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PROPERTIES OF A GENERALIZED UNIVERSAL COVERING SPACE OVER A DIGITAL WEDGE

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Abstract. The paper studies an existence problem of a (generalized) universal covering space over a digital wedge with a compatible adjacency. In algebraic topology it is well-known that a connected, locally path connected, semilocally simply connected space has a universal covering space. Unlike this property, in digital covering theory we need to investigate its digital version which remains open.

1. Introduction

Useful tools from algebraic topology for studying digital topological properties of a (binary) digital space include a digital covering space. This has been studied in many papers including [2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Motivated by the study of a *covering space* over a *figure eight* in algebraic topology [28], the recent papers [6] (see also [3, 11, 16, 17, 18, 19]) studied its digital version, which plays an important role in classifying digital spaces. In algebraic topology, it is also well known that *a universal covering space over the figure eight is an infinite tree with a fractal structure.* But such a kind of approach cannot be available in digital covering theory. Indeed, we can find some intrinsic features of an infinite fold covering space over a digital wedge consisting of two simple closed k-curves. By using intrinsic features of a digital covering of a digital wedge, the papers [16, 18, 19] study the *generalized universal property* which is the digital version of a universal covering space in algebraic topology. As shown in [16, 18, 19], compared with the algebraic topological version, the generalized universal property in digital covering theory has own properties.

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In the study of a digital wedge consisting of two simple closed k-curves, since we have mainly studied digital wedges consisting of two simple closed k-curves in $\mathbf{Z}^2$, $k \in \{4, 8\}$, the other cases remain unsolved. Thus we need to expand the knowledge of a covering space over a digital wedge with a compatible adjacency.

This paper is organized as follows. Section 2 provides some basic notions and terminology. Section 3 reviews properties of a covering space over a digital wedge consisting of two simple closed k-curves with a compatible adjacency. Section 4 investigates properties of a generalized universal covering space over a digital wedge. Section 5 studies an existence problem of a generalized universal covering over a digital wedge.

2. Preliminaries

Let $\mathbf{Z}^n$ and $\mathbf{N}$ denote the sets of points in the Euclidean nD space with integer coordinates and the set of natural numbers $n \in \mathbf{N}$, respectively. Since a digital image in $\mathbf{Z}^n$ can be regarded as a set with one of the k-adjacency relations of $\mathbf{Z}^n$ or a digital k-graph [27], in this paper we use the terminology *digital space* instead of *digital image*.

As a generalization of the k-adjacency relations of 2D and 3D digital space in [25, 27], we have used *k-adjacency relations* of $\mathbf{Z}^n$ for studying a multi-dimensional digital space $X \subset \mathbf{Z}^n$ induced from the following criterion [4] (see also [6, 9]):
For a natural number m with $1 \leq m \leq n$, two distinct points $p = (p_i)_{i \in [1,n]_{\mathbf{Z}}}$ and $q = (q_i)_{i \in [1,n]_{\mathbf{Z}}}$ are k_m- (or $k(m,n)$-)adjacent if
• there are at most m indices i such that $|p_i - q_i| = 1$ and
• for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

By using this operator, we established the following $k := k_m := k(m,n)$-adjacency relations of $\mathbf{Z}^n$.

Proposition 2.1. [15] *In $\mathbf{Z}^n$ we obtain the following k-adjacency relations.*
$k := k(m,n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n$, *where* $C_i^n = \frac{n!}{(n-i)!\ i!}$.

We say that two subsets (A, k) and (B, k) of (X, k) are k-adjacent to each other if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are k-adjacent to each other [25]. We say that a set $X \subset \mathbf{Z}^n$ is k-connected if it is not a union of two disjoint non-empty sets that are not k-adjacent to each other [25]. For an adjacency relation k of $\mathbf{Z}^n$, a simple k-path with $l + 1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i+1$ or $i = j+1$ [25]. If $x_0 = x$ and $x_l = y$, then we say that the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [10], is the simple k-path $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $j = i + 1(mod\, l)$ or $i = j + 1(mod\, l)$ [25].

In order to study both digital continuity and various properties of a digital k-surface [9, 10], we have used the following digital k-neighborhood.

