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SUBMANIFOLDS OF AN ALMOST r-PARACONTACT RIEMANNIAN MANIFOLD ENDOWED WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. We define a semi-symmetric metric connection in an almost r-paracontact Riemannian manifold and we consider submanifolds of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection and obtain Gauss and Codazzi equations, Weingarten equation and curvature tensor for submanifolds of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection.

1. Introduction

The almost complex and almost contact submanifolds were studied by many authors including R. S. Mishra[11]. In [12], R. Nivas considered submanifolds of a Riemannian manifold with a semi-symmetric connection. Some properties of submanifolds of a Riemannian manifold with semi-symmetric semi-metric connection were studied in [3] by B. Barua. Moreover, in [10], I. Mihai and K. Matsumoto studied submanifolds of an almost r-paracontact Riemannian manifold of P-Sasakian type.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmeteic. The connection ∇ is a metric connection if

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there is a Riemannian metric g in M is such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

In [5] and [13], A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be *semi-symmetric connection* if its torsion tensor T is of the form

$$T(X,Y) = u(Y)X - u(X)Y,$$

where u is a 1-form. In [16], K. Yano considered a semi-symmetric metric connection and studied some of its properties. In [1], [2], [3], [6], [7], [8], [9] and [14], some kinds of semi-symmetric metric or non-metric connections were studied.

Let M be an n-dimensional Riemannian manifold with a positive definite metric g. If there exist a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, \xi_3, ...\xi_r$ $(n > r)$, r 1-forms $\eta^1, \eta^2, \eta^3, ..., \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \alpha, \beta \in (r) = \{1, 2, 3, \ldots r\} \qquad (i)$$

$$\phi^2(X) = X - \eta^\alpha(X)\xi_\alpha \qquad (ii)$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \quad \alpha \in (r) \qquad (iii)$$

$$g(\phi X, \phi Y) = g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \qquad (iv)$$

where X and Y are vector fields on M, then the structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be an *almost r-paracontact Riemannian structure* and M is an *almost r-paracontact Riemannian manifold* [8].

From (i) through (iv), we have

$$\phi(\xi_\alpha) = 0, \quad \alpha \in (r) \qquad (v)$$

$$\eta^\alpha \circ \phi = 0, \quad \alpha \in (r) \qquad (vi)$$

$$\Phi(X,Y) \overset{\text{def}}{=} g(\phi X, Y) = g(X, \phi Y), \qquad (vii)$$

where Φ is a semi-symmetric $(0,2)$-tensor field. An almost r-paracontact Riemannian manifold M with structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be *S-paracontact manifold* if

$$\Phi(X,Y) = (\dot{\nabla}_Y \eta^\alpha)(X), \qquad \text{for all} \quad \alpha \in (r).$$

On an almot r-paracontact Riemannian manifold M with structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be P-*Sasakian manifold* if it also satisfies

$$\dot{\nabla}_Z \Phi(X,Y) = -\sum_\alpha \eta^\alpha(X)[g(Y,Z) - \sum_\beta \eta^\beta(Y)\eta^\beta(Z)]$$

$$-\sum_\alpha \eta^\alpha(Y)[g(X,Z) - \sum_\beta \eta^\beta(X)\eta^\beta(Z)]$$

for all vector fields X, Y and Z on M, where $\dot{\nabla}$ denotes the Riemannian connection with respect to g [10].

The above two conditions are respectively equivalent to

$$\phi X = \dot{\nabla}_X \xi_\alpha \qquad \text{for all} \quad \alpha \in (r) \quad \text{and}$$

$$\dot{\nabla}_Y \phi(X) = -\sum_\alpha \eta^\alpha(X)[Y - \eta^\alpha(Y)\xi_\alpha]$$

$$-[g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)] \sum_\beta \xi_\beta.$$

In this paper, we study semi-symmetric metric connection in an almost r-paracontact Riemannian manifold. We consider hypesurfaces and submanifolds of almost r-paracontact Riemannian manifold endowed with a semi-symmetric metric connection. We also obtain Gauss and Codazzi equations for hypersurfaces and curvature tensor and Weingarten equation for submanifolds of almost r-paracontact Riemannian manifold with respect to semi-symmetric metric connection.

