KYUNGPOOK Math. J. 50(2010), 337-343

Affine Translation Surfaces with Constant Gaussian Curvature

YU FU* and ZHONG-HUA HOU
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P. R. China
e-mail : yu_fu@yahoo.cn *and* zhhou@dlut.edu.cn

ABSTRACT. We study affine translation surfaces in $\mathbb{R}^3$ and get a complete classification of such surfaces with constant Gauss-Kronecker curvature.

1. Introduction

A surface in $\mathbb{E}^3$ is called a translation surface if it is obtained as a graph of a function $F(x,y) = p(x)+q(y)$, where $p(x)$ and $q(y)$ are differentiable functions. It's well known that a minimal translation surface in the Euclidean space $\mathbb{E}^3$ must be a plane or a Scherk surface, which is the graph of the function $F(x,y) = \ln(\cos x/\cos y)$, the only doubly periodic minimal translation surface.

In this note, we study nondegenerate translation surfaces in affine space $\mathbb{R}^3$. This class of surfaces has been studied previously by many geometers. F. Manhart [3] classified all the nondegenerate affine minimal translation surfaces in affine space $\mathbb{R}^3$. Further treatments are due to H. F. Sun [5], who classified the nondegenerate affine translation surface with nonzero constant mean curvature in $\mathbb{R}^3$. Later on, Sun and Chen extended this into the case of hypersurfaces [6]. On the other hand, Binder [1] classified locally symmetric affine translation surfaces in $\mathbb{R}^3$. Here we give a complete classification of nondegenerate affine translation surfaces with constant Gaussian curvature in $\mathbb{R}^3$. Precisely, we will prove the following theorems.

Theorem 1.1. *Let M be a nondegenerate affine translation surface in $\mathbb{R}^3$ with vanishing Gaussian curvature. Then M is affinely equivalent to one of the graph of the following functions:*

$$z = x^2 + q(y); \tag{1.1}$$
$$z = e^x \pm y^{\frac{1}{2}}; \tag{1.2}$$
$$z = x\ln x \pm y\ln y; \tag{1.3}$$
$$z = \ln x \pm \ln y; \tag{1.4}$$
$$z = x^{\frac{3-2\lambda}{1-\lambda}} \pm y^{\frac{3-2\lambda}{5-3\lambda}}, \tag{1.5}$$

* Corresponding Author.

Received January 25, 2010; accepted March 4, 2010.
2000 Mathematics Subject Classification: Primary 53A15.
Key words and phrases: Translation surface; Affine Gauss-Kronecker curvature.

where $q(y)$ is an arbitrary function and λ is a constant satisfying $\lambda \neq 1, \frac{3}{2}, \frac{5}{3}, 2$.

Theorem 1.2. *Let M be a nondegenerate affine translation surface in $\mathbb{R}^3$ with nonzero constant Gaussian curvature. Then M is affinely equivalent to the graph given by:*

$$z = \frac{1}{x^2} + q(y), \tag{1.6}$$

where $q(y)$ satisfies $(q''')^2 = q''^{\frac{13}{4}}(aq''^{-\frac{3}{4}} + b)$ for constants a, b and $a \neq 0$.

2. Preliminaries

Concerning the following basic facts of affine differential geometry, we refer to [4]. Let $f : M \to \mathbb{R}^3$ be an immersion of a connected, orientable 2-dimensional differentiable manifold into the affine space $\mathbb{R}^3$ equipped with usual flat connection D, a parallel volume element ω, and ξ be an arbitrary local field of transversal vector to $f(M)$. Thus we have the decomposition

$$D_X(f_*Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$D_X\xi = -f_*(SX) + \tau(X)\xi. \tag{2.2}$$

Thus we have an induced affine connection ∇, a symmetric tensor h of type (0,2), a tensor S of type (1,1) and 1-form τ on M and we call h, S and τ the affine second fundamental form, the affine shape operator and the affine transversal connection form, respectively. The affine mean curvature H and the affine Gaussian curvature K are defined by

$$H = \frac{1}{2}\operatorname{Tr} S, \quad K = \det S. \tag{2.3}$$

We define a volume element θ on M by

$$\theta(X_1, X_2) = \omega(f_*(X_1), f_*(X_2), \xi),$$

for any tangent vector fields X_1, X_2 of M.

We say that f is nondegenerate if h is nondegenerate. This condition does not depend on choice of ξ. It's well known that there exists unique choice of ξ such that the corresponding induced connection ∇, the nondegenerate metric h, and the induced volume θ satisfy

(1) (∇, θ) is an equiaffine structure, that is, $\nabla\theta = 0$.