Definition 1. [6] *For a digital space (X, k) in $\mathbf{Z}^n$, the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X*

$$N_k(x_0, \varepsilon) = \{x \in X \,|\, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$.

Motivated by both the digital continuity in [27] and the (k_0, k_1)-continuity in [2] (see also [6]), the following notion of digital continuity has been often used for the study of multi-dimensional digital spaces.

Proposition 2.2. [13] *Let (X, k_0) and (Y, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

Since a digital space can be considered to be a digital k-graph, we can use a (k_0, k_1)-*isomorphism* instead of a (k_0, k_1)-*homeomorphism* in [2], as follows.

Definition 2. [7] *For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-isomorphism and use the notation $X \approx_{k_0} Y$.*

For a digital space (X, k) and $A \subset X$, (X, A) is called a *digital space pair* with k-adjacency [9]. Furthermore, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a *pointed digital space* [25]. Motivated by the *pointed digital homotopy* in [2], the following notion of *relative digital homotopy* to a subset $A \subset X$ is often used for studying a digital space (X, k) in $\mathbf{Z}^n$ in terms of the k-homotopic thinning and the strong k-deformation retract in [9, 10] (see also [15]). If the identity map 1_X is (k, k)-homotopic relative to $\{x_0\}$ in X to a constant map with space consisting of some $x_0 \in X$, then we say that (X, x_0) is *pointed k-contractible* [2]. Indeed, the notion of k-contractility is slightly different from the contractility in Euclidean topology [2] (see also [12]).

Unlike the two digital fundamental groups [1, 24], motivated by Khalimsky's digital k-fundamental group in [23], for a digital space (X, x_0) the paper [2] establishes the digital k-fundamental group $\pi^k(X, x_0)$ which is a group [2], where the base point is assumed as a point which is not deletable by a strong deformation retract [11]. Besides, if X is pointed k-contractible, then $\pi^k(X, x_0)$ is proved trivial [2]. Let $((X, A), k)$ be a digital space pair with k-adjacency. A map $f : ((X, A), k_0) \to ((Y, B), k_1)$ is called (k_0, k_1)-continuous if f is (k_0, k_1)-continuous and $f(A) \subset B$ [8]. If $A = \{a\}$, $B = \{b\}$, we write $(X, A) = (X, a)$, $(Y, B) = (Y, b)$, and we say that f is a *pointed* (k_0, k_1)-continuous map [25]. Besides, a (k_0, k_1)-continuous map $f : ((X, x_0), k_0) \to ((Y, y_0), k_1)$ induces a group homomorphism $f_* : \pi^{k_0}(X, x_0) \to \pi^{k_1}(Y, y_0)$ given by $f_*([\alpha]) = [f \circ \alpha]$, where $[\alpha] \in \pi^{k_0}(X, x_0)$ [2].

The following notion has been often used in digital k-homotopy theory and digital covering theory.

Definition 3. [6] *A pointed k-connected digital space (X, x_0) is called simply k-connected if $\pi^k(X, x_0)$ is a trivial group.*

Theorem 2.3. [6] *(see also* [11]*) $\pi^k(SC_k^{n,l})$ is an infinite cyclic group. Precisely, $\pi^k(SC_k^{n,l}) \simeq (l\mathbf{Z}, +)$, where $SC_k^{n,l}$ is not k-contractible and "$\simeq$" means a group isomorphism.*

3. Generalized Universal Covering Space over a Digital Wedge

Some properties of a digital covering space including the unique lifting property [6] and digital homotopy lifting theorem [5] have been substantially used in calculating $\pi^k(X, x_0)$ and classifying digital spaces [9, 10, 11, 12, 13, 14, 15, 16, 17], proving an existence of a universal covering space [17] and studying the Cartesian product of universal covering property [16]. Let us now recall the axiom of a digital covering space which is equivalent to the earlier version in [5, 6]. In this section we will refer to a simpler form of a digital covering space (see Proposition 3.2).