2. Preliminaries

Let M^{n+1} be an $(n+1)$-dimensional differentiable manifold of class C^∞ and M^n be the hypersurface in M^{n+1} by the immersion τ: $M^n \to M^{n+1}$. The differential $d\tau$ of the immersion τ is denoted by B. The vector field X in the tangent space of M^n corresponds to a vector field BX in that of M^{n+1}. Suppose that the enveloping manifold M^{n+1} is an almost r-paracontact Riemannian manifold with metric $\tilde{g}$. Then the hypersurface M^n is also an almost r-paracontact Riemannian manifold with induced metric g defined by

$$g(\phi X, Y) = \tilde{g}(B\phi X, BY),$$

where X and Y are the arbitrary vector fields and ϕ is a tensor of type (1,1) on M^n. If the Riemannian manifolds M^{n+1} and M^n are both orientable, we can choose a unique vector field N defined along M^n such that $\tilde{g}(BX, N) = 0$ and $\tilde{g}(N, N) = 1$ for arbitrary vector field N in M^n. We call this vector field the normal vector field to the hypersurface M^n.

We now define a semi-symmetric metric connection $\tilde{\nabla}$ by ([2], [9])

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \dot{\tilde{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\eta}^{\alpha}(\tilde{Y})\tilde{X} \tag{2.1}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ tangent to M^{n+1}, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to Riemannian metric $\tilde{g}$, $\tilde{\eta}^{\alpha}$ is a 1-form, and $\tilde{\xi_{\alpha}}$ is the vector field defined by

$$\tilde{g}(\tilde{\xi_{\alpha}}, \tilde{X}) = \tilde{\eta}^{\alpha}(\tilde{X})$$

for arbitrary vector fields $\tilde{X}$ on M^{n+1}. Also

$$\tilde{g}(\tilde{\phi}\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{\phi}\tilde{Y}),$$

where $\tilde{\phi}$ is a (1,1)-tensor field.

Now, suppose that $\sum = (\tilde{\phi}, \tilde{\xi}_{\alpha}, \tilde{\eta}^{\alpha}, \tilde{g})_{\alpha \in (r)}$ is an almost r-paracontact Riemannian structure on M^{n+1}. Then every vector field $\tilde{X}$ on M^{n+1} is decomposed as

$$\tilde{X} = BX + \lambda(\tilde{X})N,$$

where λ is a 1-form on M^{n+1} and for any vector field X on M^n and normal N. Also we have $b(BX) = b(X)$, $\phi(BX) = B\phi(X)$ and $\eta^{\alpha}(BX) = \eta^{\alpha}(X)$, where b is a 1-form on M^n.

For each $\alpha \in (r)$, we have [4]

$$\tilde{\phi}BX = B\phi X + b(X)N \tag{2.2}$$

$$\tilde{\xi}_{\alpha} = B\xi_{\alpha} + a_{\alpha}N, \tag{2.3}$$

where ξ_{α} is a vector field and a_{α} is defined as

$$a_{\alpha} = m(\xi_{\alpha}) = \eta^{\alpha}(N) \tag{2.4}$$

for each $\alpha \in (r)$ on M^n. Now we defined $\tilde{\eta}^{\alpha}$ as

$$\tilde{\eta}^{\alpha}(BX) = \eta^{\alpha}(X). \tag{2.5}$$

Then we can know the following.