(2) θ coincides with the volume element ω_h of the nondegenerate metric h, where $\omega_h = |\det(h(X_i, X_j))|^{\frac{1}{2}}$. We call such a pair (f, ξ) a Blaschke immersion, ∇ the induced connection and h the affine metric. Condition (2) implies that $\tau = 0$.

Let $z = F(x^1, x^2)$ be a differential function on a domain G in $\mathbb{R}^3$ and consider the immersion

$$f : (x^1, x^2) \in G \mapsto (x^1, x^2, F(x^1, x^2)) \in \mathbb{R}^3.$$

We start with a tentative choice of transversal field $\xi = (0,0,1)$. Since $D_{\partial_i}\xi = 0$, we have $\tau = 0$. Denoting by ∂_i the coordinate vector field ∂/∂_i we have

$$f_*(\partial_1) = (1,0,F_1), \quad f_*(\partial_2) = (0,1,F_2),$$

where $F_j = \partial F/\partial x^j$. Thus

$$D_{\partial_i} f_*(\partial_j) = (0,0,F_{ij}) = F_{ij}\xi, \quad F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}, \tag{2.4}$$

which implies

$$\nabla_{\partial_i}\partial_j = 0, \quad h(\partial_i,\partial_j) = F_{ij}.$$

Thus the immersion is nondegenerate if and only if $\det(F_{ij}) \neq 0$. Since

$$\theta(\partial_1,\partial_2) = \det(f_*(\partial_1), f_*(\partial_2), \xi) = 1,$$

by taking $\phi = |\det(F_{ij})|^{\frac{1}{4}}$, we can find the affine normal field $\overline{\xi}$ in the form

$$\overline{\xi} = -\sum_{j,k}(F^{kj}\phi_j) f_*(\partial_k) + \phi\xi,$$

where $\phi_j = \partial\phi/\partial x_j$, (F^{ij}) is the inverse of the matrix (F_{ij}). It follows that

$$D_{\partial_i}\overline{\xi} = -\sum_{j,k}\partial_i(F^{kj}\phi_j) f_*(\partial_k), \quad S(\partial_i) = \sum_{j,k}\partial_i(F^{kj}\phi_j)\partial_k. \tag{2.5}$$

3. Proof of the theorems

Throughout this section, we assume that M is a translation surface, which is obtained by the graph of function $F(x,y) = p(x) + q(y)$ for some differential functions $p(x)$ and $q(y)$. Hence, we have

$$(F_{ij}) = (h_{ij}) = \begin{pmatrix} p''(x) & 0 \\ 0 & q''(y) \end{pmatrix}, \quad (F^{ij}) = (F_{ij})^{-1} = \begin{pmatrix} p''(x)^{-1} & 0 \\ 0 & q''(y)^{-1} \end{pmatrix},$$

and

$$\phi = |\det(F_{ij})|^{\frac{1}{4}} = |p''(x)q''(y)|^{\frac{1}{4}} \neq 0.$$

It follows from (2.4) and (2.5) that the Gaussian curvature satisfies

$$\begin{aligned} K &= \partial_1(F^{11}\phi_1)\partial_2(F^{22}\phi_2) - \partial_1(F^{22}\phi_2)\partial_2(F^{11}\phi_1) \\ &= (-\frac{7}{16}p'''^2 + \frac{1}{4}p^{(4)}p'')(-\frac{7}{16}q'''^2 + \frac{1}{4}q^{(4)}q'')(p''q'')^{-\frac{5}{2}} - \frac{1}{256}p'''^2 q'''^2 (p''q'')^{-\frac{5}{2}} \\ &= (\frac{12}{64}p'''^2 q'''^2 - \frac{7}{64}p'''^2 q'' q^{(4)} - \frac{7}{64}p'' p^{(4)} q'''^2 + \frac{1}{16}p''q''p^{(4)}q^{(4)})(p''q'')^{-\frac{5}{2}}. \end{aligned}$$

If we put $f(x) = p''(x)$, $g(y) = q''(y)$, then we have

$$64K = [f'^2(12g'^2 - 7gg'') + ff''(4gg'' - 7g'^2)](fg)^{-\frac{5}{2}}. \tag{3.1}$$

Firstly we consider the case when Gaussian curvature K vanishes identically, then

$$f'^2(12g'^2 - 7gg'') + ff''(4gg'' - 7g'^2) = 0. \tag{3.2}$$

From (3.2), it follows that f and g can be interchanged with each other. If $f'(x) = 0$, we can easily get that $p(x) = ax^2 + bx + c$, where a, b, c are constant. By applying an affine transformation, we get the graph of function in the form (1.1). If $g'(y) = 0$, after interchanging x and y, we also obtain (1.1).