Proposition 3.1. [6] *(see also* [9]*) Let (E, k_0) and (B, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $p : E \to B$ be a (k_0, k_1)-continuous surjection. Suppose, for any $b \in B$, there exists $\varepsilon \in \mathbf{N}$ such that*
(1) for some index set M, $p^{-1}(N_{k_1}(b, \varepsilon)) = \cup_{i \in M} N_{k_0}(e_i, \varepsilon)$ with $e_i \in$

$p^{-1}(b)$;
(2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, \varepsilon) \cap N_{k_0}(e_j, \varepsilon)$ is an empty set; and
(3) the restriction map p on $N_{k_0}(e_i, \varepsilon)$ is a (k_0, k_1)-isomorphism for all $i \in M$.
Then, the map p is called a (k_0, k_1)-covering map and (E, p, B) is said to be a (k_0, k_1)-covering. The digital space E is called a (k_0, k_1)-covering space over B.

In Proposition 3.1 we may take $\varepsilon = 1$ [9] (see also [3]). Recently, by using the *surjection* instead of the (k_0, k_1)-*continuous surjection* of Proposition 3.1, the paper [18] improves the axiom of a digital (k_0, k_1)-covering, as follows.

Proposition 3.2. [22] *Let us replace the (k_0, k_1)-continuous surjection of Proposition 3.1 by a surjection. Then the map p is a (k_0, k_1)-covering map.*

Definition 4. [9] *A (k_0, k_1)-covering (E, p, B) is called a radius n-(k_0, k_1)-covering if $\varepsilon \geq n$.*

According to Definition 4, we clearly observe that a (k_0, k_1)-covering of Proposition 3.1 is obviously a *radius* 1-(k_0, k_1)-*covering* [9].

For three digital spaces (E, k_0) in $\mathbf{Z}^{n_0}$, (B, k_1) in $\mathbf{Z}^{n_1}$, and (X, k_2) in $\mathbf{Z}^{n_2}$, let $p : E \to B$ be a (k_0, k_1)-continuous map. For a (k_2, k_1)-continuous map $f : (X, k_2) \to (B, k_1)$, as the digital analogue of the lifting in [26], we say that a *digital lifting* of f is a (k_2, k_0)-continuous map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$ [6]. Thus, the *unique digital lifting theorem* in [6] (see also [9, 12]) and *digital homotopy lifting theorem* was introduced in [5], which plays an important role in studying digital covering theory.

Although in algebraic topology it is well-known that a simply connected and locally path connected covering space is a universal covering space [28], in digital covering theory we can propose that a generalized universal covering space has its intrinsic feature. The following theorem has been often used in studying the digital lifting theorem.

Unlike the non-2-contractibility of $\mathbf{Z}$, we can observe the simply 2-connectedness of $\mathbf{Z}$ [13], which can be essential to the proof of the generalized lifting theorem in [3, 13]. While the universal property of a digital covering in [3] was studied for a radius 2-(k_0, k_1)-covering with some hypothesis. As a generalization of the *universal covering property* of [3], we obtain the following.

Definition 5. [16] *We say that $(\tilde{E}, p, B)$ is a universal $(\tilde{k}, k)$-covering if for any radius 2-(k_1, k)-covering map $q : \tilde{X} \to B$, there is always a $(\tilde{k}, k_1)$-continuous map $f : (\tilde{E}, \tilde{k}) \to (\tilde{X}, \tilde{k}_1)$ such that $q \circ f = p$.*

In Definition 5, the space $(\tilde{E}, \tilde{k})$ is called a *universal* $(\tilde{k}, k)$*-covering space* of (B, k) and $(\tilde{E}, p, B)$ is called a *universal* $(\tilde{k}, k)$*-covering*. In addition, we say that $(\tilde{E}, \tilde{k})$ has the *universal* $(\tilde{k}, k)$*-covering property.* Indeed, the paper [3] studied the universal $(2, k)$-covering property.

