s)\,\Gamma(c-s)}\int_0^{\frac{\pi}{2}}\left(\sin^2\xi\right)^{s-\frac{1}{2}}\left(\cos^2\xi\right)^{c-s-\frac{1}{2}}$$
$$\cdot X_5\left(a_1,a_2,a_3;s;x\sin^2\xi,y\sin^2\xi,z\sin^2\xi\right)d\xi\quad(\Re(c)>\Re(s)>0);\tag{2.12}$$

$$X_5\left(a_1,a_2,a_3;c;x,y,z\right)=\frac{2\Gamma\left(c\right)\left(1+\lambda\right)^s}{\Gamma\left(s\right)\Gamma\left(c-s\right)}$$
$$\cdot\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{s-\frac{1}{2}}\left(\cos^2\xi\right)^{c-s-\frac{1}{2}}}{\left(1+\lambda\sin^2\xi\right)^c}X_5\left(a_1,a_2,a_3;s;\sigma x,\sigma y,\sigma z\right)d\xi$$
$$\left(\sigma:=\frac{\left(1+\lambda\right)\sin^2\xi}{1+\lambda\sin^2\xi};\ \Re(c)>\Re(s)>0;\ \lambda>-1\right);\tag{2.13}$$

$$X_5\left(a_1,a_2,a_3;c;x,y,z\right)=\frac{2\Gamma\left(c\right)\lambda^s}{\Gamma\left(s\right)\Gamma\left(c-s\right)}$$
$$\cdot\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{s-\frac{1}{2}}\left(\cos^2\xi\right)^{c-s-\frac{1}{2}}}{\left(\cos^2\xi+\lambda\sin^2\xi\right)^c}X_5\left(a_1,a_2,a_3;s;\sigma x,\sigma y,\sigma z\right)d\xi$$
$$\left(\sigma:=\frac{\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi};\ \Re(c)>\Re(s)>0;\ \lambda>0\right);\tag{2.14}$$

$$X_5\left(a_1,a_2,a_3;c;x,y,z\right)=\frac{\Gamma\left(s\right)}{\Gamma\left(a_2\right)\Gamma\left(s-a_2\right)}\int_0^1\xi^{a_2-1}\left(1-\xi\right)^{s-a_2-1}$$
$$\cdot X_5\left(a_1,s,a_3;c;x,y\xi,z\right)d\xi\quad\left(\Re(s)>\Re(a_2)>0\right);\tag{2.15}$$

$$X_5\left(a_1,a_2,a_3;\varepsilon+a_3;x,y,z\right)$$
$$=\frac{\Gamma\left(\varepsilon+a_3\right)}{\Gamma\left(\varepsilon\right)\Gamma\left(a_3\right)}\int_0^1\xi^{\varepsilon-1}\left(1-\xi\right)^{a_3-1}\left(1-z+z\xi\right)^{-a_1}$$
$$\cdot H_3\left(a_1,a_2;\varepsilon;\frac{x\xi}{\left(1-z+z\xi\right)^2},\frac{y\xi}{1-z+z\xi}\right)d\xi\quad\left(\Re(\varepsilon)>0;\ \Re(a_3)>0\right);\tag{2.16}$$

$$X_5\left(a_1,a_2,a_3;\varepsilon+a_3;x,y,z\right)=\frac{\Gamma\left(\varepsilon+a_3\right)\Gamma\left(\varepsilon\right)}{\Gamma\left(\varepsilon\right)\Gamma\left(a_3\right)\Gamma\left(a_1\right)\Gamma\left(\varepsilon-a_1\right)}$$
$$\cdot\int_0^1\int_0^1\xi^{\varepsilon-1}\eta^{a_1-1}\left(1-\xi\right)^{a_3-1}\left(1-\eta\right)^{\varepsilon-a_1-1}\left(1-z+z\xi\right)^{-a_1}\tag{2.17}$$
$$\cdot H_4\left(a_1,a_2;a_1,\varepsilon-a_1;\frac{x\xi\eta}{\left(1-z+z\xi\right)^2},\frac{y\xi\left(1-\eta\right)}{1-z+z\xi}\right)d\xi d\eta$$
$$\left(\Re(\varepsilon)>\Re(a_1)>0;\ \Re(a_3)>0\right);$$