Theorem 2.1. *The connection induced on the hypersurface of an almost r-paracontact Riemannian manifold with a semi-symmetric metric connection with respect to the unit normal is also a semi-symmetric metric connection.*

Proof: Let $\dot{\nabla}$ be the induced connection from $\dot{\tilde{\nabla}}$ on the hypersurface M^n with respect to the unit normal N, then we have

$$\dot{\tilde{\nabla}}_{BX}BY = B\dot{\nabla}_X Y + h(X,Y)N \tag{2.6}$$

for arbitrary vector fields X and Y of M^n, where h is a second fundamental tensor of the hypersurface M^n. Let ∇ be connection induced on the hypersurface from $\tilde{\nabla}$ with respect to the normal N, then we have

$$\tilde{\nabla}_{BX}BY = B\nabla_X Y + m(X,Y)N \tag{2.7}$$

for arbitrary vector fields X and Y of M^n, where m being a tensor field of type (0,2) on the hypersurface M^n.

From equation (2.1), we have

$$\tilde{\nabla}_{BX}BY = \dot{\tilde{\nabla}}_{BX}BY + \tilde{\eta}^\alpha(BY)\tilde{\phi}BX - \tilde{g}(BX,BY)(B\xi_\alpha + a_\alpha N).$$

Using (2.5), (2.6) and (2.7) in the above equation, we get

$$B(\nabla_X Y) + m(X,Y)N = B\dot{\nabla}_X Y + h(X,Y)N + \eta^\alpha(Y)BX \tag{2.8}$$
$$-g(X,Y)(B\xi_\alpha + a_\alpha N).$$

Comparision of tangential and normal vector fields yields,

$$\nabla_X Y = \dot{\nabla}_X Y + \eta^\alpha(Y)X - g(X,Y)\xi_\alpha \tag{2.9}$$

and

$$m(X,Y) = h(X,Y) - a_\alpha g(X,Y). \tag{2.10}$$

Thus

$$\nabla_X Y - \nabla_Y X - [X,Y] = \eta^\alpha(Y)X - \eta^\alpha(X)Y. \tag{2.11}$$

Hence the connection ∇ induced on M^n is a semi-symmetric metric connection [6]. □

3. Totally geodesic and totally umbilical hypersurfaces

We define $\dot{\nabla}B$ and ∇B respectively by

$$(\dot{\nabla}B)(X,Y) = (\dot{\nabla}_X B)(Y) = \dot{\tilde{\nabla}}_{BX}BY - B(\dot{\nabla}_X Y)$$

and

$$(\nabla B)(X,Y) = (\nabla_X B)(Y) = (\tilde{\nabla}_{BX}BY) - B(\nabla_X Y),$$

where X and Y being arbitrary vector fields on M^n. Then equations (2.6) and (2.7) take the form

$$(\dot{\nabla}_X B)(Y) = h(X, Y)N$$

and

$$(\nabla_X B)(Y) = m(X, Y)N.$$

These are the Gauss equations with respect to the induced connection $\dot{\nabla}$ and ∇ respectively.

Let X_1, X_2, X_3,....,X_n be n-orthonormal vector fields, then the function

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i, X_i)$$

is called the mean curvature of M^n with respect to Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{n}\sum_{i=1}^{n} m(X_i, X_i)$$

is called the mean curvature of M^n with respect to the semi-symmetric metric connection ∇.
From these we define the followings:
Definition 3.1. The hypersurface M^n is called totally geodesic hypersurface of M^{n+1} with respect to the Riemannian connection $\dot{\nabla}$ if h vanishes.
Definition 3.2. The hypersurface M^n is called totally umbilical with respect to connection $\dot{\nabla}$ if h is proportional to the metric tensor g.

We call M^n is totally geodesic and totally umbilical with respect to semi-symmetric metric connection ∇ according as the function m vanishes and proportional to the metric g respectively.