As a generalization of the universal $(\tilde{k}, k)$-covering of Definition 5, we can established the following:

Definition 6. [19] *Let $((E, e_0), \tilde{k})$ and $((B, b_0), k)$ be two digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A $(\tilde{k}, k)$-covering map $p : ((E, e_0), \tilde{k}) \to ((B, b_0), k)$ is called generalized universal if for any pointed (k', k)-covering map $q : ((X, x_0), k') \to ((B, b_0), k)$, there exists a pointed $(\tilde{k}, k')$-continuous map $\phi : (E, e_0) \to (X, x_0)$ such that $q \circ \phi = p$. Then, $((E, e_0), k_0)$ is called a generalized universal $(\tilde{k}, k)$-covering space (briefly, GU-$(\tilde{k}, k)$-covering space) of $((B, b_0), k)$. Furthermore, we say that this $(\tilde{k}, k)$-covering map p has the generalized universal $(\tilde{k}, k)$-covering property (briefly, GU-$(\tilde{k}, k)$-covering property). Besides, $((E, e_0), p, (B, b_0))$ is called a generalized universal $(\tilde{k}, k)$-covering (briefly, GU-$(\tilde{k}, k)$-covering).*

The current *universal* (k_0, k_1)*-covering* has no limitation of both the radius 2-(k_0, k_1)-covering and the adjacency relations of (E, k_0) in $\mathbf{Z}^{n_0}$ and (B, k_1) in $\mathbf{Z}^{n_1}$ related to the (k_0, k_1)-covering $((E, k_0), p, (B, k_1))$, where $((E, e_0), k_0)$ and $((B, b_0), k_1)$ are two digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. In other words, in view of Definition 6, if a given (k_0, k_1)-covering does not satisfy a radius 2-(k_1, k)-isomorphism, then we cannot study further the universal $(\tilde{k}, k)$-covering property of $(\tilde{E}, p, B)$ in [3]. Thus, the paper [19] generalizes the universal $(2, k)$-covering without any limitation of $SC_k^{n,l}$: Let $((\mathbf{Z}, 0), p, (SC_k^{n,l}, c_0))$ be a $(2, k)$-covering. Then for any (k_0, k)-covering $((X, x_0), q, (SC_k^{n,l}, c_0))$, there is always a $(2, k_0)$-continuous map $f : (\mathbf{Z}, 0) \to (X, x_0)$ such that $q \circ f = p$. As an example of the GU-$(\tilde{k}, k)$-covering of Definition 6, we obtained the following:

Theorem 3.3. [19] *Consider a (k', k)-covering $(E, p, SC_k^{n,l})$, where $SC_k^{n,l}$ need not be k-contractible. Then, $(E, p, SC_k^{n,l})$ has the GU-(k', k)-covering property, where (E, k') is $(k', 2)$-isomorphic to $(\mathbf{Z}, 2)$.*

Since the study of a digital covering space over a digital wedge is very important in digital covering theory, let us now recall a compatible adjacency of a digital wedge which is an advance form of the former in [6] (see also [10, 17]).

Definition 7. [21] *For pointed digital spaces $((X, x_0), k_0)$ in $\mathbf{Z}^{n_0}$ and $((Y, y_0), k_1)$ in $\mathbf{Z}^{n_1}$, the wedge of (X, k_0) and (Y, k_1), written $(X \vee Y, (x_0, y_0))$, is the digital space in $\mathbf{Z}^n$*

$$\{(x, y) \in X \times Y | \quad x = x_0 \quad \text{or} \quad y = y_0\} \tag{3.1}$$

with the following compatible $k(m, n)$(or k)-adjacency relative to both (X, k_0) and (Y, k_1), and the only one point (x_0, y_0) in common such that

(W1) the $k(m, n)$ (or k)-adjacency is determined by the numbers m and n with $n = \max\{n_0, n_1\}$, $m = \max\{m_0, m_1\}$ satisfying $(W1-1)$ below, where the numbers m_i are taken from the k_i(or $k(m_i, n_i)$)-adjacency relations of the given digital spaces $((X, x_0), k_0)$ and $((Y, y_0), k_1)$, $i \in \{0, 1\}$.