$$X_5\left(a_1,a_2,a_3;\varepsilon+a_3;x,y,z\right)=\frac{\Gamma\left(\varepsilon+a_3\right)}{\Gamma\left(a_2\right)\Gamma\left(a_3\right)\Gamma\left(\varepsilon-a_2\right)}$$
$$\cdot\int_0^1\int_0^1\xi^{\varepsilon-1}\eta^{a_2-1}\left(1-\xi\right)^{a_3-1}\left(1-\eta\right)^{\varepsilon-a_2-1}\left(1-z+z\xi-y\xi\eta\right)^{-a_1}$$
$$\cdot F\left(\frac{a_1}{2},\frac{1}{2}+\frac{a_1}{2};\varepsilon-a_2;\frac{4x\xi\left(1-\eta\right)}{\left(1-z+z\xi-y\xi\eta\right)^2}\right)d\xi d\eta$$
$$\left(\Re(\varepsilon)>\Re(a_2)>0;\ \Re(a_3)>0\right);\tag{2.18}$$

$$X_5\left(a_1,a_2,a_3;c+s;x,y,z\right)=\frac{\Gamma\left(c+s\right)}{\Gamma\left(c\right)\Gamma\left(s\right)}\int_0^1\xi^{c-1}\left(1-\xi\right)^{s-1}$$
$$\cdot X_6\left(a_1,a_2,a_3;c,s;x\xi,y\xi,z\left(1-\xi\right)\right)d\xi\quad\left(\Re(c)>0;\ \Re(s)>0\right);\tag{2.19}$$

$$X_5\left(a_1,a_2,a_3;c+s;x,y,z\right)=\frac{\Gamma\left(c+s\right)\left(1+\lambda\right)^c}{\Gamma\left(c\right)\Gamma\left(s\right)}$$
$$\cdot\int_0^1\xi^{c-1}\left(1-\xi\right)^{s-1}\left(1+\lambda\xi\right)^{-c-s}X_6\left(a_1,a_2,a_3;c,s;\sigma_1x,\sigma_1y,\sigma_2z\right)d\xi$$
$$\left(\sigma_1:=\frac{\left(1+\lambda\right)\xi}{1+\lambda\xi};\ \sigma_2:=\frac{\left(1-\xi\right)}{1+\lambda\xi};\ \Re(c)>0;\ \Re(s)>0;\ \lambda>-1\right);\tag{2.20}$$

$$X_5\left(a_1,a_2,a_3;c+s;x,y,z\right)=\frac{\Gamma\left(c+s\right)\left(\beta-\gamma\right)^c\left(\alpha-\gamma\right)^s}{\Gamma\left(c\right)\Gamma\left(s\right)\left(\beta-\alpha\right)^{c+s-1}}$$
$$\cdot\int_\alpha^\beta\left(\beta-\xi\right)^{s-1}\left(\xi-\alpha\right)^{c-1}\left(\xi-\gamma\right)^{-c-s}$$
$$X_6\left(a_1,a_2,a_3;c,s;\sigma_1x,\sigma_1y,\sigma_2z\right)d\xi$$
$$\left(\sigma_1:=\frac{\left(\beta-\gamma\right)\left(\xi-\alpha\right)}{\left(\beta-\alpha\right)\left(\xi-\gamma\right)};\ \sigma_2:=\frac{\left(\alpha-\gamma\right)\left(\beta-\xi\right)}{\left(\beta-\alpha\right)\left(\xi-\gamma\right)};\ \Re(c)>0;\ \Re(s)>0\right);$$
$$\left(\gamma<\alpha<\beta\right);\tag{2.21}$$

$$X_5\left(a_1,a_2,a_3;c+s;x,y,z\right)=\frac{\Gamma\left(c+s\right)}{\Gamma\left(c\right)\Gamma\left(s\right)}\frac{\left(\gamma-\beta\right)^c\left(\gamma-\alpha\right)^s}{\left(\beta-\alpha\right)^{c+s-1}}$$
$$\cdot\int_\alpha^\beta\left(\beta-\xi\right)^{s-1}\left(\xi-\alpha\right)^{c-1}\left(\gamma-\xi\right)^{-c-s}$$
$$X_6\left(a_1,a_2,a_3;c,s;\sigma_1x,\sigma_1y,\sigma_2z\right)d\xi$$
$$\left(\sigma_1:=\frac{\left(\gamma-\beta\right)\left(\xi-\alpha\right)}{\left(\beta-\alpha\right)\left(\gamma-\xi\right)};\ \sigma_2:=\frac{\left(\gamma-\alpha\right)\left(\beta-\xi\right)}{\left(\beta-\alpha\right)\left(\gamma-\xi\right)};\ \Re(c)>0;\ \Re(s)>0\right);$$
$$\left(\alpha<\beta<\gamma\right);\tag{2.22}$$

$$X_5\left(a_1,a_2,a_3;c+s;x,y,z\right)=\frac{2\Gamma\left(c+s\right)}{\Gamma\left(c\right)\Gamma\left(s\right)}\int_0^{\frac{\pi}{2}}\left(\sin^2\xi\right)^{c-\frac{1}{2}}\left(\cos^2\xi\right)^{s-\frac{1}{2}}$$