Then we can state the following theorems:

Theorem 3.1. *In order that the mean curvature of the hypersurface M^n of an almost r-paracontact Riemannian manifold M^{n+1} with respect to the Riemannian connection $\dot{\nabla}$ coincides with that of M^n with respect to semi-symmetric metric connection ∇, it is necessary and sufficient that the vector field $\tilde{\xi_\alpha}$ is tangent to M^n.*
Proof: In view of (2.10) we have

$$m(X_i, X_i) = h(X_i, X_i) - a_\alpha g(X_i, X_i).$$

Summing up for $i = 1, 2, 3, ..., n$ and dividing by n, we obtain

$$\frac{1}{n}\sum_{i=1}^{n} m(X_i, X_i) = \frac{1}{n}\sum_{i=1}^{n} h(X_i, X_i)$$

if and only if $a_\alpha = 0$. Hence from (2.3), we have

$$\tilde{\xi_\alpha} = B\xi_\alpha.$$

Thus the vector field $\tilde{\xi_\alpha}$ is tangent to M^{n+1}, which proves the theorem. □

Corollary 3.2. *The hypersurface M^n of an almost r-paracontact Riemannian manifold M^{n+1} is totally umbilical with respect to the Riemannian connection $\dot{\nabla}$ if and only if it is also totally umbilical with respect to the semi-symmetric metric connection ∇.*

Proof: The proof follows from (2.10) easily. □

4. Gauss, Weingarten and Codazzi equations

In this section we shall obtain Weingarten equation with respect to the semi-symmetric metric connection $\tilde{\nabla}$. For the Riemannian connection $\tilde{\dot{\nabla}}$, these equations are given by

$$\tilde{\dot{\nabla}}_{BX} N = -BHX \tag{4.1}$$

for any vector field X in M^n, where h is a tensor field of type (1,1) of M^n defined by

$$g(HX, Y) = h(X, Y). \tag{4.2}$$

From equations (2.1), (2.2) and (2.4) we have

$$\tilde{\nabla}_{B\tilde{X}} N = \tilde{\dot{\nabla}}_{B\tilde{X}} N + a_\alpha BX. \tag{4.3}$$

Using (4.1) we have

$$\tilde{\nabla}_{B\tilde{X}} N = -BMX, \tag{4.4}$$

where $MX = (H - a_\alpha)X$ for any vector field X in M^n. Equation (4.4) is the Weingarten equation with respect to the semi-symmetric metric connection.

We shall find the equations of Gauss and those of Codazzi with respect the semi-symmetric metric connection. The curvature tensor with respect to semi-symmetric metric connection $\tilde{\nabla}$ of M^{n+1} is, by definition,

$$\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}. \tag{4.5}$$

Putting $\tilde{X} = BX$, $\tilde{Y} = BY$ and $\tilde{Z} = BZ$, we have

$$\tilde{R}(BX, BY)BZ = \tilde{\nabla}_{BX}\tilde{\nabla}_{BY}BZ - \tilde{\nabla}_{BY}\tilde{\nabla}_{BX}BZ - \tilde{\nabla}_{[BX,BY]}BZ$$

By virtue of (2.7), (2.11) and (4.4), we get

$$\begin{aligned}\tilde{R}(BX, BY)BZ = B\{R(X, Y)Z + m(X, Z)MY - m(Y, Z)MX\} \\ +\{(\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z)\}N \\ +\{m(\eta^\alpha(Y)X - \eta^\alpha(X)Y, Z)\}N,\end{aligned} \tag{4.6}$$

where

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the curvature tensor of the semi-symmetric metric connection ∇. We now put

$$\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{U}) = g(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{U})$$

and

$$R(X, Y, Z, U) = g(R(X, Y)Z, U).$$

Then from (4.6), we can easily show that

$$\begin{aligned}\tilde{R}(BX, BY, BZ, BU) = R(X, Y, Z, U) + m(X, Z)m(Y, U) \\ -m(Y, Z)m(X, U)\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\tilde{R}(BX, BY, BZ, U) = (\nabla_X m)(Y, Z) - (\nabla_Y m)(X, Z) \\ +m(\eta^\alpha(Y)X - \eta^\alpha(X)Y, Z).\end{aligned} \tag{4.8}$$

Equation (4.7) and (4.8) are respectively the equations of Gauss and those of Codazzi with respect to the semi-symmetric metric connection.