From now on, we assume that $f'g' \neq 0$. From (3.2), we get

$$\frac{ff''}{f'^2} = \frac{12g'^2 - 7gg''}{7g'^2 - 4gg''} = \lambda, \tag{3.3}$$

which is equivalent to

$$ff'' = \lambda f'^2, \tag{3.4}$$

$$12g'^2 - 7gg'' = \lambda(7g'^2 - 4gg''), \tag{3.5}$$

where λ is a constant.

We consider the equation (3.4), which splits into two cases:

$$\lambda \neq 1, \quad f = (C_1x + C_2)^{\frac{1}{1-\lambda}}, \quad C_1 \in \mathbb{R} \setminus 0, \quad C_2 \in \mathbb{R}. \tag{3.6}$$

$$\lambda = 1, \quad f = C_3e^{C_4x}, \quad C_3 \in \mathbb{R} \setminus 0, \quad C_4 \in \mathbb{R}. \tag{3.7}$$

If $\lambda = 1$, (3.5) gives $5g'^2 = 3gg''$. Using (3.6), we get that

$$g = (C_5y + C_6)^{-\frac{3}{2}}, \quad C_5 \in \mathbb{R} \setminus 0, \quad C_6 \in \mathbb{R}.$$

After a further integral computation, we obtain the graph of the function in the form (1.2). Especially, if $\lambda = \frac{5}{3}$, the same graph can be obtained.

If $\lambda = 2$, $f = (C_1x + C_2)^{-1}$. Integrating twice, we get

$$p(x) = \frac{(C_1x + C_2)\ln|C_1x + C_2|}{{C_1}^2} - \frac{x}{C_1} + C_3x + C_4,$$

where both C_3 and C_4 are constant. And $2g'^2 = gg''$, similarly, we can get

$$q(y) = \frac{(D_1y + C_2)\ln|D_1y + D_2|}{{D_1}^2} - \frac{y}{D_1} + D_3y + D_4,$$

where all D_i are constant and $D_1 \neq 0$. By applying an affine transformation, we have the graph of function in the form (1.3).

If $\lambda = \frac{3}{2}$, similarly, after some integral computation, we obtain the function in the form (1.4).

In general cases, i.e., $\lambda \neq 1, \frac{3}{2}, \frac{5}{3}, 2$, by an appropriate affine transformation, equations (3.4)-(3.6) immediately yield case (1.5). This completes the proof of Theorem 1.1.

Next we assume that K is a nonzero constant. Differentiating (3.1) with respect to x and y, we get

(3.8) $(4ff'f'' - 5f'^3)(12g'^2 - 7gg'') + (2f^2f''' - 3ff'f'')(4gg'' - 7g'^2) = 0,$

(3.9) $(4gg'g'' - 5g'^3)(12f'^2 - 7ff'') + (2g^2g''' - 3gg'g'')(4ff'' - 7f'^2) = 0.$

In order to prove Theorem 1.2, we need the following lemma:

Lemma 3.1. *If $4ff'' \neq 5f'^2$ and $4gg'g'' \neq 5g'^3$, then $K = 0$.*

Proof. Under the hypothesis, $4ff'' \neq 5f'^2$ and $4gg'g'' \neq 5g'^3$, from (3.8) and (3.9) we see that there exist two constants λ and μ such that

$$
\begin{aligned}
&(3.10) & 2f^2f''' - 3ff'f'' &= \lambda(4ff'f'' - 5f'^3),\\
&(3.11) & 7gg'' - 12g'^2 &= \lambda(4gg'' - 7g'^2),\\
&(3.12) & 2g^2g''' - 3gg'g'' &= \mu(4gg'g'' - 5g'^3),\\
&(3.13) & 7ff'' - 12f'^2 &= \mu(4ff'' - 7f'^2).
\end{aligned}
$$

Differentiating (3.11) with respect to y, we get

$$(3.14) \qquad (4\lambda - 7)gg''' = (10\lambda - 17)g'g'',$$

clearly $\lambda \neq \frac{7}{4}, \lambda \neq \frac{12}{7}$. Substituting (3.11) and (3.14) into (3.12), we get

$$(3.15) \qquad \frac{2(10\lambda - 17)}{4\lambda - 7} - (4\mu + 3) + \frac{5\mu(4\lambda - 7)}{7\lambda - 12} = 0,$$

i.e.