$$\cdot X_6\left(a_1,a_2,a_3;c,s;x\sin^2\xi,y\sin^2\xi,z\cos^2\xi\right)d\xi\quad\left(\Re(c)>0;\ \Re(s)>0\right); \tag{2.23}$$

$$X_5\left(a_1,a_2,a_3;c+s;x,y,z\right)=\frac{2\Gamma\left(c+s\right)\left(1+\lambda\right)^c}{\Gamma\left(c\right)\Gamma\left(s\right)}$$

$$\cdot\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{c-\frac{1}{2}}\left(\cos^2\xi\right)^{s-\frac{1}{2}}}{\left(1+\lambda\sin^2\xi\right)^{c+s}}X_6\left(a_1,a_2,a_3;c,s;\sigma_1x,\sigma_1y,\sigma_2z\right)d\xi$$

$$\left(\sigma_1:=\frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi};\ \sigma_2:=\frac{\cos^2\xi}{1+\lambda\sin^2\xi};\ \Re(c)>0;\ \Re(s)>0;\ \lambda>-1\right); \tag{2.24}$$

$$X_5\left(a_1,a_2,a_3;c+s;x,y,z\right)=\frac{2\Gamma\left(c+s\right)\lambda^c}{\Gamma\left(c\right)\Gamma\left(s\right)}$$

$$\cdot\int_0^{\frac{\pi}{2}}\frac{\left(\sin^2\xi\right)^{c-\frac{1}{2}}\left(\cos^2\xi\right)^{s-\frac{1}{2}}}{\left(\cos^2\xi+\lambda\sin^2\xi\right)^{c+s}}X_6\left(a_1,a_2,a_3;c,s;\sigma_1x,\sigma_1y,\sigma_2z\right)d\xi$$

$$\left(\sigma_1:=\frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi};\ \sigma_2:=\frac{\cos^2\xi}{\cos^2\xi+\lambda\sin^2\xi};\ \Re(c)>0;\ \Re(s)>0;\right)$$

$$(\lambda>0), \tag{2.25}$$

where $F={}_2F_1$, H_3, H_4, and X_6 denote the Gauss hypergeometric function, Horn's functions, and an Exton function defined, respectively, by

$$H_3\left(\alpha,\beta;\gamma;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}\,(\beta)_n}{(\gamma)_{m+n}}\,\frac{x^m\,y^n}{m!\,n!},$$

$$H_4\left(\alpha,\beta;\gamma,\varepsilon;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}\,(\beta)_n}{(\gamma)_m\,(\varepsilon)_n}\,\frac{x^m\,y^n}{m!\,n!},$$

and

$$X_6\left(a_1,a_2,a_3;c_1,c_2;x,y,z\right)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{2m+n+p}\,(a_2)_n\,(a_3)_p}{(c_1)_{m+n}\,(c_2)_p}\,\frac{x^m\,y^n\,z^p}{m!\,n!\,p!}.$$

3. Proof of results

It is noted that each of the integral representations in Section 2 can be proved mainly by expressing the series definition of the involved special function in each integrand and changing the order of the integral sign and the summation, and finally using the following well-known relationship between the Beta function $B(\alpha, \beta)$ and the Gamma function Γ:

$$B(\alpha,\, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha) > 0;\ \Re(\beta) > 0), \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases} \tag{3.1}$$

As an illustration, we try to prove only (2.1). By applying the definition of X_5 to the integrand of (2.1), and and changing the order of the integral sign and the summation, we find from (3.1) that

$$X_5\left(a_1, a_2, a_3; c; x, y, z\right) = \frac{\Gamma\left(s\right)}{\Gamma\left(a_1\right)\Gamma\left(s-a_1\right)}$$
$$\cdot \sum_{m,n,p=0}^{\infty} \frac{(s)_{2m+n+p}\,(a_2)_n\,(a_3)_p}{(c)_{m+n+p}\,m!n!p!} x^m y^n z^p\, B\left(a_1+2m+n+p, s-a_1\right)$$
$$\left(\Re(s) > \Re(a_1) > 0\right),$$

which, upon employing the second identity of (3.1), yields

$$\sum_{m,n,p=0}^{\infty} \frac{(s)_{2m+n+p}\,(a_2)_n\,(a_3)_p}{(c)_{m+n+p}\,m!n!p!} x^m y^n z^p \frac{(a_1)_{2m+n+p}}{(s)_{2m+n+p}} = X_5\left(a_1, a_2, a_3; c; x, y, z\right).$$

This completes the proof of (2.1).

Acknowledgments. This research was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (2010-0011005).

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