5. Submanifolds of codimension 2

Let M^{n+1} be an $(n+1)$-dimensional differentiable manifold of differentiability class C^∞ and M^{n-1} be an $(n-1)$-dimensional submanifold immersed in M^{n+1} by immersion τ: $M^{n-1} \to M^{n+1}$. We denote the differential $d\tau$ of the immersion τ by B, so that the vector field X in the tangent space of M^{n-1} corresponds to a vector field BX in that of M^{n+1}. Suppose that M^{n+1} is an almost r-paracontact Riemannian manifold with metric $\tilde{g}$. Then the submanifold M^{n-1} is also an almost r-paracontact Riemannian manifold with metric tensor g such that

$$\tilde{g}(B\phi X, BY) = g(\phi X, Y)$$

for any arbitrary vector fields X and Y in M^{n-1} [12]. Let the manifolds M^{n+1} and M^{n-1} be both orientable such that

$$\tilde{g}(B\phi X, N_1) = \tilde{g}(B\phi X, N_2) = \tilde{g}(N_1, N_2) = 0$$

and

$$\tilde{g}(N_1, N_1) = \tilde{g}(N_2, N_2) = 1$$

for arbitrary vector field X in M^{n-1} and two unit normals N_1 and N_2 to M^{n-1} [3].

We suppose that the enveloping manifold M^{n+1} admits a semi-symmetric metric connection $\tilde{\nabla}$ given by [4]

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \tilde{\dot{\nabla}}_{\tilde{X}}\tilde{Y} + \tilde{\eta}^{\alpha}(\tilde{Y})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Y})\tilde{\xi\alpha}$$

for arbitrary vector fields $\tilde{X}$ and $\tilde{Y}$ in M^{n+1}, $\tilde{\dot{\nabla}}$ denotes the Levi-Civita connection with respect to the Riemannian metric $\tilde{g}$ and $\tilde{\eta}^{\alpha}$ is a 1-form. Let us now put

$$\tilde{\phi}BX = B\phi X + a(X)N_1 + b(X)N_2 \tag{5.1}$$

$$\tilde{\xi}_{\alpha} = B\xi_{\alpha} + a_{\alpha}N_1 + b_{\alpha}N_2, \tag{5.2}$$

where $a(X)$ and $b(X)$ are 1-forms on M^{n-1}, ξ_{α} is a vector field in the tangent space on M^{n-1}, and a_{α}, b_{α} are functions on M^{n-1} defined by

$$\eta^{\alpha}(N_1) = a_{\alpha}, \quad \eta^{\alpha}(N_2) = b_{\alpha}. \tag{5.3}$$

Then we can prove the following.

Theorem 5.1. *The connection induced on the submanifold M^{n-1} of codimension 2 of an almost r-paracontact Riemannian manifold M^{n+1} with semi-symmetric metric connection ∇ is also a semi-symmetric metric connection.*

Proof: Let $\dot{\nabla}$ be the connection induced on the submanifolds M^{n-1} from the connection $\tilde{\dot{\nabla}}$ on the enveloping manifold with respect to unit normals N_1 and N_2, then we have [9]

$$\tilde{\dot{\nabla}}_{BX}BY = B(\dot{\nabla}_X Y) + h(X, Y)N_1 + k(X, Y)N_2 \tag{5.4}$$

for arbitrary vector fields X and Y of M^{n-1}, where h and k are second fundamental tensors of M^{n-1}. Similarly, if ∇ is the connection induced on M^{n-1} from the semi-symmetric metric connection $\tilde{\nabla}$ on M^{n+1}, we have

$$\tilde{\nabla}_{BX}BY = B(\nabla_X Y) + m(X, Y)N_1 + n(X, Y)N_2, \tag{5.5}$$

where m and n being tensor fields of type (0,2) of the submanifold M^{n-1}. In view of equation (2.1), we have

$$\tilde{\nabla}_{BX}BY = \tilde{\dot{\nabla}}_{BX}BY + \tilde{\eta}^{\alpha}(BY)(BX) - \tilde{g}(BX, BY)\tilde{\xi_{\alpha}}.$$