(W 1-1) In view of (3.1), induced from the projection maps, we can consider the natural projection maps $W_X : (X \vee Y, (x_0, y_0)) \to (X, x_0)$ and $W_Y : (X \vee Y, (x_0, y_0)) \to (Y, y_0)$. In relation to the establishment of a compatible k-adjacency of the digital wedge $(X \vee Y, (x_0, y_0))$, the following restriction maps of W_X and W_Y on $(X \times \{y_0\}, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ and $(\{x_0\} \times Y, (x_0, y_0)) \subset (X \vee Y, (x_0, y_0))$ satisfy the following properties, respectively:

$$\begin{cases} (1)\, W_X|_{X \times \{y_0\}} : (X \times \{y_0\}, k) \to (X, k_0) \text{ is a } (k, k_0)\text{-isomorphism; and} \\ (2)\, W_Y|_{\{x_0\} \times Y} : (\{x_0\} \times Y, k) \to (Y, k_1) \text{ is a } (k, k_1)\text{-isomorphism.} \end{cases}$$

(W2) Any two distinct elements $x(\neq x_0) \in X \subset X \vee Y$ and $y(\neq y_0) \in Y \subset X \vee Y$ are not $k(m, n)$ (or k)-adjacent to each other.

Example 3.4. [22] *Consider the following three simple closed k-curves in* [4, 8, 15].
$MSC_{18} := ((0, 0, 0), (1, -1, 0), (1, -1, 1), (2, 0, 1), (1, 1, 1), (1, 1, 0)) \subset \mathbf{Z}^3$,
$SC_8^{2,6} \approx_8 ((0, 0), (1, 1), (1, 2), (0, 3), (-1, 2), (-1, 1))$ *and*
$SC_{26}^{3,4} := ((0, 0, 0), (1, 1, 1), (0, 2, 2), (-1, 1, 1))$.

Then we can consider digital wedges with compatible adjacency, as follows.
(1) $(MSC_{18} \vee SC_8^{2,6}, 18)$ and $(SC_{26}^{3,4} \vee SC_8^{2,6}, 26)$.
(2) No existence of compatible k-adjacency of $SC_{26}^{3,4} \vee MSC_{18}$.

Comparing with the former adjacency of a digital wedge in [12, 15], we obtain the following.

Remark 3.5. *The compatible adjacency of Definition 7 is a generalization of the former version in* [3, 12, 14].

4. Some Properties of Infinite Fold Covering Spaces over a Digital Wedge Consisting of Two Simple Closed k-Curves

In the study of an existence of an infinite fold covering space over $SC_8^{2,6} \vee SC_8^{2,6}$, the recent papers [3, 6, 13, 16, 18, 19] suggested two types of infinite fold covering spaces over $SC_8^{2,6} \vee SC_8^{2,6}$. Now we have the following question: How many kinds of infinite fold covering spaces over $SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$ with a compatible k-adjacency in $\mathbf{Z}^n$. Thus, in this paper by using compatible adjacency of a digital wedge, we can study infinite fold digital covering spaces over a digital wedge consisting of two simple closed k-curves without any limitation of both a dimension and a digital wedge consisting of two simple closed k-curves.

Theorem 4.1. *Assume $SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$ with a compatible k-adjacency in $\mathbf{Z}^n$, where $n = max\{n_1, n_2\}$. Then there are countably many infinite fold (k', k)-covering spaces (E', k') in $\mathbf{Z}^m$ over $(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}, k)$, $m \geq n$ such that*
(1) each of them is the type 1 of Figure 1 and
(2) (E', k') is $(k', 8)$-isomorphic to an infinite fold $(8, 8)$-covering space in $\mathbf{Z}^2$ over $(SC_8^{2,l_1} \vee SC_8^{2,l_2}, 8)$.