$$(3.16) \qquad (13 - 8\lambda)(12 - 7\mu + \lambda(4\mu - 7)) = 0.$$

Similarly, differentiating (3.13) with respect to x, we can get

$$(3.17) \qquad (13 - 8\mu)(12 - 7\mu + \lambda(4\mu - 7)) = 0,$$

where $\mu \neq \frac{7}{4}, \mu \neq \frac{12}{7}$. If $\lambda = \mu = \frac{13}{8}$, (3.11) and (3.13) give $4ff'' = 5f'^2$ and $4gg'g'' = 5g'^3$, thus we have

$$(3.18) \qquad 12 - 7\mu + \lambda(4\mu - 7) = 0.$$

Substituting (3.18) into (3.11) we get $\mu g'^2 = gg''$. Then substituting (3.13) into (3.1) we can find K=0. □

Proof of Theorem 1.2. Under the hypothesis, K is a nonzero constant. From Lemma 3.1, we have either $4ff'' = 5f'^2$ or $4gg'' = 5g'^2$. Without loss of generality, we assume $4ff'' = 5f'^2$. Hence (3.1) reduces to

$$256K = [f'^2(13g'^2 - 8gg'')](fg)^{-\frac{5}{2}}. \tag{3.19}$$

Thus there exists a nonzero constant C such that

$$13g'^2 - 8gg'' = Cg^{\frac{5}{2}}. \tag{3.20}$$

If we put

$$g' = \frac{dg}{dy} = h,$$

then

$$g'' = \frac{dh}{dg}h = \frac{1}{2}\frac{dh^2}{dg}.$$

It follows from (3.20) that

$$\frac{dh^2}{dg} = \frac{13h^2}{4g} - \frac{C}{4}g^{\frac{3}{2}}. \tag{3.21}$$

Solving (3.21) gives

$$h^2 = g^{\frac{13}{4}}(ag^{-\frac{3}{4}} + b), \tag{3.22}$$

where $a = \frac{1}{3}C$ and b is a constant. Hence, by applying an affine transformation, we have the graph of the function in the form (1.6) of Theorem 1.2. This completes the proof of Theorem 1.2. □

4. A special example

In this section, we give a special example of affine translation surfaces in $\mathbb{R}^3$ with nonzero constant affine Gaussian curvature.

In view of equation (3.22), it is equivalent to

$$\frac{dg}{dy} = \pm g^{\frac{13}{8}}(ag^{-\frac{3}{4}} + b)^{\frac{1}{2}}. \tag{4.1}$$

If $b = 0$, then (4.1) becomes

$$g' = \pm a^{\frac{1}{2}} g^{\frac{5}{4}} \tag{4.2}$$

for $a > 0$. Solving (4.2) gives

$$g = \big(c \mp \frac{1}{4}a^{-\frac{1}{2}}y\big)^{-4}. \tag{4.3}$$

Integrating (4.3) twice with respect to y and by applying an affine transformation, we obtain a graph

$$z = \frac{1}{x^2} + \frac{1}{y^2}, \tag{4.4}$$

which is a special example of affine translation surfaces with nonzero constant affine Gaussian curvature.

If $b \neq 0$, we can not obtain a explicit solution of (4.1). Assume that $g^{-\frac{3}{4}} = t$, (4.1) implies that

$$\int \frac{t^{-\frac{1}{6}}}{(at+b)^{\frac{1}{2}}} dt = \int \pm\frac{3}{4} dy. \tag{4.5}$$

The left of equality (4.5) is a binomial calculous,

$$\int \frac{t^{-\frac{1}{6}}}{(at+b)^{\frac{1}{2}}} dt = \int t^{-\frac{1}{6}-\frac{1}{2}} (\frac{at+b}{t})^{-\frac{1}{2}} dt. \tag{4.6}$$

As is well known, Tchebyshev proved that this kind of integration's primary functions are not elementary functions.

Acknowledgments

We would like to thank the referee for giving us very valuable advice and comments.

References

[1] T. Binder, *Projectively flat affine surfaces*, J. Geom., **79**(2004), 31-45.

[2] W. Blaschke, Vorlesungen Uber Differentialgeometrie II, Berlin, 1923.

[3] F. Manhart, *Die affineminimalruckungfachen*, Arch. Math., **44**(1985), 547-556.

[4] K. Nomizu and T. Sasaki, Affine differential geometry: geometry of affine immersions, Cambridge University Press, 1994.

[5] H. Sun, *On affine translation surfaces of constant mean curvature*, Kumamoto J. Math., **13**(2000), 49-57.

[6] H. Sun and C. Chen, *On affine translation hypersurfaces of constant mean curvature*, Publ. Math. Debrecen, **64**(2004), 3-4, 381-390.