In view of equations (5.1), (5.2) and (5.5), we have

$$B(\nabla_X Y)+m(X,Y)N_1+n(X,Y)N_2 = B(\dot{\nabla}_X Y)+h(X,Y)N_1+k(X,Y)N_2 \tag{5.6}$$
$$-g(X,Y)(B\xi_\alpha + a_\alpha N_1 + b_\alpha N_2),$$

where $\tilde{\eta}^\alpha(BY) = \tilde{\eta}^\alpha(Y)$ and $\tilde{g}(BX, BY) = g(X,Y)$.
Comparing tangential and normal vector fields to M^{n-1}, we get

$$\nabla_X Y = \dot{\nabla}_X Y + \eta^\alpha(Y)X - g(X,Y)\xi_\alpha \tag{5.7}$$

$$m(X,Y) = h(X,Y) - a_\alpha g(X,Y), \tag*{$(5.8)_a$}$$

and

$$n(X,Y) = k(X,Y) - b_\alpha g(X,Y). \tag*{$(5.8)_b$}$$

Thus

$$\nabla_X Y - \nabla_Y X - [X,Y] = \eta^\alpha(Y)X - \eta^\alpha(X)Y. \tag{5.9}$$

Hence the connection ∇ induced on M^{n-1} is semi-symmetric metric connection [16]. □

6. Totally geodesic and totally umbilical submanifolds

Let X_1, X_2, X_3,....,X_n be $(n-1)$-orthonormal vector fields on the submanifold M^{n-1}. Then the function

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{h(X_i,X_i)+k(X_i,X_i)\}$$

is the mean curvature of M^{n-1} with respect to the Riemannian connection $\dot{\nabla}$ and

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{m(X_i,X_i)+n(X_i,X_i)\}$$

is the mean curvature of M^{n-1} with respect to ∇ [12].
Now we define the followings:
Definition 6.1. If h and k vanish separately, the submanifold M^{n-1} is called totally geodesic with respect to the Riemannian connection $\dot{\nabla}$.
Definition 6.2. The submanifold M^{n-1} is called totally umbilical with respect to the Riemannian connection $\dot{\nabla}$ if h and k are proportional to the metric g.

We call M^{n-1} is totally geodesic and totally umbilical with respect to the semi-symmetric metric connection ∇ according as the functions m and n vanish separately and are proportional to metric tensor g respectively. Then we can prove the following.

Theorem 6.1. *In order that the mean curvature of submanifold M^{n-1} of an almost r-paracontact Riemannian manifold M^{n+1} with respect to the Riemannian connection $\dot{\nabla}$ coincides with that of M^{n-1} with respect to the semi-symmetric metric connection ∇, it is necessary and sufficient that $\tilde{\xi_\alpha}$ is in the tangent space of M^{n+1}.*

Proof: In view of (5.8) we have

$$m(X_i, X_i) + n(X_i, X_i) = h(X_i, X_i) + k(X_i, X_i) - (a_\alpha + b_\alpha)g(\phi X_i, X_i).$$

Summing up for $i = 1, 2, 3, ..., n-1$ and dividing by $2(n-1)$, we get

$$\frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{m(X_i, X_i)+n(X_i, X_i)\} = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}\{h(X_i, X_i)+k(X_i, X_i)\}$$

if and only if $a_\alpha = b_\alpha = 0$, which proves our assertion. □

Theorem 6.2. *The submanifold M^{n-1} of an almost r-paracontact Riemannian manifold M^{n+1} is totally umbilical with respect to the Riemannian connection $\dot{\nabla}$ if and only if it is totally umbilical with respect to the semi-symmetric metric connection ∇.*

Proof: The proof follows easily from equations $(5.8)_a$ and $(5.8)_b$. □

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