Proof: If $SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$ has a compatible k-adjacency in $\mathbf{Z}^n$, $n = \max\{n_1, n_2\}$, then it is $(k, 8)$-isomorphic to $(SC_8^{2,l_1} \vee SC_8^{2,l_2}, 8)$ because each point $x \in SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$, which is not the common point, has $N_k(x, 1) \subset SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$ such that $N_k(x, 1) \approx_{(k,2)} N_2(0, 1)$ in $\mathbf{Z}$ and further, the common point denoted by $x_0 \in SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$ has $N_k(x_0, 1) \approx_{(k,8)} N_8(v_0, 1)$, where $N_8(v_0, 1) \subset SC_8^{2,l_1} \vee SC_8^{2,l_2}$ and v_0 is the common point of the digital wedge of $SC_8^{2,l_i}, i \in \{1, 2\}$. Therefore, it implies that there is a local $(k, 8)$-isomorphic bijection between $(SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}, k)$ and $(SC_8^{2,l_1} \vee SC_8^{2,l_2}, 8)$. In view of this assertion, for some $m \in \mathbf{N}$ with $m \geq n$ (here the number m need not be equal to n), we obtain an infinite fold $(k', 8)$-covering space $(E', k') \subset \mathbf{Z}^m$ over $(SC_8^{2,l_1} \vee SC_8^{2,l_2}, 8)$ having the type 1 of Figure 1, and further, $(E', k') \approx_{(k',8)} (F, 8)$, where $(F, 8)$ is an infinite fold $(8, 8)$-covering space over $SC_8^{2,l_1} \vee SC_8^{2,l_2}$ (see the covering space over $SC_8^{2,l_1} \vee SC_8^{2,l_2}$ in Figure 1(b)). As presented in (a) of Figure 1, since the object (E', k') is assumed in $\mathbf{Z}^m$ with $m \geq n$ with $n = \max\{n_1, n_2\}$, the existence of (E', k') in $\mathbf{Z}^m$ is valid. For instance, for any finite numbers l_i of $SC_{k_i}^{n_i, l_i}$, $i \in \{0, 1\}$ we can figure out the two digital spaces (E', k') and $(F, 8)$ presented in (a) and (b) of Figure 1. $\square$

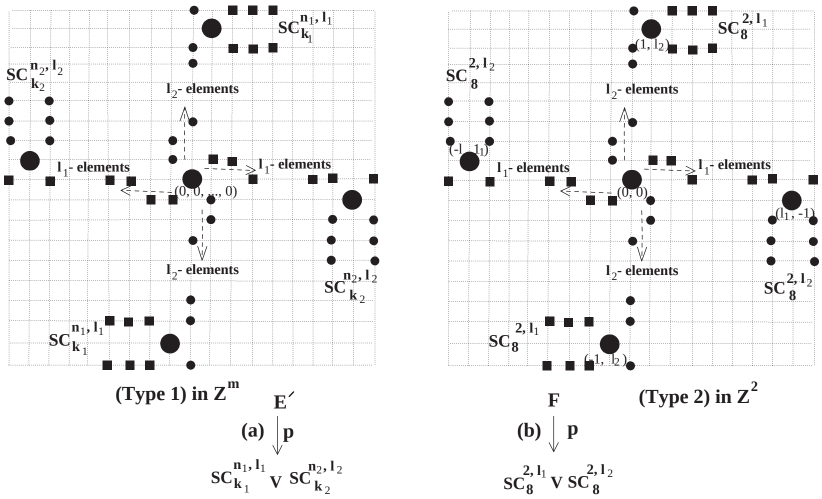

FIGURE 1. (a) Infinite fold (k', k)-covering $(E', p, SC_{k_1}^{n_1, l_1} \vee SC_{k_2}^{n_2, l_2})$ in $\mathbf{Z}^m, m \geq n,\ n = \max\{n_1, n_2\}$ (b) Infinite fold $(8, 8)$-covering $(F, p, SC_8^{2, l_1} \vee SC_8^{2, l_2})$.

Example 4.2. *Consider the two digital wedges* $(MSC_{18} \vee SC_8^{2,6}, 18)$ *and* $(SC_{26}^{3,4} \vee SC_8^{2,6}, 26)$ *in Example 3.4. By using the same method as Theorem 3.1, we can find countably many infinite fold* $(3^n - 2^n - 1, 18)$*-covering spaces* $(E', p, MSC_{18} \vee SC_8^{2,6}, 18)$, $n \geq 3$ *such that* $(E', 3^n - 2^n - 1)$ *is* $(3^n - 2^n - 1, 8)$*-isomorphic to an infinite fold* $(8, 18)$*-covering space* $(F_1, 8)$ *in* $\mathbf{Z}^2$ *over* $MSC_{18} \vee SC_8^{2,6}$ *having the type* 1 *of Figure 1. Besides, by the same method as above, we can find countably many infinite fold* $(3^m - 1, 26)$*-covering spaces over* $(SC_{26}^{3,4} \vee SC_8^{2,6}, 26)$ *having the type* 1 *of Figure 1.*

5. Remark on a Generalized Universal Covering Space

In algebraic topology it is well-known that the existence problem of a universal covering space [28]: A simply connected and locally path connected covering space is a universal covering space. For a digital wedge consisting of two simple closed 8-curves, the papers [3, 16] deals with a generalized universal covering space in $\mathbf{Z}^2$. But in this paper, by using a compatible adjacency of a digital wedge of Definition 7, we can discuss an existence problem of a generalized universal covering space over a digital wedge consisting of two simple closed k-curves without any limitation of dimensions of both a digital covering space and a digital wedge of two simple closed k-curves.

Remark 5.1. *As discussed in* [16]*, in view of each of digital covering spaces over $SC_{k_1}^{n_1,l_1} \vee SC_{k_2}^{n_2,l_2}$ in Figure 1, we obtain that the digital covering space $(E, k') \subset \mathbf{Z}^m$ in Theorem 4.1 cannot be a generalized universal covering space.*

In addition, we can observe that the *simply k-connected* of a base space (B, k) need not guarantee the existence of a GU-(k', k)-covering, as follows.

Theorem 5.2. *Consider a $(k, 8)$-covering $(E, p, SC_8^{2,4} \vee SC_8^{2,4})$ such that E is simply k-connected. Then, $(E, p, SC_8^{2,4} \vee SC_8^{2,4})$ need not have the GU-$(k, 8)$-covering property.*

Before proving this theorem, we had better comment the proof of the assertion of [18], as follows. In Theorem 5.2, we have corrected by replacing the word "cannot" of [18] by "need not". In order to prove the former version of Theorem 5.2, the paper [18] used an infinite fold covering space over $SC_8^{2,4} \vee SC_8^{2,4}$ (see Figure 1 of [18]). However, the picture is ambiguous and is not clear. Thus, the present paper will use a finite and simple example instead.

Proof: Consider an $(8, 8)$-covering (E, p, B) such that E is simply 8-connected, where $B = SC_8^{2,4} \vee SC_8^{2,4}$. Then, (E, p, B) need not have the GU-$(8, 8)$-covering property. In order to prove the assertion, we suffice to give the following counter example. Consider the identity map $1_B : B \to B$ so that $E = B$. Further, consider the $(8, 8)$-covering map $r : F \to B$ in Figure 2. Then, it is clear that the given identity map $1_B : B \to B$ cannot have an $(8, 8)$-continuous map $q : B \to F$ such that $r \circ q = 1_B$. □

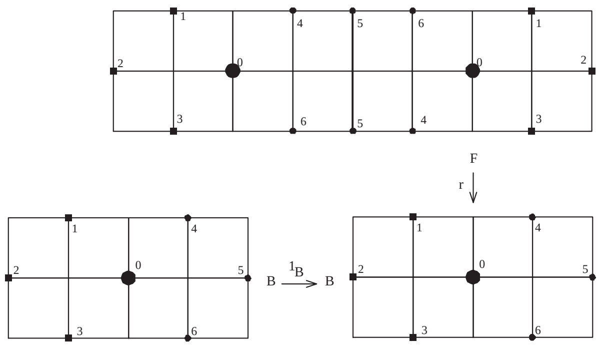

FIGURE 2. Configuration of a non-existence of the universal covering property.

In view of Theorem 5.2, we may have the following question.
[Question A] Does the digital wedge $(SC_8^{2,4} \vee SC_8^{2,4}, 8)$ have the universal covering space?
As a general form of Question A, we can suggest the following question.
[Open question] Let $B := SC_{k_0}^{n_0,l_0} \vee SC_{k_1}^{k_1,l_1}$ be a digital wedge with some compatible k-adjacency, where $SC_{k_i}^{n_i,l_i}$ need not be k_i-contractible, $i \in \{0,1\}$. Then we propose an open problem: under what condition of B is there a universal (k', k)-covering space over B?

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